Adam Bowers and Nigel Kalton

An Introductory Course in Functional Analysis

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To the memory of Nigel J. Kalton
(1946-2010)

Foreword

Mathematicians are peculiar people who spend their life struggling to understand the great book of Mathematics, and find it rewarding to master a few pages of its daunting and ciphered chapters. But this book was wide open in front of Nigel Kalton, who could browse through it with no apparent effort, and share with his colleagues and students his enlightening vision.

That book is now closed and we are left with the grief, and the duty to follow Nigel's example and to keep working, no matter what. Fortunately, his mathematical legacy is accessible, which includes the last graduate course he taught in Columbia during the Academic Year 2009–2010. Adam Bowers, a post-doctoral student during that year, gathered very careful notes of all classes and agreed with Nigel that these notes would eventually result in a textbook. Fate had in store that he completed this work alone.

Adam Bowers decided that this tribute to Nigel should be co-authored by the master himself. All functional analysts should be grateful to Adam for his kind endeavour, and for the splendid textbook he provides. Indeed this book is a smooth and well-balanced introduction to functional analysis, constantly motivated by applications which make clear not only how but why the field developed. It will therefore be a perfect base for teaching a one-semester (or two) graduate course in functional analysis. A cascade falling from so high is a powerful force, and a beautiful sight. Please open this book, and enjoy.

Paris, France, January 2014 *Gilles Godefroy*

Preface

During the Spring Semester of 2010, Nigel Kalton taught what would be his final course. At the time, I was a postdoctoral fellow at the University of Missouri-Columbia and Nigel was my mentor. I sat in on the course, which was an introduction to functional analysis, because I simply enjoyed watching him lecture. No matter how well one knew a subject, Nigel Kalton could always show something new, and watching him present a subject he loved was a joy in itself.

Over the course of the semester, I took notes diligently. It had occurred to me that someday, if I had the good fortune to teach a functional analysis course of my own, Nigel Kalton's notes would make the best foundation. When I happened to mention to Nigel what I was doing, he suggested we turn the notes into a textbook. Sadly, Nigel was unexpectedly taken from us before the text was complete. Without him, this work—and indeed mathematics itself—suffers from a terrible loss.

About this book

This book, as the title suggests, is meant as an introduction to the topic of functional analysis. It is not meant to function as a reference book, but rather as a first glimpse at a vast and ever-deepening subject. The material is meant to be covered from beginning to end, and should fit comfortably into a one-semester course. The text is essentially self-contained, and all of the relevant theory is provided, usually as needed. In the cases when a complete treatment would be more of a distraction than a help, the necessary information has been moved to an appendix.

The book is designed so that a graduate student with a minimal amount of advanced mathematics can follow the course. While some experience with measure theory and complex analysis is expected, one need not be an expert, and all of the advanced theory used throughout the text can be found in an appendix.

The current text seeks to give an introduction to functional analysis that will not overwhelm the beginner. As such, we begin with a discussion of normed spaces and define a Banach space. The additional structure in a Banach space simplifies many proofs, and allows us to work in a setting which is more intuitive than is necessary for the development of the theory. Consequently, we have sacrificed some generality for the sake of the reader's comfort, and (hopefully) understanding. In Chapter 2, we meet the key examples of Banach spaces—examples which will appear again and again throughout the text. In Chapter 3, we introduce the celebrated Hahn-Banach Theorem and explore its many consequences.

Banach spaces enjoy many interesting properties as a result of having a complete norm. In Chapter 4, we investigate some of the consequences of completeness, including the Baire Category Theorem, the Open Mapping Theorem, and the Closed Graph Theorem. In Chapter 5, we relax our requirements and consider a broader class of objects known as locally convex spaces. While these spaces will lack some of the advantages of Banach spaces, considerable and interesting things can and will be said about them. After a general discussion of topological preliminaries, we consider topics such as Haar measure, extreme points, and see how the Hahn-Banach Theorem appears in this context.

The origins of functional analysis lie in attempts to solve differential equations using the ideas of linear algebra. We will glimpse these ideas in Chapter 6, where we first meet compact operators. We will continue our discussion of compact operators in Chapter 7, where we see an example of how techniques from functional analysis can be used to solve a system of differential equations, and we will encounter results which allow us to do unexpected things, such as sum the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

We conclude the course in Chapter 8, with a discussion of Banach algebras. We will meet the spectrum of an operator and see how it relates to the seemingly unrelated concept of maximal ideals of an algebra. As a final flourish, we will prove the Wiener Inversion Theorem, which provides a nontrivial result about Fourier series.

At the end of each chapter, the reader will find a collection of exercises. Many of the exercises are directly related to topics in the chapter and are meant to complement the discussion in the textbook, but some introduce new concepts and ideas and are meant to expose the reader to a broader selection of topics. The exercises come in varying degrees of difficulty. Some are very straightforward, but some are quite challenging.

It is hoped that the reader will find the material intriguing and seek to learn more. The inquisitive mind would do well with the classic text *Functional Analysis* by Walter Rudin [34], which covers the material of this text, and more. For further study, the reader might wish to peruse *A Course in Operator Theory* by John B. Conway [8] or (moving in another direction) *Topics in Banach Space Theory* by Albiac and Kalton [2].

About Nigel Kalton

Nigel Kalton was born on 20 June 1946 in Bromley, England. He studied mathematics at Trinity College Cambridge, where he took his Ph.D. in 1970. His thesis was awarded the Rayleigh Prize for research excellence. He held positions at Lehigh University, Warwick University, University College of Swansea, the University of Illinois, and Michigan State University before taking a permanent position in the mathematics department at the University of Missouri in 1979.

In 1984, Kalton was appointed the Luther Marion Defoe Distinguished Professor of Mathematics and then he was appointed to the Houchins Chair of Mathematics in 1985. In 1995, he was appointed a Curators' Professor, the highest recognition bestowed by the University of Missouri.

Among the many honors Nigel Kalton received, he was awarded the Chancellor's Award for outstanding research (at the University of Missouri) in 1984, the Weldon Springs Presidential Award for outstanding research (at the University of Missouri) in 1987, and the Banach Medal from the Polish Academy of Sciences in 2005 (the highest honor in his field).

During his career, he wrote over 270 articles and books, mentored 14 Ph.D. students, served on many editorial boards, and inspired countless mathematicians. For more information about Nigel Kalton, please visit the the Nigel Kalton Memorial Website developed by Fritz Gesztesy and hosted by the University of Missouri:

http://kaltonmemorial.missouri.edu/

A very nice tribute to Nigel Kalton appeared in the Notices of the American Mathematics Society with contributions from Peter Casazza, Joe Diestel, Gilles Godefroy, Aleksander Pełczyński, and Roman Vershynin:

A Tribute to Nigel J. Kalton (1946–2010), Peter G. Casazza, Coordinating Editor, Notices of the AMS, Vol. 59, No. 7 (2012), pp. 942–951.

The article (which is [6] in the references) is a good starting point to learn about the life and work of Nigel Kalton.

Acknowledgements

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Of course, I owe my deepest gratitude to Professor Nigel Kalton, who was both an inspiration and mentor to me. Since I cannot show him the gratitude and admiration I feel, I offer this text as my humble tribute to his memory.

I assure the reader that if there are any errors in this book, they belong to me.

La Jolla, California *Adam Bowers*
January 2014

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