# Classification of Three-Dimensional Real Lie Algebras

Adam Bowers

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## 1 Introduction

In this paper, we will classify (up to isomorphism) all real Lie algebras of three-dimensions. To that end, we will need the following proposition.

**Proposition 1** *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras of dimension $n < \infty$. Suppose each has a basis with respect to which the structure constants are the same. Then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.*

Proof. By assumption, a basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}_1$ and a basis $\{Y_1, \ldots, Y_n\}$ for $\mathfrak{g}_2$ may be chosen so that the structure constants are the same; that is,

$$[X_i, X_j] = \sum_{k=1}^{n} C_{ijk} X_k \text{ for all } i, j = 1, \ldots, n,$$

and

$$[Y_i, Y_j] = \sum_{k=1}^{n} \tilde{C}_{ijk} Y_k \text{ for all } i, j = 1, \ldots, n,$$

and

$$C_{ijk} = \tilde{C}_{ijk} \text{ for all } i, j, k = 1, \ldots, n.$$

Define a linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ by mapping basis elements to basis elements,

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$$\phi(X_i) = Y_i \text{ for all } i = 1, \ldots, n,$$

and linearly extending to $\mathfrak{g}_1$. By construction, $\phi$ is a bijection. It only remains to check that $\phi$ is a homomorphism.

By linearity, it suffices to check the homomorphism property on basis elements. Let $i, j \in \{1, \ldots, n\}$. Then

$$\phi\Big([X_i, X_j]\Big) = \phi\left(\sum_{k=1}^{n} C_{ijk} X_k\right) = \sum_{k=1}^{n} C_{ijk}\, \phi(X_k) = \sum_{k=1}^{n} C_{ijk} Y_k.$$

But the structure constants are the same, and so

$$\sum_{k=1}^{n} C_{ijk} Y_k = \sum_{k=1}^{n} \tilde{C}_{ijk} Y_k = [Y_i, Y_j] = [\phi(X_i), \phi(X_j)].$$

Therefore, the map $\phi$ is indeed a homomorphism, and hence an isomorphism. $\diamond$

Using this result, we can classify a given Lie algebra by finding a suitable basis so that its Lie bracket multiplication table will be in a "canonical" form. These canonical Lie algebras will be shown to be non-isomorphic by considering certain invariants of isomorphic mappings. One such invariant is the dimension of the *derived algebra*.

**Definition 1** *Let $\mathfrak{g}$ be a Lie algebra. The* **derived algebra** *of $\mathfrak{g}$ is the set of all linear combinations of Lie brackets in $\mathfrak{g}$. It is denoted $[\mathfrak{g}, \mathfrak{g}]$ or $\mathfrak{g}^{(1)}$.*

**Claim 2** *The derived algebra of the Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$.*

PROOF. By definition, $\mathfrak{g}^{(1)}$ is a vector space. Let $X, Y \in \mathfrak{g}^{(1)}$. Then

$$[X, Y] \in [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^{(1)},$$

so $\mathfrak{g}^{(1)}$ is closed under the Lie bracket. $\diamond$

**Corollary 3** *If two Lie algebras are isomorphic, then their derived algebras have the same dimension.*

This is the only use we will be making of derived algebras, but the idea can be extended to something much more general and useful. For more information on this topic (including some insight into the notation), see [4] or [6].

The classification presented here (which parallels that of [2], but for real Lie algebras instead of complex) makes heavy use of Corollary 3. We will refer to the dimension of the derived algebra as the *rank of the Lie bracket*, which is somewhat an abuse of terminology (see [2]).

## 2 Lie Algebras of 1 and 2 Dimensions

Although the purpose of this paper is to classify the Lie algebras of dimension three, it will be helpful to begin with those of dimensions one and two.

First, suppose $\mathfrak{g}$ is a one-dimensional Lie algebra. Then any basis will consist of one element, $e_1$. Since any element of $\mathfrak{g}$ is a multiple of $e_1$ and $[e_1, e_1] = 0$, the rank of the Lie bracket is zero. We call such a Lie algebra *abelian*. It is a trivial consequence of Proposition 1 that there is only one abelian Lie algebra of any dimension. We denote the one-dimensional abelian Lie algebra by L(1,0) (see Section 6).

Now suppose $\mathfrak{g}$ is a two-dimensional Lie algebra. Let $\mathfrak{g}$ have any basis, say $\{X_1, X_2\}$. Then

$$[X_1, X_2] = \alpha X_1 + \beta X_2 \text{ for some } \alpha, \beta \in \mathbb{R}.$$

By linearity and skew-symmetry, the non-zero structure constants are determined by $\alpha$ and $\beta$. Therefore the derived algebra can only have dimension zero or one.

If $\alpha = 0$ and $\beta = 0$ (i.e., the dimension of the derived algebra is zero), then $\mathfrak{g}$ is the unique two-dimensional abelian Lie algebra, denoted by L(2,0). Suppose $\alpha^2 + \beta^2 \neq 0$ (so the derived algebra is one-dimensional). We will show that $\mathfrak{g}$ has a basis $\{e_1, e_2\}$ such that $[e_1, e_2] = e_1$. It will follow from Proposition 1 that any two such Lie algebras are isomorphic, thus completing the classification of two-dimensional Lie algebras.

At least one of $\alpha, \beta$ is not zero. If $\alpha \neq 0$, then let

$$e_1 = X_1 + \frac{\beta}{\alpha} X_2 \quad \text{and} \quad e_2 = \frac{1}{\alpha} X_2.$$

This is a basis for $\mathfrak{g}$, and

$$\begin{aligned} [e_1, e_2] &= \left[X_1 + \tfrac{\beta}{\alpha} X_2, \tfrac{1}{\alpha} X_2\right] &&= \tfrac{1}{\alpha}\,[X_1, X_2] \\ &= \tfrac{1}{\alpha}\,(\alpha X_1 + \beta X_2) &&= e_1. \end{aligned}$$

Therefore, $[e_1, e_2] = e_1$, as required.

Now suppose $\alpha = 0$. Then $\beta \neq 0$, so let

$$e_1 = X_2 \quad \text{and} \quad e_2 = -\frac{1}{\beta}X_1.$$

This is a basis for $\mathfrak{g}$ and

$$\begin{aligned} [e_1, e_2] &= \left[X_2, -\tfrac{1}{\beta}X_1\right] &&= \tfrac{1}{\beta}[X_1, X_2] \\ &= \tfrac{1}{\beta}(\beta X_2) &&= e_1. \end{aligned}$$

So, once again, $[e_1, e_2] = e_1$, as required.

As claimed, there is only one non-abelian two-dimensional Lie algebra. This algebra will be called L(2,1).

# 3 Dimension 3, Ranks 0 and 1

Now we turn our attention to Lie algebras of three dimensions, the main topic of this paper. As indicated above, we will be considering the dimension of the derived algebra as a starting point. In this section, we consider the cases when the derived algebra is of dimension zero or one.

If the dimension of the derived algebra is zero, then the Lie algebra in question is abelian. As before, it is a trivial consequence of Proposition 1 that there is only one abelian Lie algebra of dimension three. This Lie algebra will be called L(3,0).

Now suppose the derived algebra has dimension 1. Then there exits some non-zero $X_1 \in \mathfrak{g}$ such that $\mathfrak{g}^{(1)} = \text{span}\{X_1\}$. Extend this to a basis $\{X_1, X_2, X_3\}$ for $\mathfrak{g}$. Then there exist scalars $\alpha, \beta, \gamma \in \mathbb{R}$ (not all zero) such that

$$\begin{aligned} [X_1, X_2] &= \alpha X_1, \\ [X_1, X_3] &= \beta X_1, \\ [X_2, X_3] &= \gamma X_1. \end{aligned}$$

Suppose $\alpha \neq 0$. Then construct a new basis, as follows:

$$\begin{aligned} e_1 &= X_1, \\ e_2 &= \frac{1}{\alpha}X_2, \\ e_3 &= \alpha X_3 - \beta X_2 + \gamma X_1. \end{aligned}$$

Since $\alpha \neq 0$, by assumption, this is a basis for the Lie algebra $\mathfrak{g}$. Let us calculate the Lie brackets for this basis:

$$[e_1, e_2] = \left[X_1, \tfrac{1}{\alpha}X_2\right] = X_1 = e_1,$$

$$[e_1, e_3] = [X_1, \alpha X_3 - \beta X_2 + \gamma X_1] = \alpha\beta X_1 - \beta\alpha X_1 = 0,$$

$$[e_2, e_3] = \left[\tfrac{1}{\alpha}X_2, \alpha X_3 - \beta X_2 + \gamma X_1\right] = \gamma X_1 - \gamma X_1 = 0.$$

This Lie algebra is seen to be the direct sum of two Lie algebras, L(2,1) $\oplus$ L(1,0), and will be denoted L(3,−1). (The minus indicates the Lie algebra can be decomposed as a direct sum of other Lie algebras.)

If $\beta \neq 0$, then we can switch the labels of $X_2$ and $X_3$ and we will have the same Lie brackets as before, so we might as well assume $\alpha = 0$, $\beta = 0$, and $\gamma \neq 0$. In this case let $e_1 = X_2, e_2 = X_3$, and $e_3 = \gamma X_1$. Then the Lie brackets are:

$$[e_1, e_2] = [X_2, X_3] = \gamma X_1 = e_3,$$

$$[e_1, e_3] = [X_2, \gamma X_1] = 0,$$

$$[e_2, e_3] = [X_3, \gamma X_1] = 0.$$

This Lie algebra will be denoted L(3,1).

# 4 Dimension 3, Rank 2

In the case when the dimension of $\mathfrak{g}^{(1)}$ is two, we can find a basis $\{Y, Z\}$ for the derived algebra, and extend this to a basis for $\mathfrak{g}$, say $\{X, Y, Z\}$. By assumption,

$$[Y, Z] = \alpha Y + \beta Z$$

for some scalars $\alpha, \beta \in \mathbb{R}$.

Consider the adjoint map, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, defined by

$$\mathrm{ad}_{X'}(Y') = [X', Y'], \qquad X', Y' \in \mathfrak{g}.$$

For any fixed $X' \in \mathfrak{g}$, the map $\mathrm{ad}_{X'}$ takes values only in $\mathfrak{g}^{(1)}$, so the restriction of $\mathrm{ad}_{X'}$ to the derived algebra is well-defined. In particular, we want to consider

$$\mathrm{ad}_Y : \mathfrak{g}^{(1)} \to \mathfrak{g}^{(1)},$$

where $Y$ is the element in our basis, chosen above.

With respect to our chosen basis for $\mathfrak{g}^{(1)}$, $\{Y, Z\}$, the matrix associated with $\mathrm{ad}_Y$ (restricted to $\mathfrak{g}^{(1)}$) is as follows:

$$[ad_Y]_{\{Y,Z\}} = \begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix}.$$

**Claim 4** *For any $Y \in \mathfrak{g}^{(1)}$,* $\mathrm{trace}\Big(\mathrm{ad}_Y\Big) = 0$.

Proof. By assumption, there exist $Y_1, Y_2 \in \mathfrak{g}$ such that $[Y_1, Y_2] = Y$. Thus

$$\mathrm{ad}_Y = \mathrm{ad}_{[Y_1,Y_2]}.$$

But

$$\mathrm{ad}_{[Y_1,Y_2]} = [\mathrm{ad}_{Y_1}, \mathrm{ad}_{Y_2}] = \mathrm{ad}_{Y_1}\mathrm{ad}_{Y_2} - \mathrm{ad}_{Y_2}\mathrm{ad}_{Y_1},$$

and so

$$\mathrm{trace}\Big(\mathrm{ad}_Y\Big) = \mathrm{trace}\Big(\mathrm{ad}_{Y_1}\mathrm{ad}_{Y_2}\Big) - \mathrm{trace}\Big(\mathrm{ad}_{Y_2}\mathrm{ad}_{Y_1}\Big) = 0.$$

This proves the claim. $\diamond$

Applying the above claim to

$$[\mathrm{ad}_Y]_{\{Y,Z\}} = \begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix},$$

we see that $\beta = 0$. Similarly,

$$[\mathrm{ad}_Z]_{\{Y,Z\}} = \begin{pmatrix} -\alpha & 0 \\ -\beta & 0 \end{pmatrix},$$

so that, again using the claim, $\alpha = 0$.

Having shown that both $\alpha = 0$ and $\beta = 0$, we see that $[Y, Z] = 0$. But the derived algebra is two dimensional, so the image of $\mathrm{ad}_X$ must be two-dimensional. Therefore,

$$\mathrm{ad}_X : \mathfrak{g}^{(1)} \to \mathfrak{g}^{(1)}$$

must be an isomorphism. It is the properties of this isomorphism that we will exploit for our classification.

**Proposition 5** *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two isomorphic Lie algebras of dimension 3, with Lie brackets of rank 2. Let $X_1 \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$ be chosen as in the preceding paragraph. The matrices representing $\mathrm{ad}_{X_1}$ and $\mathrm{ad}_{X_2}$ restricted to the derived algebras are similar matrices.*

PROOF. Since the two Lie algebras are isomorphic, we can find a change of basis so that they have the same Lie brackets, hence the same ad matrices. A change of basis is represented by conjugation of the ad matrix, so the matrices are similar. ◇

Because $\mathrm{ad}_X$ is an isomorphism, it must have two non-zero eigenvalues. Since similar matrices have the same eigenvalues, we can, using Proposition 5, distinguish between three separate cases:

1. $\mathrm{ad}_X$ has two distinct real eigenvalues, or
2. $\mathrm{ad}_X$ has one real eigenvalue of multiplicity two, or
3. $\mathrm{ad}_X$ has two complex conjugate eigenvalues.

For the final item in this list, we remark that, if $\mathrm{ad}_X$ has one complex eigenvalue $\lambda$, then the complex conjugate $\overline{\lambda}$ is also an eigenvalue. This is clear if we consider the characteristic polynomial, $P(x)$ for $\mathrm{ad}_X$. Any eigenvalue of $\mathrm{ad}_X$ will be a zero of the characteristic polynomial, so

$$P(\overline{\lambda}) = \overline{P(\lambda)} = \overline{0} = 0.$$

**Case 1: Two distinct real eigenvalues.**

Suppose $\mathrm{ad}_X$ has two distinct real eigenvalues, $\lambda_1$ and $\lambda_2$. Then $\mathrm{ad}_X$ is diagonalizable over $\mathbb{R}$, so

$$[\mathrm{ad}_X]_{\{Y,Z\}} \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Therefore, there exits a basis of eigenvectors for $\mathfrak{g}^{(1)}$, call it $\{e_1, e_2\}$, such that

$$[X, e_1] = \lambda_1 e_1, \qquad [X, e_2] = \lambda_2 e_2, \qquad [e_1, e_2] = 0.$$

Let $e_3 = -\dfrac{1}{\lambda_1} X$ and let $x = \dfrac{\lambda_2}{\lambda_1}$. Then

$$[e_1, e_3] = e_1, \qquad [e_2, e_3] = x e_2, \qquad [e_1, e_2] = 0. \tag{1}$$

The structure equations in (1) describe a continuously varying family of Lie algebras, which will be called L(3,2,$x$), where $x$ is the parameter appearing in (1). Observe that $x \neq 0$, because neither of the eigenvalues was 0.

**Theorem 6** *Two Lie algebras L(3,2,$x$) and L(3,2,$x'$) corresponding to two different scalars in the structure equations in (1) are isomorphic if and only if either $x = x'$ or $x = 1/x'$*

PROOF. I will prove only that L(3,2,$x$) $\cong$ L(3,2,$1/x$). The other direction, while not difficult, is a bit technical and time consuming.

Consider the Lie algebra L(3,2,$1/x$) with basis $\{e_1, e_2, e_3\}$ and structure equations

$$[e_1, e_3] = e_1, \qquad [e_2, e_3] = \frac{1}{x} e_2, \qquad [e_1, e_2] = 0,$$

where $x \in \mathbb{R}$ is nonzero. Let

$$\tilde{e_1} = e_2, \;\; \tilde{e_2} = e_1, \;\; \tilde{e_3} = x e_3.$$

Then

$$\begin{aligned} [\tilde{e_1}, \tilde{e_3}] &= [e_2, x e_3] = e_2 = \tilde{e_1}, \\ [\tilde{e_2}, \tilde{e_3}] &= [e_1, x e_3] = x e_1 = x\tilde{e_2}, \\ [\tilde{e_1}, \tilde{e_2}] &= [e_2, e_1] = 0. \end{aligned}$$

This change of basis gives the Lie brackets for the Lie algebra L(3,2,$x$), as required. ◇

**Case 2: One real eigenvalue of multiplicity two.**

Suppose $\mathrm{ad}_X$ has one real eigenvalue, $\lambda$, of multiplicity 2. If $\mathrm{ad}_X$ has two distinct eigenvectors, then we can use the same argument as in the first case and we end up with the Lie algebra L(3,2,$x = 1$). We may suppose, then, that $\mathrm{ad}_X$ does not have two distinct eigenvectors. Then

$$[\mathrm{ad}_X]_{\{Y,Z\}} \sim \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

so there exists a basis for $\mathfrak{g}^{(1)}$ of so-called *generalized eigenvectors*, call it $\{e_1, e_2\}$, such that

$$[X, e_1] = \lambda e_1, \qquad [X, e_2] = e_1 + \lambda e_2, \qquad [e_1, e_2] = 0.$$

Let $e_3 = -\frac{1}{\lambda}X$ and replace $e_1$ with $\frac{1}{\lambda}e_1$. Then

$$[e_1, e_3] = e_1, \qquad [e_2, e_3] = e_1 + e_2, \qquad [e_1, e_2] = 0. \tag{2}$$

The structure equations in (2) describe a Lie algebra which will be called L(3,3).

For more information on the theory of generalized eigenvectors and generalized eigenspaces, and how this leads to the Jordan form of a matrix, see [1] for a delightful discussion.

**Case 3: Two complex conjugate eigenvalues.**

Suppose $\mathrm{ad}_X$ has two complex conjugate eigenvalues, $\lambda$ and $\overline{\lambda}$. Then $\mathrm{ad}_X$ is diagonalizable over $\mathbb{C}$, and so

$$[\mathrm{ad}_X]_{\{Y,Z\}} \sim \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}.$$

Therefore, there exit vectors $e_1, e_2 \in \mathfrak{g}^{(1)}$ such that

$$\mathrm{ad}_X(e_1 + ie_2) = \lambda(e_1 + ie_2). \tag{3}$$

Let $\lambda = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Without loss of generality, we may assume $\mathrm{sign}(\beta) = \mathrm{sign}(\alpha)$ (otherwise, we may replace $\lambda$ with $\overline{\lambda}$ and have our desired property). Observe that $\beta \neq 0$, since $\lambda \notin \mathbb{R}$, by assumption.

By linearity of the adjoint map,

$$\mathrm{ad}_X(e_1 + ie_2) = \mathrm{ad}_X(e_1) + i\mathrm{ad}_X(e_2).$$

Since

$$\begin{aligned} \lambda(e_1 + ie_2) &= (\alpha + i\beta)e_1 + i(\alpha + i\beta)e_2 \\ &= (\alpha e_1 - \beta e_2) + i(\beta e_1 + \alpha e_2), \end{aligned}$$

we can equate real and imaginary parts in (3), whence

$$\mathrm{ad}_X(e_1) = \alpha e_1 - \beta e_2, \;\; \mathrm{ad}_X(e_2) = \beta e_1 + \alpha e_2.$$

Observe, if we replace $X$ with $\frac{1}{\beta}X$, we have

$$\mathrm{ad}_{\frac{1}{\beta}X}(e_1) = \frac{\alpha}{\beta}e_1 - e_2, \qquad \mathrm{ad}_{\frac{1}{\beta}X}(e_2) = e_1 + \frac{\alpha}{\beta}e_2.$$

Let $e_3 = -\dfrac{1}{\beta}X$ and define $x = \dfrac{\alpha}{\beta}$. Then

$$[e_1, e_3] = xe_1 - e_2, \qquad [e_2, e_3] = e_1 + xe_2, \qquad [e_1, e_2] = 0. \tag{4}$$

The structure equations in (4) describe another continuously varying family of Lie algebras, which will be called L(3,4,$x$), where $x$ is the parameter appearing in (4). Note, by construction, $x \geq 0$.

**Remark 1.** The adjoint matrix from Case 1 was diagonalizable over $\mathbb{R}$, whereas the one in Case 3 was not. We could therefore conclude that they would lead to non-isomorphic Lie algebras (by Proposition 5). If we were considering *complex* Lie algebras instead of real ones, we could not make such a distinction. In fact, the same arguments used in Case 1 would apply to Case 3, so there would be only one family of Lie algebras instead of two. Indeed, for a fixed $x$,

$$\mathrm{L}(3,\,4,\,x) \cong \mathrm{L}\Big(3, 2,\, \tfrac{x-i}{x+i}\Big).$$

To see this isomorphism, construct a new basis for L(3,4,$x$) as follows:

$$\begin{aligned} e_1' &= e_1 + ie_2, \\ e_2' &= e_1 - ie_2, \\ e_3' &= \frac{1}{x+1}e_3. \end{aligned}$$

Note that $x \in \mathbb{R}$, by construction, and so $\left|\dfrac{x-i}{x+i}\right| = 1$.

For L(3,2,$x$), we saw that distinct values of $x$ could lead to isomorphic Lie algebras. The same is not true for L(3,4,$x$), a fact made precise in the following theorem, given without proof.

**Theorem 7** *Two Lie algebras L(3,4,x) and L(3,4,x′) corresponding to two different scalars in the structure equations in (4) are isomorphic if and only if* $x = x'$.

**Remark 2.** If we had allowed the choice of complex eigenvalue with real and imaginary parts of different sign, we would have ended up with $x < 0$. In this case we would have L(3,4,$-x$) $\cong$ L(3,4,$x$).

## 5 Dimension 3, Rank 3

Finally, we consider the case where the Lie bracket has full rank, that is, the derived algebra is three-dimensional. For the moment, let the Lie algebra $\mathfrak{g}$ have as basis $\{X', Y', Z'\}$. Since the derived algebra $\mathfrak{g}^{(1)}$ is three-dimensional, each bracket $[X', Y'], [X', Z']$, and $[Y', Z']$ must be non-zero. Since each basis element appears twice, the adjoint maps $\mathrm{ad}_{X'}$, $\mathrm{ad}_{Y'}$, and $\mathrm{ad}_{Z'}$ have rank 2. Therefore, for any non-zero $H \in \mathfrak{g}$, the adjoint map $\mathrm{ad}_H$ must also have rank 2. Fix such an $H \in \mathfrak{g}$. As in the previous section, we will consider the eigenvalues of $\mathrm{ad}_H$. Either

1. $\mathrm{ad}_H$ has two real eigenvalues, or

2. $\mathrm{ad}_H$ has two distinct complex conjugate eigenvalues.

We can be certain that these are the only possibilities, because, as we saw before, if $\mathrm{ad}_H$ has one complex eigenvalue $\lambda$, then the complex conjugate $\overline{\lambda}$ is also an eigenvalue.

**Case 1: Two real eigenvalues.**

We begin by making a claim.

**Claim 8** *There is an element in $\mathfrak{g}$ such that the corresponding adjoint map has a non-zero eigenvalue.*

Proof. Let $H$ be the element we picked in $\mathfrak{g}$ before making the claim. If $\mathrm{ad}_H$ does not have a non-zero eigenvalue, then, since it's eigenvalues are real, it is nilpotent, and so there is some basis $B$ such that

$$[\mathrm{ad}_H]_B = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix}.$$

Neither $\alpha$ nor $\gamma$ is zero in this matrix, since the rank of $\mathrm{ad}_H$ is two. We see that there is some vector $H'$ such that $\mathrm{ad}_H(H') = \alpha H$. But then

$$\mathrm{ad}_{H'}(H) = -\alpha H,$$

and so $H$ is an eigenvector for $\mathrm{ad}_{H'}$ with non-zero eigenvalue. $\diamond$

By the above claim, we may assume, without loss of generality, that $H$ is such that $\mathrm{ad}_H$ has a non-zero eigenvalue. Let $\alpha$ be the non-zero eigenvalue and let $X$ be the corresponding eigenvector. Since $\mathfrak{g}^{(1)} = \mathfrak{g}$, $H$ is a commutator, whence the trace of $\mathrm{ad}_H$ is zero (see Claim 4). Then $\mathrm{ad}_H$ must have another eigenvector, $Y$, with eigenvalue $-\alpha$. So we have

$$[H, X] = \alpha X, \;\; [H, Y] = -\alpha Y.$$

It only remains to find the bracket of $X$ and $Y$. By the *Jacobi identity*,

$$\begin{aligned} [H, [X, Y]] &= -[X, [Y, H]] - [Y, [H, X]] \\ &= -[X, \alpha Y] - [Y, \alpha X] \; = \; 0. \end{aligned}$$

Since $\mathrm{ad}_H$ has rank two, $\ker(\mathrm{ad}_H) = \mathrm{span}\{H\}$. Therefore,

$$[X, Y] = \beta H,$$

for some $\beta \neq 0$. We make a slight change of basis, for aesthetic purposes:

$$X' = \frac{1}{\alpha\beta} X, \;\; Y' = Y, \;\; H' = \frac{1}{\alpha} H.$$

Now our structure equations are

$$[H', X'] = X', \;\; [H', Y'] = -Y', \;\; [X', Y'] = H'. \tag{5}$$

Let

$$e_1 = Y', \;\; e_2 = H', \;\; e_3 = 2X'.$$

Then we have our final structure equations,

$$\begin{aligned} [e_1, e_2] &= [Y', H'] &= Y' &= e_1, \\ [e_1, e_3] &= [Y', 2X'] &= -2H' &= -2e_2, \\ [e_2, e_3] &= [H', 2X'] &= 2X' &= e_3. \end{aligned}$$

This Lie algebra will be called L(3,5).

**Case 2: Two distinct complex conjugate eigenvalues.**

In this, our last case, the map we started with, $\mathrm{ad}_H$, has two distinct complex conjugate eigenvalues, $\lambda$ and $\overline{\lambda}$. By Claim 4, the trace of $\mathrm{ad}_H$ is zero, and so the eigenvalues must be purely imaginary. Let $\lambda = i\alpha$ for some non-zero $\alpha \in \mathbb{R}$. We may assume $\alpha > 0$, since otherwise we can consider the conjugate, $\overline{\lambda}$. There exists some vector $X + iY$ such that

$$\mathrm{ad}_H(X + iY) = i\alpha(X + iY).$$

Equating real and imaginary parts in this expression, we have

$$[H, X] = -\alpha Y, \;\; [H, Y] = \alpha X.$$

In a calculation similar to the one in the previous case, we find, using the Jacobi identity,

$$[X, Y] = \beta H,$$

for some $\beta \neq 0$.

Let

$$X' = \frac{1}{\sqrt{|\alpha\beta|}}X, \;\; Y' = -\frac{1}{\sqrt{|\alpha\beta|}}Y, \;\; H' = \frac{1}{\alpha}H.$$

Now, our structure equations are

$$[H', X'] = Y', \;\; [H', Y'] = -X', \;\; [X', Y'] = -\frac{\beta}{|\beta|}H'. \tag{6}$$

If $\beta > 0$ in (6), then $[X', Y'] = -H'$. Let

$$e_1 = X' - H', \;\; e_2 = Y', \;\; e_3 = X' + H'.$$

Then

$$\begin{aligned}
[e_1, e_2] &= [X' - H', Y'] &&= -H' + X' &&= e_1, \\
[e_1, e_3] &= [X' - H', X' + H'] &&= -2Y' &&= -2e_2, \\
[e_2, e_3] &= [Y', X' + H'] &&= H' + X' &&= e_3.
\end{aligned}$$

Checking back to the previous case, we see these are once again the structure equations for the Lie algebra we called L(3,5).

Now suppose $\beta < 0$ in (6). Then $[X', Y'] = H'$. Let

$$e_1 = X', \;\; e_2 = Y', \;\; e_3 = H'.$$

This time, our structure equations become

$$\begin{aligned}
[e_1, e_2] &= [X', Y'] &&= H' &&= e_3, \\
[e_1, e_3] &= [X', H'] &&= -Y' &&= -e_2, \\
[e_2, e_3] &= [Y', H'] &&= X' &&= e_1.
\end{aligned}$$

This, our final Lie algebra, will be called L(3,6).

**Remark 3.** Consider the structure equations in (6) above. The choice between $\beta > 0$ and $\beta < 0$ changed only the sign of the bracket $[X', Y']$. This simple change in sign produced two different (non-isomorphic) real Lie algebras. It may come as no surprise that these two Lie algebras, L(3,5) and L(3,6), are isomorphic as *complex* Lie algebras. One can simply multiply both $X'$ and $Y'$ by $i$ in (6) and change the sign from negative to positive.

## 6 The Winternitz Classification

The choice of canonical basis for each Lie algebra in this paper, as well as the name given to each Lie algebra, was taken from a classification scheme introduced by Dr. Pavel Winternetz, of the Département de mathématiques et de statistique, Université de Montréal, in the early 2000's. To my knowledge, his classification has never been published.[1]

Below is listed the Lie bracket multiplication table for each of the nonabelian Lie algebras discussed in this paper.

Table 1: Nonabelian Lie Algebras of Dimensions 2, 3

| | |
|---|---|
| L(2,1) | $\begin{array}{c\|cc} & e_1 & e_2 \\ \hline e_1 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{array}$ |
| L(3,1) | $\begin{array}{c\|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & 0 \\ e_2 & 0 & 0 & e_1 \\ e_3 & 0 & -e_1 & 0 \end{array}$ |

*Continued on next page.*

[1] As of 2010, this classification still has not been published. However, we refer the reader to [5] and [7].

Table 1: *Continued.*

| | | |
|---|---|---|
| L(3,2,$x$) | $\begin{array}{c\|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & e_1 \\ e_2 & 0 & 0 & xe_2 \\ e_3 & -e_1 & -xe_2 & 0 \end{array}$ | $0 < \|x\| \leq 1$ |
| L(3,3) | $\begin{array}{c\|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & e_1 \\ e_2 & 0 & 0 & e_1 + e_2 \\ e_3 & -e_1 & -e_1 - e_2 & 0 \end{array}$ | |
| L(3,4,$x$) | $\begin{array}{c\|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & xe_1 - e_2 \\ e_2 & 0 & 0 & e_1 + xe_2 \\ e_3 & -xe_1 + e_2 & -e_1 - xe_2 & 0 \end{array}$ | $x \geq 0$ |
| L(3,5) | $\begin{array}{c\|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 & -2e_2 \\ e_2 & -e_1 & 0 & e_3 \\ e_3 & 2e_2 & -e_3 & 0 \end{array}$ | |
| L(3,6) | $\begin{array}{c\|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_3 & -e_2 \\ e_2 & -e_3 & 0 & e_1 \\ e_3 & e_2 & -e_1 & 0 \end{array}$ | |
| L(3,-1) | $\begin{array}{c\|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 & 0 \\ e_2 & -e_1 & 0 & 0 \\ e_3 & 0 & 0 & 0 \end{array}$ | L(2,1) $\oplus$ L(1,1) |

## 7 Examples

In this, the final section, we look at some examples of each nonabelian three-dimensional Lie algebra isomorphism class.

### 7.-1 L(3,-1)

Consider the following basis for the vector space $\mathfrak{t}(2)$ of $2\times 2$ upper-triangular real matrices:

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

With this basis, the structure equations are

$$[e_1, e_2] = e_1, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

### 7.1 L(3,1)

For this example, we consider the vector space $\mathfrak{n}(3)$ of $3 \times 3$ strictly upper-triangular real matrices, which is the Lie algebra of the Heisenberg group. If we use the basis

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

then the structure equations for $\mathfrak{n}_3$ are

$$[e_1, e_2] = 0, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = e_1.$$

### 7.2 L(3,2,$x = -1$)

The Poincaré group, P(1,1) – sometimes denoted ISO(2) – is the group of affine transformations of $\mathbb{R}^2$ which preserve the Lorentz metric (see [3]). It is isomorphic to the group of $3 \times 3$ matrices of the form

$$\begin{pmatrix} A & \mathbf{x} \\ 0 & 1 \end{pmatrix},$$

where $A \in \mathrm{O}(1,1)$ and $\mathbf{x} \in \mathbb{R}^2$. The associated Lie algebra, $\mathfrak{p}(1,1)$ or $\mathfrak{iso}(2)$, has as a basis

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The structure equations with respect to this basis are

$$[e_1, e_2] = 0, \;\; [e_1, e_3] = e_1, \;\; [e_2, e_3] = -e_2.$$

### 7.3 L(3,3)

Consider the Lie algebra $\mathfrak{lg}(3)$ with basis

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \;\; e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \;\; e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The structure equations with respect to this basis are

$$[e_1, e_2] = 0, \;\; [e_1, e_3] = e_1, \;\; [e_2, e_3] = e_1 + e_2.$$

This Lie algebra is the so-called "lucky guess" algebra, so named because the author does not know of any standard examples of this Lie algebra in the literature and found this one by chance.

### 7.4 L(3,4,$x = 0$)

The Euclidean group, E(2), is the group of affine transformations of $\mathbb{R}^2$ which preserve the Euclidean metric (see [3]). It is isomorphic to the group of $3 \times 3$ matrices of the form

$$\begin{pmatrix} A & \mathbf{x} \\ 0 & 1 \end{pmatrix},$$

where $A \in \mathrm{O}(2)$ and $\mathbf{x} \in \mathbb{R}^2$. The associated Lie algebra, $\mathfrak{e}(2)$, has as a basis

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \;\; e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \;\; e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The structure equations with respect to this basis are

$$[e_1, e_2] = 0, \;\; [e_1, e_3] = -e_2, \;\; [e_2, e_3] = e_1.$$

### 7.5 L(3,5)

For this example, we consider the well-known Lie algebra of the special linear group, $\mathfrak{sl}(2)$, the Lie algebra of $2 \times 2$ trace-free real matrices. Let $\mathfrak{sl}(2)$ have the following basis:

$$e_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

With this basis, the structure equations are

$$[e_1, e_2] = e_1, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = 0.$$

This is not the standard basis for $\mathfrak{sl}(2)$, which is the primary example of this Lie algebra. The reasons for this choice of basis are not clear to the author; however, in the words of Professor Ian Anderson (the author's master's thesis advisor) of Utah State University, it likely is "well understood by those who understand it well."

### 7.6 L(3,6)

Finally, we consider the Lie algebra of the special orthogonal group, $\mathfrak{so}(3)$, the Lie algebra of $3 \times 3$ trace-free, *skew-symmetric* real matrices. Let $\mathfrak{so}(3)$ have the following basis:

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With this basis, the structure equations are

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1.$$

Fin

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