

Elementary Point-Set Topology

A Transition to Advanced Mathematics

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Preface

About this book

As the title indicates, this book is about topology. In particular, this book is an introduction to the basics of what is often called *point-set* topology (also known as *general* topology). However, as the subtitle suggests, this book is intended to serve another purpose as well. A primary goal of this text, in addition to introducing students to an interesting subject, is to bridge the gap between the elementary calculus sequence and more advanced mathematics courses. For this reason, the focus of the text is on learning to read and write proofs rather than providing an advanced treatment of the subject itself.

During its infancy, this book consisted of a set of notes covering the basic topics of point-set topology and proof writing. These early topics make up the content of Chapters 1, 2, 3, and selected sections from Chapters 4, 5, and 6. For many years, the notes were used in the topology classes at Seattle University. The students had completed their calculus sequence and were using the topology course to help them make the transition into courses on abstract mathematics—in particular, the analysis sequence.

The desire to make this introduction to topology intuitive and accessible to our students has led to several innovations that we feel make our approach to the subject unique. Over the years, we collected feedback from many students and found they had difficulty seeing the connection between a product of n sets as a set of ordered n -tuples and as a set of functions. So we revised Chapter 4 several times until we had it in a form accessible to most students. We stressed the fact that it is desirable to have the projection functions be continuous, which helped the students appreciate the definition of the product topology. We introduced in Chapter 5 the idea of a function splitting a set which resulted in giving simpler proofs of theorems on connectedness.

Another aspect of this text which distinguishes it from most introductory topology textbooks is the content of Chapter 7, which demonstrates applications of topological concepts to other areas of mathematics. We decided to choose applications based on a single concept: the fixed point property. The applications are quite far-reaching, and include solving differential equations and proving the Fundamental Theorem of Algebra.

Layout and style

This text contains the standard content for an introductory point-set topology class, as well as an introduction to techniques of proof writing. This means the text can be used either for a “transitions to advanced mathematics” course or a standard topology course. The additional topics, which are not generally included in introductory topology books, make this text suitable also as a supplement for a more advanced topology course.

Chapter 1 introduces some elementary concepts in logic and basic techniques of proof, some elementary set theory, and an introduction to cardinal arithmetic. In Chapter 2, topological spaces and metric spaces are defined and a brief treatment of Euclidean space is given. The motivation for the definition of a topology is based on the notion of open sets in basic calculus. The need for the definition of Hausdorff space is shown by stressing that the concept is essential to prove that the limit of a convergent sequence is unique. Continuity and homeomorphism are presented in Chapter 3. Again the definitions are based on the familiar concept of continuity in calculus. Product spaces are discussed in Chapter 4. In Chapter 5, we treat connectedness and consider the special case of connectedness on the real line, leading to the proof of the Intermediate Value Theorem. Different forms of compactness are treated in Chapter 6, where we try to make the student appreciate that compactness is the vehicle that takes us from the infinite to the finite. Using compactness in the space of real numbers, we prove some important theorems from calculus. Our aim in Chapter 7 is to give the student an appreciation of the fact that topology can serve as a powerful tool in other branches of mathematics. We present a proof of the Fundamental Theorem of Algebra using Brouwer’s Fixed-Point Theorem. We also prove Picard’s Existence Theorem for Differential Equations using the Banach Fixed-Point Theorem for Contractions.

Although in principle we would avoid introducing a term, or a topic, in a textbook unless that term, or topic, is used later in the book, we have gone against this general rule. We feel that it is extremely useful and important for the student to be given ample opportunity to provide simple proofs as often as possible. It is our experience that one of the greatest difficulties encountered by students is to write a proof, even if it does not require much more than checking that some definitions are satisfied. Therefore, we introduce a number of terms in the main text, or in the exercises, even though those terms will not be used later, simply to give the students an opportunity to write proofs that are straightforward.

Whenever possible, we stress that the concepts covered in topology are abstractions of what the student learned in calculus. We strongly feel that the student will have a better understanding of abstract notions if concrete, familiar examples are tied to them. Throughout the book, we have used both a narrative and a formal symbolic style. We believe it is important for the student to be exposed, as often as possible, to both styles. It is our hope that the student will then be at ease when reading or writing mathematical proofs, using either sentences or mathematical symbols.

How to use this book

This book could be used in a number of different courses.

- *Elementary Topology, a Transition to Abstract Mathematics*
(One quarter/semester.) Intended for students who have not been exposed to writing proofs. This course would cover Chapters 1, 2, 3, 4 and Sections 5.1, 5.3, 6.1, and 6.2. (These are the topics covered in the original set of notes at Seattle University.)
- *Introduction to Topology*
(One quarter/semester.) Intended for students who are familiar with proof writing. This course would cover Chapters 2, 3, 4, 5, Sections 6.1, 6.2, 6.4, 6.5, and selected sections from Chapter 7.
- *Introduction to Abstract Mathematics via Topology*
(Two quarters/semesters.) Intended for students whose only mathematical exposure has been the Calculus sequence, Linear Algebra, Differential Equations. This sequence would cover the whole book.
- A reference for a number of Topology courses. Since the treatment of product spaces (Chapter 4), connectedness (Chapter 5), and applications (Chapter 7) is very likely different from the standard approach of those topics in other topology texts, this book could serve as an excellent reference for a number of Topology courses.

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Contents

Preface	3
List of Figures	8
List of Symbols	10
1 Mathematical Proofs and Sets	13
1.1 Introduction to Elementary Logic	13
1.2 More Elementary Logic	17
1.3 Quantifiers	23
1.4 Methods of Mathematical Proof	29
1.5 Introduction to Elementary Set Theory	40
1.6 Cardinality	53
1.7 Cardinal Arithmetic	61
2 Topological Spaces	67
2.1 Introduction	67
2.2 Topologies	69
2.3 Bases	74
2.4 Subspaces	79
2.5 Interior, Closure, and Boundary	82
2.6 Hausdorff spaces	92
2.7 Metric Spaces	96
2.8 Euclidean Spaces	105
3 Continuous Functions	111
3.1 Review of the Function Concept	111
3.2 More on Image and Inverse Image	118
3.3 Continuous Functions	124
3.4 More on Continuous Functions	134
3.5 More on Homeomorphism	143
4 Product Spaces	149
4.1 Products of Sets	149
4.2 Product Spaces	156
4.3 More on Product Spaces	162
5 Connectedness	167

CONTENTS

7

5.1	Introduction to Connectedness	167
5.2	Products of Connected Spaces	173
5.3	Connected Subsets of the Real Line	176
6	Compactness	185
6.1	Introduction to Compactness	185
6.2	Compactness in the Space of Real Numbers	191
6.3	The Product of Compact Spaces	194
6.4	Compactness in Metric Spaces	198
6.5	More on Compactness in Metric Spaces	205
6.6	The Cantor Set	210
7	Fixed Point Theorems and Applications	217
7.1	Sperner's Lemma	217
7.2	Brouwer's Fixed Point Theorem	221
7.3	The Fundamental Theorem of Algebra	226
7.4	Function Spaces	233
7.5	Contractions	239

List of Figures

1.4.a	An increasing function	31
1.4.b	The Pigeonhole Principle	35
1.5.a	Venn diagram: subset	41
1.5.b	Venn diagram: complement of a set	44
1.5.c	Venn diagram: difference of sets	44
1.5.d	Venn diagrams: unions and intersections	46
1.5.e	Venn diagrams: De Morgan's Laws	48
1.5.f	Venn diagrams: Distributive Law	49
1.5.g	Venn diagram: symmetric difference	53
1.6.a	Relations on subsets of the unit interval	55
2.3.a	Equivalent topologies	78
2.5.a	Interior of a set	82
2.5.b	Neighborhood of a point	85
2.5.c	Boundary point	85
2.5.d	Interior point	86
2.6.a	Hausdorff space	93
2.7.a	Open ball in a metric space	99
2.7.b	Bounded set	101
2.7.c	Open ball in the space of continuous functions	104
2.8.a	Norm of a vector in \mathbb{R}^2	106
2.8.b	Illustration for the Cauchy-Schwarz Inequality	107
3.1.a	The function concept	111
3.1.b	Restriction of a function	115
3.3.a	Illustration to accompany Exercise 17 in Section 3.3	133
3.4.a	The Pasting Lemma (Part 1)	142
3.4.b	The Pasting Lemma (Part 2)	142
3.5.a	The Fixed Point Property	144
3.5.b	Regular topological space	146
3.5.c	Normal topological space	147
3.5.d	An example of a retract	148
4.1.a	Projections in \mathbb{R}^2	150
4.1.b	Illustrations for Example 4.1.4	154
4.2.a	The product topology	157

LIST OF FIGURES

9

5.1.a	A function that splits a topological space	170
5.1.b	Connected subsets and splitting functions	171
5.3.a	Connectedness for intervals	178
5.3.b	The Intermediate Value Theorem	179
5.3.c	Illustration to accompany Exercise 18 in Section 5.3	183
5.3.d	Illustration to accompany Exercise 19 in Section 5.3	183
6.1.a	A cover which admits a subcover	186
6.3.a	Illustration to accompany Exercise 8 in Section 6.3.	198
6.6.a	The Cantor Set	211
7.1.a	A graph with five vertices, four edges, and one loop.	218
7.1.b	A graph with two vertices and two edges.	219
7.1.c	Partitioning a triangle	220
7.1.d	Sperner's Lemma	221
7.2.a	Linear dependence	222
7.2.b	Linear independence	223
7.2.c	A point in a triangle	225
7.2.d	Centroid of a triangle	226
7.3.a	Complex numbers	227

List of Symbols

$\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ Collections of sets are denoted by upper case letters in script

$\mathcal{R}, \mathcal{I}, \mathcal{T}, \dots$ Common letters used for topologies

$\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$ Vectors are denoted by lower case letters in bold

\mathbb{N}	The set of natural numbers (or positive integers)
\mathbb{E}	The set of even positive integers
\mathbb{Z}	The set of integers
\mathbb{Q}	The set of rational numbers
\mathbb{R}	The set of real numbers
\mathbb{R}^+	The set of positive real numbers
\mathbb{R}^n	The set of n -dimensional vectors (ordered n -tuples) with entries in \mathbb{R}
\mathbb{C}	The set of complex numbers
\mathcal{H}	Hilbert space
\mathcal{K}	Cantor set
\emptyset	The empty set
\in	Membership of an element in a set
\subseteq	Inclusion of one set in another
\subset	Proper inclusion of one set in another
$=$	Equality
\cup	Union of sets

\cap	Intersection of sets
\forall	Universal quantifier (“for all”)
\exists	Existential quantifier (“there exists at least one”)
$\exists!$	Unique existence quantifier (“there exists a unique”)
\wedge	Conjunction (“and”)
\vee	Disjunction (“inclusive or”)
\neg	Negation
\Rightarrow	Implication
\Leftrightarrow	Double implication, equivalence (“if and only if”)
\oplus	Exclusive or
\sim	Equivalence relation
$[x]$	Equivalence class represented by x
A'	Complement of the set A
$A \setminus B$	Complement of B with respect to A (“ A without B ”)
$A \Delta B$	Symmetric difference of the sets A and B
$A \times B$	Cartesian product of the sets A and B
$\mathcal{P}(A)$	Power set of A (the collection of all subsets of A)
2^A	Power set of A (the collection of all subsets of A)
A^B	The set of functions from B to A
$ A $	Cardinality of the set A
\aleph_0	Aleph-naught, the cardinality of the set \mathbb{N}
$ A ^{ B }$	Cardinality of the set A^B
$d(A)$	Diameter of the set A
sup	Supremum of a set
inf	Infimum of a set

\mathcal{T}_d	Metric topology generated by the metric d
$n!$	n -factorial (the product of the first n positive integers)
$\binom{n}{k}$	Binomial coefficient (“ n choose k ”)
$\mathbf{x} \cdot \mathbf{y}$	Dot product (or scalar product) of the vectors \mathbf{x} and \mathbf{y}
$\ \mathbf{x}\ $	Norm (or length) of the vector \mathbf{x}

Q.E.D. *Quod erat demonstrandum* (denotes the end of a proof)