Elementary Point-Set Topology

A Transition to Advanced Mathematics

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Preface

About this book

As the title indicates, this book is about topology. In particular, this book is an introduction to the basics of what is often called *point-set* topology (also known as *general* topology). However, as the subtitle suggests, this book is intended to serve another purpose as well. A primary goal of this text, in addition to introducing students to an interesting subject, is to bridge the gap between the elementary calculus sequence and more advanced mathematics courses. For this reason, the focus of the text is on learning to read and write proofs rather than providing an advanced treatment of the subject itself.

During its infancy, this book consisted of a set of notes covering the basic topics of point-set topology and proof writing. These early topics make up the content of Chapters 1, 2, 3, and selected sections from Chapters 4, 5, and 6. For many years, the notes were used in the topology classes at Seattle University. The students had completed their calculus sequence and were using the topology course to help them make the transition into courses on abstract mathematics—in particular, the analysis sequence.

The desire to make this introduction to topology intuitive and accessible to our students has led to several innovations that we feel make our approach to the subject unique. Over the years, we collected feedback from many students and found they had difficulty seeing the connection between a product of n sets as a set of ordered n-tuples and as a set of functions. So we revised Chapter 4 several times until we had it in a form accessible to most students. We stressed the fact that it is desirable to have the projection functions be continuous, which helped the students appreciate the definition of the product topology. We introduced in Chapter 5 the idea of a function splitting a set which resulted in giving simpler proofs of theorems on connectedness.

Another aspect of this text which distinguishes it from most introductory topology textbooks is the content of Chapter 7, which demonstrates applications of topological concepts to other areas of mathematics. We decided to choose applications based on a single concept: the fixed point property. The applications are quite far-reaching, and include solving differential equations and proving the Fundamental Theorem of Algebra.

Layout and style

This text contains the standard content for an introductory point-set topology class, as well as an introduction to techniques of proof writing. This means the text can be used either for a "transitions to advanced mathematics" course or a standard topology course. The additional topics, which are not generally included in introductory topology books, make this text suitable also as a supplement for a more advanced topology course.

Chapter 1 introduces some elementary concepts in logic and basic techniques of proof, some elementary set theory, and an introduction to cardinal arithmetic. In Chapter 2, topological spaces and metric spaces are defined and a brief treatment of Euclidean space is given. The motivation for the definition of a topology is based on the notion of open sets in basic calculus. The need for the definition of Hausdorff space is shown by stressing that the concept is essential to prove that the limit of a convergent sequence is unique. Continuity and homeomorphism are presented in Chapter 3. Again the definitions are based on the familiar concept of continuity in calculus. Product spaces are discussed in Chapter 4. In Chapter 5, we treat connectedness and consider the special case of connectedness on the real line, leading to the proof of the Intermediate Value Theorem. Different forms of compactness are treated in Chapter 6, where we try to make the student appreciate that compactness is the vehicle that takes us from the infinite to the finite. Using compactness in the space of real numbers, we prove some important theorems from calculus. Our aim in Chapter 7 is to give the student an appreciation of the fact that topology can serve as a powerful tool in other branches of mathematics. We present a proof of the Fundamental Theorem of Algebra using Brouwer's Fixed-Point Theorem. We also prove Picard's Existence Theorem for Differential Equations using the Banach Fixed-Point Theorem for Contractions.

Although in principle we would avoid introducing a term, or a topic, in a textbook unless that term, or topic, is used later in the book, we have gone against this general rule. We feel that it is extremely useful and important for the student to be given ample opportunity to provide simple proofs as often as possible. It is our experience that one of the greatest difficulties encountered by students is to write a proof, even if it does not require much more that checking that some definitions are satisfied. Therefore, we introduce a number of terms in the main text, or in the exercises, even though those terms will not be used later, simply to give the students an opportunity to write proofs that are straightforward.

Whenever possible, we stress that the concepts covered in topology are abstractions of what the student learned in calculus. We strongly feel that the student will have a better understanding of abstract notions if concrete, familiar examples are tied to them. Throughout the book, we have used both a narrative and a formal symbolic style. We believe it is important for the student to be exposed, as often as possible, to both styles. It is our hope that the student will then be at ease when reading or writing mathematical proofs, using either sentences or mathematical symbols.

How to use this book

This book could be used in a number of different courses.

- *Elementary Topology, a Transition to Abstract Mathematics*
  (One quarter/semester.) Intended for students who have not been exposed to writing proofs. This course would cover Chapters 1, 2, 3, 4 and Sections 5.1, 5.3, 6.1, and 6.2. (These are the topics covered in the original set of notes at Seattle University.)

- *Introduction to Topology*
  (One quarter/semester.) Intended for students who are familiar with proof writing. This course would cover Chapters 2, 3, 4, 5, Sections 6.1, 6.2, 6.4, 6.5, and selected sections from Chapter 7.

- *Introduction to Abstract Mathematics via Topology*
  (Two quarters/semesters.) Intended for students whose only mathematical exposure has been the Calculus sequence, Linear Algebra, Differential Equations. This sequence would cover the whole book.

- A reference for a number of Topology courses. Since the treatment of product spaces (Chapter 4), connectedness (Chapter 5), and applications (Chapter 7) is very likely different from the standard approach of those topics in other topology texts, this book could serve as an excellent reference for a number of Topology courses.

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$\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$ Collections of sets are denoted by upper case letters in script

$\mathscr{R}, \mathscr{S}, \mathscr{T}, \ldots$ Common letters used for topologies

$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots$ Vectors are denoted by lower case letters in bold

$\mathbb{N}$ The set of natural numbers (or positive integers)

$\mathbb{E}$ The set of even positive integers

$\mathbb{Z}$ The set of integers

$\mathbb{Q}$ The set of rational numbers

$\mathbb{R}$ The set of real numbers

$\mathbb{R}^+$ The set of positive real numbers

$\mathbb{R}^n$ The set of n-dimensional vectors (ordered n-tuples) with entries in $\mathbb{R}$

$\mathbb{C}$ The set of complex numbers

$\mathscr{H}$ Hilbert space

$\mathscr{K}$ Cantor set

$\emptyset$ The empty set

$\in$ Membership of an element in a set

$\subseteq$ Inclusion of one set in another

$\subset$ Proper inclusion of one set in another

$=$ Equality

$\cup$ Union of sets

$\cap$ Intersection of sets

$\forall$ Universal quantifier ("for all")

$\exists$ Existential quantifier ("there exists at least one")

$\exists!$ Unique existence quantifier ("there exists a unique")

$\wedge$ Conjunction ("and")

$\vee$ Disjunction ("inclusive or")

$\neg$ Negation

$\Rightarrow$ Implication

$\Leftrightarrow$ Double implication, equivalence ("if and only if")

$\oplus$ Exclusive or

$\sim$ Equivalence relation

$[x]$ Equivalence class represented by x

A' Complement of the set A

$A \setminus B$ Complement of B with respect to A ("A without B")

$A \triangle B$ Symmetric difference of the sets A and B

$A \times B$ Cartesian product of the sets A and B

$\mathscr{P}(A)$ Power set of A (the collection of all subsets of A)

2^A Power set of A (the collection of all subsets of A)

A^B The set of functions from B to A

$|A|$ Cardinality of the set A

$\aleph_0$ Aleph-naught, the cardinality of the set $\mathbb{N}$

$|A|^{|B|}$ Cardinality of the set A^B

$d(A)$ Diameter of the set A

sup Supremum of a set

inf Infimum of a set

$\mathscr{T}_d$ Metric topology generated by the metric d

$n!$ n-factorial (the product of the first n positive integers)

$\binom{n}{k}$ Binomial coefficient ("n choose k")

$\boldsymbol{x} \cdot \boldsymbol{y}$ Dot product (or scalar product) of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$

$\|\boldsymbol{x}\|$ Norm (or length) of the vector $\boldsymbol{x}$

Q.E.D. *Quod erat demonstrandum* (denotes the end of a proof)