Correction to "Cohomological obstructions to lifting properties for full group C*-algebras of property (T) groups"

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Introduction. The proof of [ISW20, Lemma 2.6, part (1)] is incorrect as it is based on an incorrect application of [AM13, Theorem A]. The statement itself is true however, and we give a corrected proof below.

Remark 6.3 in [ISW20] asserts that [ISW20, Theorem 6.1] holds when $\mathbb{R}$ is replaced by $K = \mathbb{R}^n \times \Lambda$, for any $n \geq 0$ and countable discrete torsion-free abelian group Λ. This assertion is false. Since $\mathrm{H}^2(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \neq \{0\}$ and $\mathrm{H}^2(\mathbb{Z}/m\mathbb{Z}, \mathbb{T}) = \{0\}$, the assertion fails when $\Gamma = \mathbb{Z}/m\mathbb{Z}$, $m \geq 2$, $K = \mathbb{Z}$ and X consists of one point. Note that [ISW20, Remark 6.3] is not used anywhere in the paper.

We also note that [ISW20, Theorem 6.1] holds when $\mathbb{R}$ is replaced (in the sense explained in [ISW20, Remark 6.3]) by any locally compact Polish abelian group K, if we additionally assume that $\mathrm{H}^1(\Gamma, \mathrm{L}^0(X, \mathbb{T})) = \{0\}$. In particular, as observed by Alon Dogon, [ISW20, Corollary 6.2] holds if we replace $\mathbb{R}$ by any such K, under the additional assumption that $\mathrm{Char}(\Gamma) = \{0\}$.

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Corrected proof of [ISW20, Lemma 2.6, part (1)]. Part (1) of [ISW20, Lemma 2.6] asserts that if A is a countable discrete abelian group endowed with the trivial $\mathbb{R}$-action, then any symmetric Borel 2-cocycle $c : \mathbb{R} \times \mathbb{R} \to A$ is a Borel 2-coboundary. We justify this assertion in two ways.

First, the proof of [AM13, Proposition 72] shows that if $G = K \times V$, where K is a compact group and V is a finite dimensional real vector space, then $\mathrm{H}^p(G, A) \cong \mathrm{H}^p(K, A)$, for $p \geq 1$. Letting $G = V = \mathbb{R}$, $K = \{e\}$, $p = 2$, we get $\mathrm{H}^2(\mathbb{R}, A) = \{0\}$. This implies that c is a Borel 2-coboundary.

Second, by [Mo76a, Theorem 10], there are an extension $0 \mapsto A \xrightarrow{i} H \xrightarrow{\pi} \mathbb{R} \to 0$, where H is a Polish group and i, π are continuous homomorphisms, and a Borel section $d : \mathbb{R} \to H$ of π such that $c(x, y) = d(x)d(y)d(x + y)^{-1}$, for every $x, y \in \mathbb{R}$. Since A is endowed with the trivial $\mathbb{R}$-action and c is symmetric, H is abelian. Since A and $\mathbb{R}$ are locally compact, so is H. Thus, the structure of locally compact abelian groups implies that H admits an open subgroup of the form $L = \mathbb{R}^n \times K$, where $n \geq 0$ and K is compact. Since $\mathbb{R}$ has no nontrivial compact subgroups, $\{0\} \times K \subset A$, hence K is finite. Since $H/L = H/(\mathbb{R}^n \times K)$ is countable and H is uncountable, we get that $n \geq 1$. Let ρ denote the restriction of π to $\mathbb{R}^n \times \{e\}$. Since the homomorphism $\rho : \mathbb{R}^n \times \{e\} \to \mathbb{R}$ is continuous and has countable kernel, we get that $n = 1$ and ρ is an isomorphism of the form $\rho(x, e) = tx$, for some $t \in \mathbb{R}$. Thus, the homomorphism $\rho^{-1} : \mathbb{R} \to \mathbb{R} \times \{e\} \subset H$ is a Borel section of π. Hence, $e(x) = d(x)(\rho^{-1}(x))^{-1}$ defines a Borel map $e : \mathbb{R} \to H$ such that $e(x) \in A$ and $c(x, y) = e(x)e(y)e(x + y)^{-1}$, for every $x, y \in \mathbb{R}$. This proves that c is a Borel 2-coboundary. □

Remark. In the proof of [ISW20, Lemma 2.6, part (2)], "Then E is a locally compact second countable group and we have a short exact sequence $0 \mapsto A \mapsto E \mapsto \mathbb{R} \mapsto 0$" should instead be "Then E is a Polishable group such that the short exact sequence $0 \mapsto A \mapsto E \mapsto \mathbb{R} \mapsto 0$ consists of continuous maps, see [Mo76a, Theorem 10]". The last line before [ISW20, Corolary 6.2] should be replaced by "Finally, the proof of Theorem 6.1 only uses that $\mathrm{H}^1(\Gamma, \mathrm{L}^0(X, \mathbb{T}))$ is a countable group, which trivially holds if X consists of one point (i.e., $\mathrm{L}^0(X, \mathbb{T}) = \mathbb{T}$), leading to the following:".

References

[AM13] T. Austin and C.C. Moore: *Continuity properties of measurable group cohomology*, Math. Ann. **356** (2013), no. 3, 885-937.

[ISW20] A. Ioana, P. Spaas and M. Wiersma: *Cohomological obstructions to lifting properties for full C^*-algebras of property (T) groups*, Geom. Funct. Anal. **30** (2020), no. 5, 1402-1438.

[Mo76a] C.C. Moore: *Group extensions and cohomology for locally compact groups*, III. Trans. Amer. Math. Soc. **221** (1976), no. 1, 1-33.