Practice problems

Problem ① Let A be an infinite set and B a set with at least two elements. Prove that the set of all functions $f: A \to B$ is uncountable.

Problem ② Let A be a set of real numbers such that $\mathbb{R} \setminus A = \{x \in \mathbb{R} \mid x \notin A\}$ is countable.
Prove that $\overline{A} = \mathbb{R}$ (in other words, A is dense in $\mathbb{R}$).

Problem ③ Let A be a set of real numbers which is bounded above and does not have a maximum.
Let $\alpha = \sup A$. For every $\varepsilon > 0$, denote $A_\varepsilon = A \cap (-\infty, \alpha - \varepsilon]$.
Prove that $\lim\limits_{\varepsilon \to 0} (\sup A_\varepsilon) = \alpha$.

Problem ④ Let $\{a_n\}$ be a sequence of real numbers.
Assume that the series $\sum a_{n_k}$ converges, for any subsequence $\{a_{n_k}\}$ of $\{a_n\}$.
Prove that the series $\sum a_n$ is absolutely convergent.

Problem ⑦ Let $\{x_n\}$ a sequence and let $k \geq 0$ be an integer.
Define a new sequence $\{y_n\}$ by letting $y_n = x_{n+k}, \forall n \geq 1$.
Prove that $\{x_n\}$ is convergent if and only if $\{y_n\}$ is convergent.

Problem 8 Let $a_0, a_1, \ldots, a_{k-1} \geq 0$, for some integer $k \geq 1$.
Define the polynomial $P(X) = a_0 + a_1 X + a_2 X^2 + \cdots + a_{k-1} X^{k-1} + X^k$
Prove that $$\lim_{n \to \infty} \sqrt[n]{P(n)} = 1.$$

Problem 9 Let $\{x_n\}$ be a sequence of real numbers such that $\lim_{n \to \infty} x_n = 0$.
Prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that the series $\sum_k x_{n_k}$ is absolutely convergent.

Problem 10 Let $\{a_n\}$ be a sequence of real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$.
Assume that $a_{2^k} \leq \frac{1}{3^k}$, for all $k \geq 1$.
Prove that the series $\sum a_n$ is convergent.

Problem 11 Let X and Y be metric spaces with distance functions d_X and d_Y. Let Z be the set of all pairs (x,y), with $x \in X$ and $y \in Y$. ($Z = X \times Y$)
Define $$d_Z((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)$$
for all $(x_1,y_1) \in Z$ and $(x_2,y_2) \in Z$.
Prove that
1. d_Z is a distance function on Z.
2. If $A \subset X$ and $B \subset Y$ are open, then $A \times B = \{(x,y) \mid x \in A, y \in B\}$ is open in Z.

Problem (12) Let X and Y be metric spaces and let $f: X \rightarrow Y$ be a function which is uniformly continuous.
Let $\{x_n\}$ be a Cauchy sequence in X.
Prove that $\{f(x_n)\}$ is a Cauchy sequence in X.

Problem (13) Let $f: (0, +\infty) \rightarrow \mathbb{R}$ be a function which is uniformly continuous. Prove that

1. For any sequence $\{x_n\}$ such that $x_n > 0$ and $\lim_{n \to \infty} x_n = 0$, the sequence $\{f(x_n)\}$ is convergent.
2. Deduce that the limit $\lim_{x \to 0} f(x)$ exists.

Problem (14) Let $f: [0,1] \rightarrow \mathbb{R}$ be a continuous function.
Assume that $f(0) = -1$ and $f(1) = 0$.
Prove that there exist $x \in (0,1)$ such that $f(x) = 1 - 2x$.

Problem (15) Let X be a metric space and A, B two subsets.
Prove that

1. If A and B are compact, then $A \cup B$ is compact
2. If A and B are connected and $A \cap B \neq \emptyset$ ($A \cap B$ is nonempty), then $A \cup B$ is connected.