ARITHMETICITY OF HYPERBOLIC 3-MANIFOLDS CONTAINING INFINITELY MANY TOTALLY GEODESIC SURFACES

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In memory of Anatole Katok

Abstract. We prove that if a closed hyperbolic 3-manifold M contains infinitely many totally geodesic surfaces, then M is arithmetic.

1. Introduction

Let $\mathbf{G}$ be a connected semisimple $\mathbb{R}$-group so that $\mathbf{G}(\mathbb{R})$ has no compact factors. An irreducible lattice Γ_0 in $\mathbf{G}(\mathbb{R})$ is called *arithmetic* if there exists a connected, almost $\mathbb{Q}$-simple, $\mathbb{Q}$-group $\mathbf{F}$ and an $\mathbb{R}$-epimorphism $\varrho : \mathbf{F} \to \mathbf{G}$ such that the Lie group $(\ker \varrho)(\mathbb{R})$ is compact and Γ_0 is commensurable with $\varrho(\mathbf{F}(\mathbb{Z}))$, see [29, Ch. IX].

Margulis [28] proved the following.

Theorem A (Arithmeticity)**.** *Let $\mathbf{G}$ be a connected semisimple $\mathbb{R}$-group so that $\mathbf{G}(\mathbb{R})$ has no compact factors. Let Γ_0 be an irreducible lattice in $\mathbf{G}(\mathbb{R})$. Assume further that $\mathrm{rank}_{\mathbb{R}}\mathbf{G} \geq 2$. Then Γ_0 is arithmetic.*

Let Γ_0 and $\mathbf{G}(\mathbb{R})$ be as in Theorem A. One may reduce the proof of Theorem A to the case where $\mathbf{G}$ is a group of adjoint type defined over a finitely generated field L and $\Gamma_0 \subset \mathbf{G}(\mathsf{L})$ —indeed using *local rigidity*, one may further assume that L is a number field. The proof of Theorem A is based on applying the following supperrigidity theorem to representations obtained from different embeddings of L into local fields, which was also proved in [28].

Theorem B (Superrigidity)**.** *Let $\mathbf{G}$ be a connected semisimple $\mathbb{R}$-group. Let Γ_0 be an irreducible lattice in $\mathbf{G}(\mathbb{R})$. Assume further that $\mathrm{rank}_{\mathbb{R}}\mathbf{G} \geq 2$. Let $\mathfrak{l}$ be a local field, and let $\mathbf{H}$ be a connected, adjoint, absolutely simple $\mathfrak{l}$-group. Let $\rho : \Gamma_0 \to \mathbf{H}(\mathfrak{l})$ be a homomorphism so that*

$$\rho(\Gamma_0) \text{ is Zariski dense and is not bounded in } \mathbf{H}(\mathfrak{l}).$$

Then ρ extends uniquely to a continuous homomorphism $\tilde{\rho} : \mathbf{G}(\mathbb{R}) \to \mathbf{H}(\mathfrak{l})$.

It follows from the *weak approximation theorem* that if Γ_0 is an arithmetic group, the index of Γ_0 in $\mathrm{Comm}_{\mathbf{G}(\mathbb{R})}(\Gamma_0)$ is infinite. Margulis proved the converse also holds, see [29, Ch. IX].

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Theorem C. *Let $\mathbf{G}$ be a connected semisimple $\mathbb{R}$-group so that $\mathbf{G}(\mathbb{R})$ has no compact factors. Let Γ_0 be an irreducible lattice in $\mathbf{G}(\mathbb{R})$. Then Γ_0 is arithmetic if and only if the index of Γ_0 in $\mathrm{Comm}_{\mathbf{G}(\mathbb{R})}(\Gamma_0)$ is infinite.*

Supperrigidity and arithmeticity theorems continue to hold for certain rank one Lie groups, namely $\mathrm{Sp}(n,1)$ and F_4^{-20}, [21, 9]. However, there are examples of non-arithmetic lattices in $\mathrm{SO}(n,1)$ for all $n \geq 2$ and also in $\mathrm{SU}(n,1)$ for $n = 1, 2, 3$.

Totally geodesic surfaces and arithmeticity. The connected component of the identity in the Lie group $\mathrm{SO}(3,1)$ is isomorphic to

$$\mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{PGL}_2(\mathbb{C}).$$

Let $M = \mathbb{H}^3/\Gamma$ be a closed oriented hyperbolic 3-manifold presented as a quotient of the hyperbolic space by the action of a lattice

$$\Gamma \subset \mathrm{PGL}_2(\mathbb{C}).$$

A *totally geodesic surface* in M is a proper geodesic immersion of a closed hyperbolic surface into M —note that a totally geodesic plane is compact in the setting at hand. It is well-known and easy to see that there can be at most countably many totally geodesic surfaces in M.

Reid [32] showed that if Γ is an arithmetic group, then either M contains no totally geodesic surfaces or it contains infinitely many commensurability classes of such surfaces. There are also known examples for both of these possibilities, [26]. More recently, it was shown in [14] that a large class of non-arithmetic manifolds contain only finitely many totally geodesic surfaces.

The following theorem is the main result of this paper.

1.1. **Theorem.** *Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold. If M contains infinitely many totally geodesic surfaces, then M is arithmetic. That is: Γ is an arithmetic lattice.*

The statement in Theorem 1.1 answers affirmatively a question asked by A. Reid and C. McMullen, see [30, §8.2] and [11, Qn. 7.6].

This paper is essentially a more detailed version of [27]. More explicitly, several measurability statements were taken for granted in [27], we provide their more or less standard proofs here; moreover, this paper contains a more elaborate version of the proof of Proposition 3.1 when compared to the proof given in [27]. Overall, our goal has been to make this paper as self contained as possible.

Shortly after the appearance of [27], Bader, Fisher, Miller, and Stover [1] proved that if a finite volume hyperbolic n-manifold, $\mathbb{H}^n/\Gamma$, contains infinitely many maximal totally geodesic subspaces of dimension at least 2, then Γ is arithmetic; Theorem 1.1 is a special case. Their proof and ours both use a superrigidity theorem to prove arithmeticity, but the superrigidity theorems and their proofs are quite different.

In view of Theorem C, we get the following from Theorem 1.1. If $M = \mathbb{H}^3/\Gamma$ is a closed hyperbolic 3-manifold which contains infinitely many totally geodesic surfaces, the index of Γ in its commensurator is infinite.

As was mentioned above the arithmeticity theorem for irreducible lattices in higher rank Lie groups was proved using the superrigidity Theorem B. Similarly, Theorem 1.1 follows from Theorem 1.2 below which is a rigidity type result.

A rigidity theorem. Let L be the number field and $\mathbf{G}$ the connected, semisimple L-group of adjoint type associated to Γ, see §2 also [29, 26]. In particular, $\mathsf{L} \subset \mathbb{R}$, $\Gamma \subset \mathbf{G}(\mathsf{L})$, and $\mathbf{G}$ is $\mathbb{R}$-isomorphic to $\mathrm{PO}(3,1)$ —the connected component of the identity in the Lie group $\mathbf{G}(\mathbb{R})$ is isomorphic to $\mathrm{PGL}_2(\mathbb{C})$.

Let $\mathcal{S}$ denote the set of places of L. For every $v \in \mathcal{S}$, let L_v be the completion of L at v and let Σ_v be the set of Galois embeddings $\sigma : \mathsf{L} \to \mathsf{L}_v$.

For every $v \in \mathcal{S}$ and every $\sigma \in \Sigma_v$, we let ${}^\sigma\mathbf{G}$ denote the algebraic group defined by applying σ to the equations of $\mathbf{G}$. Let $v \in \mathcal{S}$ and $\sigma \in \Sigma_v$, then $\sigma(\Gamma) \subset {}^\sigma\mathbf{G}$ is Zariski dense.

1.2. **Theorem.** *Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold. Assume further that M contains infinitely many totally geodesic surfaces. Let L and $\mathbf{G}$ be as above.*

If $v \in \mathcal{S}$ and $\sigma \in \Sigma_v$ are so that $\sigma(\Gamma) \subset {}^\sigma\mathbf{G}(\mathsf{L}_v)$ is unbounded, then σ extends to a continuous homomorphism from $\mathbf{G}(\mathbb{R})$ to ${}^\sigma\mathbf{G}(\mathsf{L}_v)$.

Theorem 1.1 follows from Theorem 1.2. We will recall the argument from [28] in §3.8 —indeed the group $\mathbf{F}$ in the definition of an arithmetic group is the Zariski closure of Γ in the restriction of scalars group $\mathrm{R}_{\mathsf{L}/\mathbb{Q}}(\mathbf{G})$, see also [29, Ch. IX].

The proof of Theorem 1.2 is based on the study of certain Γ-equivariant measurable maps from $\partial\mathbb{H}^3 = \mathbb{S}^2$ into projective spaces —equivariant maps of this kind also play a pivotal role in the proof of the strong rigidity theorem by Mostow and the proof of the superrigidity theorem by Margulis.

Indeed the proof in [28] is based on showing that an a priori only measurable *boundary map* agrees with a rational map almost surely; this rationality is then used to find the desired continuous extension. Our strategy here is to show that if M contains infinitely many totally geodesic surfaces, a certain Γ-equivariant measurable map on $\mathbb{S}^2$ is almost surely rational, see Proposition 3.1. In §3.8 we use Proposition 3.1 to complete the proof of Theorem 1.2, see [28].

We end the introduction by mentioning that in this paper the discussion is restricted to the case of closed hyperbolic 3-manifolds; however, our method extends to the case of finite volume hyperbolic 3-manifolds. Indeed our argument rests upon investigating certain properties of a cocycle which will be introduced in §5. The extension to finite volume hyperbolic 3-manifolds requires some estimates for the growth rate of this cocycle. The desired estimates may be obtained using a similar, and simpler, version of systems of inequalities in [12].

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2. Preliminaries and notation

Let $G = \mathrm{PGL}_2(\mathbb{C})$, and let $\Gamma \subset G$ be a lattice. Let $X = G/\Gamma$, and let vol_X denote the G-invariant probability measure on X. We denote by vol_G the Haar measure on G which projects to vol_X.

Let π denote the natural projection from G to X, also set $K = \mathrm{SU}(2)/\{\pm I\}$.

We let $H = \mathrm{PGL}_2(\mathbb{R})$. For every $t \in \mathbb{R}$, set

$$a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}; \tag{2.1}$$

note that $a_t \in H$ for all $t \in \mathbb{R}$. For every $\theta \in [0, 2\pi]$, $r_\theta \in \mathrm{PGL}_2(\mathbb{R})$ denotes the rotation with angle θ.

The bundle of oriented frames over $\mathbb{H}^3 = K\backslash G$ may be identified with G. The left action of $\{a_t : t \in \mathbb{R}\}$ on G and G/Γ induces the frame flow on the frame bundles of $\mathbb{H}^3$ and M, respectively. For any $g \in G$ the image of Hg in $\mathbb{H}^3$ is a geodesic embedding of $\mathbb{H}^2$ into $\mathbb{H}^3$. In this setup, a totally geodesic surface in $M = K\backslash G/\Gamma$ lifts to a closed orbit of H in X.

2.1. The number field L and the L-group $\mathbf{G}$.

Let $\mathsf{L} \subset \mathbb{R}$ be the field generated by $\{\mathrm{tr}(\mathrm{Ad}(\gamma)) : \gamma \in \Gamma\}$. Since Γ is finitely generated, L is finitely generated.

As was mentioned in the introduction, $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to the connected component of the identity in the Lie group $\mathrm{SO}(3, 1)$. Therefore, there is a connected, semisimple L-group $\mathbf{G}$ of adjoint type, $\mathbb{R}$-isomorphic to $\mathrm{PO}(3, 1)$, so that $\mathsf{L} \subset \mathbb{R}$, $\Gamma \subset \mathbf{G}(\mathsf{L})$, see [29, Ch. IX].

In view of local rigidity of Γ, [18, Thm. 0.11], L is indeed a number field, see also [33, 35, 36]. We will refer to the pair $(\mathsf{L}, \mathbf{G})$ as the number field and the group associated to Γ, see [29, 26].

Let $\mathcal{S}$ denote the set of places of L. For every $v \in \mathcal{S}$, let L_v be the completion of L at v, and let Σ_v be the set of Galois embeddings $\sigma : \mathsf{L} \to \mathsf{L}_v$.

With this notation, we let (v_0, id) be the pair which gives rise to the lattice Γ in G —recall that the connected component of the identity in the Lie group $\mathbf{G}(\mathbb{R})$ is isomorphic to $\mathrm{PGL}_2(\mathbb{C})$.

For any $v \in \mathcal{S}$ and any $\sigma \in \Sigma_v$, we let ${}^\sigma\mathbf{G}$ denote the $\sigma(\mathsf{L})$-group defined by applying σ to the coefficients of the defining equations of $\mathbf{G}$. Let $v \in \mathcal{S}$ and $\sigma \in \Sigma_v$, then $\sigma(\Gamma) \subset {}^\sigma\mathbf{G}$ is Zariski dense.

Note that $\mathbf{G}$ is isomorphic to $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ over $\mathbb{C}$. More generally, for every $v \in \mathcal{S}$, there exists an extension $\mathfrak{l}_v/\mathsf{L}_v$ of degree at most 2 so that ${}^\sigma\mathbf{G}$

splits over $\mathfrak{l}_v$. That is:

$$^{\sigma}\mathbf{G} \text{ is isomorphic to } \mathrm{PGL}_2 \times \mathrm{PGL}_2 \text{ over } \mathfrak{l}_v. \tag{2.2}$$

We note that $\mathfrak{l}_v$ depends also on the choice of $\sigma \in \Sigma_v$. Hence, a better notation would be $\mathfrak{l}_{v,\sigma}$. However, in the sequel we will work with pairs (v, σ) separately; thus in order to simplify the notation slightly, we will suppress the dependence on σ in our notation $\mathfrak{l}_v$.

Let $\Delta \subset \Gamma$ be a non-elementary Fuchsian group; in the above notation, we have the following. Let $\mathbf{H}_\Delta$ be the Zariski closure of Δ in $\mathbf{G}$. Then the Lie group $\mathbf{H}_\Delta(\mathbb{R})$ is locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$.

For every $v \in \mathcal{S}$ and every $\sigma \in \Sigma_v$, let $^{\sigma}\mathbf{H}_\Delta \subset {}^{\sigma}\mathbf{G}$ denote the $\sigma(\mathsf{L})$-group obtained by applying σ to the defining coefficients of $\mathbf{H}_\Delta$ —note that $^{\sigma}\mathbf{H}_\Delta$ is the Zariski closure of $\sigma(\Delta)$ in $^{\sigma}\mathbf{G}$.

2.2. **Lemma.** *Let $\Delta \subset \Gamma$ be a non-elementary Fuchsian group. Assume that $\sigma(\Delta)$ is unbounded in $^{\sigma}\mathbf{G}(\mathfrak{l}_v) = \mathrm{PGL}_2(\mathfrak{l}_v) \times \mathrm{PGL}_2(\mathfrak{l}_v)$. Then there exists some $g \in \mathrm{PGL}_2(\mathfrak{l}_v)$ so that*

$$^{\sigma}\mathbf{H}_\Delta(\mathfrak{l}_v) \cap \{(h, ghg^{-1}) : h \in \mathrm{PGL}_2(\mathfrak{l}_v)\}$$

is a subgroup of index at most 8 in $^{\sigma}\mathbf{H}_\Delta(\mathfrak{l}_v)$.

Proof. Since Δ is fixed throughout the proof, we omit it from the index in our notation $\mathbf{H}_\Delta$ and $^{\sigma}\mathbf{H}_\Delta$. The group $^{\sigma}\mathbf{H}$ is isogeneous to SL_2, in particular, it is a proper algebraic subgroup of $^{\sigma}\mathbf{G}$. Moreover, $^{\sigma}\mathbf{H}$ intersects each factor of $^{\sigma}\mathbf{G}$ trivially. Indeed, $^{\sigma}\mathbf{G}$ has trivial center; therefore, if $^{\sigma}\mathbf{H}$ intersects a factor non-trivially, then this intersection is normal in $^{\sigma}\mathbf{H}$ which implies that $^{\sigma}\mathbf{H} \lhd {}^{\sigma}\mathbf{G}$. Now since $\mathbf{H}(\mathsf{L})$ and $\mathbf{G}(\mathsf{L})$ are Zariski dense in $\mathbf{H}$ and $\mathbf{G}$, respectively, we get that $\mathbf{H} \lhd \mathbf{G}$. This is a contradiction as the Zariski closure of Δ in G is not a normal subgroup.

In consequence, there exists an $\mathfrak{l}_v$-group $\mathbf{L}$ which is an $\mathfrak{l}_v$-form of SL_2, i.e., either arising from a division algebra or SL_2, and an $\mathfrak{l}_v$-isogeny $\varphi : \mathbf{L} \to {}^{\sigma}\mathbf{H}$, so that $^{\sigma}\mathbf{H}(\mathfrak{l}_v)$ contains $\varphi(\mathbf{L}(\mathfrak{l}_v))$ as a subgroup of finite index. We claim $\mathbf{L}$ is indeed SL_2. The group $\sigma(\Delta)$ is unbounded, therefore, the group $^{\sigma}\mathbf{H}(\mathfrak{l}_v)$ is unbounded. Hence, $^{\sigma}\mathbf{H}$ is $\mathfrak{l}_v$-isotropic, this implies the claim.

Since $^{\sigma}\mathbf{H}$ is a group of type A_1, the isogeny φ is inner, i.e., it is induced by conjugation with an element $g \in \mathrm{PGL}_2(\mathfrak{l}_v)$.

The claim regarding the index follows as $[\mathrm{PGL}_2(\mathfrak{l}) : \mathrm{SL}_2(\mathfrak{l})] \le \mathfrak{l}^\times/(\mathfrak{l}^\times)^2 = 8$ for any local field $\mathfrak{l}$ of characteristic zero, see e.g. [5, §2] $\square$

2.3. **Lemmas from hyperbolic geometry.** In this section we will recrod some basic facts from hyperbolic geometry which will be used in the sequel.

Let $(\mathcal{T}, \mathrm{d}_\mathcal{T})$ denote either a regular tree equipped with the usual path metric or a hyperbolic space equipped with the hyperbolic metric.

We fix a base point $o \in \mathcal{T}$. Recall that the Gromov product of points $p, q \in \mathcal{T}$ with respect to o is defined by

$$(p|q)_o = \frac{1}{2}\big(\mathrm{d}_\mathcal{T}(o,p) + \mathrm{d}_\mathcal{T}(o,q) - \mathrm{d}_\mathcal{T}(p,q)\big)$$

We use the usual topology on $\overline{\mathcal{T}} = \mathcal{T} \bigcup \partial\mathcal{T}$; in the case at hand $\overline{\mathcal{T}}$ is a compact metrizable space, see [6, III.H, §3] in particular [6, III.H, 3.18].

Let us begin by recalling the hyperbolic law of cosines. Let pqr be a hyperbolic triangle and let θ be the angle opposite to the edge $\overline{qr}$. Then

$$\cosh(\overline{qr}) = \cosh(\overline{pq})\cosh(\overline{pr}) - \cos(\theta)\sinh(\overline{pq})\sinh(\overline{pr}) \tag{2.3}$$

here and in what follows, we abuse the notation, and use $\overline{qr}$ to denote the length of $\overline{qr}$ as well, see e.g. [6, II, Prop. 10.8].

If $\mathcal{T}$ is a tree and pqr is a tripod (triangles in this case are tripods), then $(q|r)_p$ counts the number of edges in $\overline{pq} \cap \overline{pr}$ and we have

$$\overline{qr} = \overline{pq} + \overline{pr} - 2(q|r)_p. \tag{2.4}$$

We begin with the following lemma.

2.4. **Lemma.** *Let $\{p_n : n = 0, 1, \ldots\} \subset \mathcal{T}$ with $p_0 = o$. Assume that there exist some $L_1, L_2, N_0 > 1$ so that $\mathrm{d}_{\mathcal{T}}(p_n, p_{(n+1)}) \leq L_1$ for all n and*

$$\mathrm{d}_{\mathcal{T}}(p_n, o) \geq n/L_2 \quad \textit{for all } n > N_0.$$

Then there exists some $\xi_\infty \in \partial\mathcal{T}$ so that $p_n \to \xi_\infty$.

Proof. This is a special case of [10, Thm. 4.4] applied with $\mathbb{Z}$ and $\mathcal{T}$. The proof in loc. cit. actually provides a more quantitative estimate, namely one has: if n and $k \leq n$ are large enough, then $(p_k|p_n)_o > \frac{k}{2L_2} - O(1)$. □

We will also need the following lemma, whose proof uses Lemma 2.4 and Egorov's theorem.

2.5. **Lemma.** *Let (Θ, ϑ) be a Borel probability space. Let $\psi : \Theta \to \partial\mathcal{T}$ and $u : \mathbb{Z}^{\geq 0} \times \Theta \to \mathcal{T}$ be two Borel maps satisfying the following.*

1. *$\vartheta(\psi^{-1}\{p\}) = 0$ for every $p \in \partial\mathcal{T}$.*
2. *$u(0, \theta) = o$ for a.e. $\theta \in \Theta$.*
3. *There exists some $L_1 \geq 1$ so that for a.e. $\theta \in \Theta$ we have*
$$\mathrm{d}_{\mathcal{T}}\big(u(n, \theta), u(n+1, \theta)\big) \leq L_1 \quad \textit{for all } n.$$
4. *There exists some $L_2 \geq 1$ and for a.e. $\theta \in \Theta$ there exists some N_θ so that*
$$\mathrm{d}_{\mathcal{T}}\big(u(n, \theta), o\big) \geq n/L_2 \quad \textit{for all } n > N_\theta.$$

In particular, by Lemma 2.4 we have $\{u(n, \theta)\}$ converges to a point in $\partial\mathcal{T}$ for a.e. $\theta \in \Theta$. Assume further that

$$u(n, \theta) \to \psi(\theta) \quad \textit{for a.e. } \theta \in \Theta. \tag{2.5}$$

Let $\{\xi_t\} \subset \mathcal{T}$ be any geodesic with $\xi_0 = o$. There exist some positive constant $c' = c'(\psi, u, L_1, L_2)$ and some $N_0 = N_0(u, L_1, L_2) \in \mathbb{N}$ so that for all $t > 0$ and all $n > N_0$ we have

$$\int_\Theta \mathrm{d}_{\mathcal{T}}(u(n, \theta), \xi_t)\, \mathrm{d}\vartheta(\theta) > t + \frac{n}{4L_2} - c'.$$

Proof. We explicate the proof when $\mathcal{T}$ is a hyperbolic space, the proof in the case of a tree is a simple modification.

For any $\theta \in \Theta$, let $q_\theta = u(n,\theta)$. By the hyperbolic law of cosines, (2.3), we have

$$\cosh(\overline{q_\theta \xi_t}) = \cosh(\overline{o\xi_t})\cosh(\overline{oq_\theta}) - \cos(\alpha)\sinh(\overline{o\xi_t})\sinh(\overline{oq_\theta}) \tag{2.6}$$

where α is the angle between $\overline{o\xi_t}$ and $\overline{oq_\theta}$ —if $\mathcal{T}$ is a tree, (2.6) gets the form (2.4).

Fix some $0 < \varepsilon < 1/(8L_1L_2)$. In view of our assumption (1) and the compactness of $\partial\mathcal{T}$, there exists some $\delta > 0$ so that for every $p \in \partial\mathcal{T}$

$$\vartheta\big(\psi^{-1}(\mathsf{N}_\delta(p))\big) < \varepsilon/2. \tag{2.7}$$

By Egorov's theorem and (2.5), there exists $\Theta' \subset \Theta$ with $\vartheta(\Theta') > 1 - \varepsilon/2$ and some $N_0 \geq 1$ so that for all $\theta \in \Theta'$ and all $n > N_0$, we have

$$\mathrm{d}_{\mathcal{T}}(u(n,\theta), o) \geq \tfrac{n}{2L_2}. \tag{2.8}$$

Let $\mathsf{E} := \{\theta \in \Theta' : \psi(\theta) \notin \mathsf{N}_\delta(\xi_\infty)\}$ where $\xi_\infty = \lim_{t\to\infty} \xi_t \in \partial\mathcal{T}$. Then thanks to (2.7) and $\vartheta(\Theta') > 1 - \varepsilon/2$ we have $\vartheta(\mathsf{E}) > 1 - \varepsilon$.

Now using (2.6) and (2.8), we conclude that

$$\mathrm{d}_{\mathcal{T}}(q_\theta, \xi_t) > t + \tfrac{n}{2L_2} - O_\delta(1) \tag{2.9}$$

for all $\theta \in \mathsf{E}$ and all $n > N_0$.

Note also that by our assumption (3), we have $\mathrm{d}_{\mathcal{T}}(u(n,\theta), o) \leq L_1 n$ for a.e. $\theta \in \Theta$. In particular, we get that

$$\mathrm{d}_{\mathcal{T}}(q_\theta, \xi_t) \geq t - L_1 n \tag{2.10}$$

for a.e. $\theta \in \Theta$ and all $n \in \mathbb{N}$.

Now, splitting the integral over E and its compliment and using the estimates in (2.9) and (2.10), we obtain the following:

$$\begin{aligned}
\int_\Theta \mathrm{d}_{\mathcal{T}}(q_\theta, \xi_t)\,\mathrm{d}\vartheta(\theta) &> (1-\varepsilon)(t + \tfrac{n}{2L_2} - O_\delta(1)) + (t - L_1 n)\varepsilon \\
{\scriptstyle \varepsilon<1/4\rightsquigarrow} &> t + \frac{n}{4L_2} + (\frac{n}{8L_2} - L_1\varepsilon n) - O_\delta(1) \\
{\scriptstyle \varepsilon<1/8L_1L_2\rightsquigarrow} &> t + \frac{n}{4L_2} - O_\delta(1).
\end{aligned}$$

This completes the proof if we let $c' = O_\delta(1)$. $\square$

It is worth mentioning that the above lemmas hold for any proper, complete, CAT(-1) space.

2.6. **Action of Γ on varieties.** Let $\mathcal{C}$ be the space of circles in $\mathbb{S}^2 = \partial\mathbb{H}^3$; the space $\mathcal{C}$ is equipped with a natural $\mathrm{PGL}_2(\mathbb{C})$-invariant measure σ. Let m denote the Lebesgue measure on $\mathbb{S}^2$. For every $C \in \mathcal{C}$, let m_C be the Lebesgue measure on C.

Here and in what follows, by a *non-atomic* measure we mean a measure without any atoms.

The following general lemma will be used in the sequel.

2.7. **Lemma.** *Let $\mathfrak{l}$ be a local field. Let $\mathbf{H}$ be a connected $\mathfrak{l}$-group which acts $\mathfrak{l}$-rationally on an irreducible $\mathfrak{l}$-variety $\mathbf{V}$. Assume further that $\mathbf{H}$ acts transitively on $\mathbf{V}$. Let $\rho : \Gamma \to \mathbf{H}(\mathfrak{l})$ be a homomorphism so that $\rho(\Gamma)$ is Zariski dense in $\mathbf{H}$ and let $\Psi : \mathbb{S}^2 \to \mathbf{V}(\mathfrak{l})$ be a ρ-equivariant measurable map. Then*

(1) $\mathsf{m}\big(\{\xi \in \mathbb{S}^2 : \Psi(\xi) \in \mathbf{W}(\mathfrak{l})\}\big) = 0$ *for any proper subvariety* $\mathbf{W} \subset \mathbf{V}$*.*
(2) *For σ-a.e. $C \in \mathcal{C}$ and every $p \in \mathbf{V}(\mathfrak{l})$ we have*

$$\mathsf{m}_C\big(\{\xi \in C : \Psi(\xi) = p\}\big) = 0.$$

Proof. Part (1) is well-known, see [20, Lemma 4.2] also [29, Ch. VI, Lemma 3.10] and [17]. We explicate the argument for the convenience of the reader.

Suppose the claim in part (1) fails. For every $d \geq 0$, let Σ_d denote the collection of irreducible subvarieties $\mathbf{W} \subset \mathbf{V}$ of dimension d so that $\mathsf{m}\big(\Psi_{\mathbf{W}}\big) > 0$ where $\Psi_{\mathbf{W}} = \{\xi \in \mathbb{S}^2 : \Psi(\xi) \in \mathbf{W}(\mathfrak{l})\}$.

By our assumption, there exists some $0 \leq d < \dim(\mathbf{V})$ so that $\Sigma_d \neq \emptyset$. Let d_0 be the smallest d so that $\Sigma_{d_0} \neq \emptyset$.

We claim that for every $\varepsilon > 0$, there are at most finitely many $\mathbf{W} \in \Sigma_{d_0}$ with $\mathsf{m}\big(\Psi_{\mathbf{W}}\big) \geq \varepsilon$. To see this note that since Σ_{d_0} consists of irreducible subvarieties, $\dim(\mathbf{W} \cap \mathbf{W}') < d_0$ for all $\mathbf{W} \neq \mathbf{W}' \in \Sigma_{d_0}$. In view of the choice of d_0, thus $\mathsf{m}\big(\Psi_{\mathbf{W}\cap\mathbf{W}'}\big) = 0$. This and the fact that m is a finite measure imply the claim.

Let $a = \sup\big\{\mathsf{m}(\Psi_{\mathbf{W}}) : \mathbf{W} \in \Sigma_{d_0}\big\}$. In view of the above, there exists some $\mathbf{W}_0 \in \Sigma_{d_0}$ so that $\mathsf{m}\big(\Psi_{\mathbf{W}_0}\big) = a$.

By a Theorem of Furstenberg, see e.g., [29, Ch. VI, Prop. 4.1], there exists a probability measure ν with $\mathrm{supp}(\nu) = \Gamma$, so that $\nu * \mathsf{m} = \mathsf{m}$. That is: $\mathsf{m} = \int \gamma_* \mathsf{m}\,\mathrm{d}\nu$.

Recall that Ψ is ρ-equivariant, hence

$$\gamma\{\xi \in \mathbb{S}^2 : \Psi(\xi) \in \mathbf{W}(\mathfrak{l})\} = \big\{\xi \in \mathbb{S}^2 : \Psi(\xi) \in \big(\rho(\gamma)\mathbf{W}\big)(\mathfrak{l})\big\}.$$

This and $\nu * \mathsf{m} = \mathsf{m}$ imply

$$a = \int \mathsf{m}\big(\gamma^{-1}\Psi_{\mathbf{W}_0}\big)\,\mathrm{d}\nu(\gamma) = \int \mathsf{m}\big(\Psi_{\rho(\gamma^{-1})\mathbf{W}_0}\big)\,\mathrm{d}\nu(\gamma).$$

In view of the definition of a, we thus have $\mathsf{m}\big(\Psi_{\rho(\gamma^{-1})\mathbf{W}_0}\big) = a$ for all $\gamma \in \Gamma$ (recall that $\mathrm{supp}(\nu) = \Gamma$). By our claim above, applied with $\varepsilon = a$, we thus conclude that

$$\{\gamma \in \Gamma : \rho(\gamma)\mathbf{W}_0 = \mathbf{W}_0\}$$

is finite index in Γ.

Since $\mathbf{H}$ is connected and $\rho(\Gamma)$ is Zariski dense in $\mathbf{H}$, we conclude that $\mathbf{W}_0$ is invariant under $\mathbf{H}$. This contradicts transitivity of the action of $\mathbf{H}$ on $\mathbf{V}$ and finishes the proof of part (1).

We prove part (2). Let $\mathcal{A} = \{(C, \xi, \xi') \in \mathcal{C} \times \mathbb{S}^2 \times \mathbb{S}^2 : \xi, \xi' \in C\}$. Equip $\mathcal{A}$ with the natural measure arising from the G-invariant measure σ on $\mathcal{C}$ and the measure m_C on $C \in \mathcal{C}$.

Define $\Phi : \mathcal{A} \to \mathbf{V}(\mathfrak{l}) \times \mathbf{V}(\mathfrak{l})$ by $\Phi(C, \xi, \xi') = \big(\Psi(\xi), \Psi(\xi')\big)$. Assume the claim in part (2) fails. Then

$$\mathcal{B} := \Phi^{-1}(\{(p,p) : p \in \mathbf{V}(\mathfrak{l})\}) \subset \mathcal{A}$$

has positive measure.

In view of Fubini's theorem then the projection of $\mathcal{B}$ onto $\mathbb{S}^2 \times \mathbb{S}^2$ contains a positive measure subset. That is: there exists a positive measure subset of $\mathbb{S}^2 \times \mathbb{S}^2$ which gets mapped into $\{(p,p) : p \in \mathbf{V}(\mathfrak{l})\}$. Thus using Fubini's theorem again, there exists some $\xi' \in \mathbb{S}^2$ so that $\{\xi \in \mathbb{S}^2 : \Psi(\xi) = \Psi(\xi')\}$ has positive measure, i.e., Ψ maps a positive measure subset of $\mathbb{S}^2$ to a point. This contradicts part (1) and finishes the proof. $\square$

Recall from §2 that for any $C \in \mathcal{C}$, the convex hull of C in $\mathbb{H}^3$ corresponds to a coset Hg for some $g \in G$. For every $E \subset G$, put

$$\mathcal{C}_E = \{C \in \mathcal{C} : \exists\, g \in E \text{ so that the convex hull of } C \text{ corresponds to } Hg\}.$$

2.8. **Lemma.** *Let the notation be as in Lemma 2.7. Further, assume that $\mathbf{V}(\mathfrak{l})$ is compact and that it is equipped with a metric. For every $p \in \mathbf{V}(\mathfrak{l})$ and every $r > 0$, let $\mathsf{N}_r(p)$ denote the open ball of radius r centered at p with respect to this metric.*

Let $E \subset G$ be a compact subset with positive measure. For every $\varepsilon > 0$, there exists a compact subset $E_\varepsilon \subset E$ with $\mathrm{vol}_G(E \setminus E_\varepsilon) \ll_E \varepsilon^{32}\mathrm{vol}_G(E)$ and some $\delta > 0$ with the following property. For every $C \in \mathcal{C}_{E_\varepsilon}$ and every $p \in \mathbf{V}(\mathfrak{l})$ we have

$$\mathsf{m}_C\big(\{\xi \in C : \Psi(\xi) \in \mathsf{N}_\delta(p)\}\big) < \varepsilon \mathsf{m}_C(C).$$

Proof. Up to a null set, we may identify the space of circles $\mathcal{C}$ with $\partial\mathbb{H}^3 \times (0,1]$ —normalize the metric on $\mathbb{S}^2$ so that the great circles have radius 1, then a circle $C \in \mathcal{C}$ is determined by its center, a point in $\partial\mathbb{H}^3$, and its radius $r \in (0,1]$.

By Lusin's theorem, there exists a compact subset $D \subset \partial\mathbb{H}^3$ with $\mathsf{m}(D) > 1 - \varepsilon^{64}$ so that $\Psi|_D$ is continuous.

Note that $\mathcal{C}_E$ is a compact subset of $\mathcal{C}$ with $\sigma(\mathcal{C}_E) > 0$. Therefore, ignoring a possible null subset, there exist $r, r' > 0$ so that

$$\mathcal{C}_E \subset \partial\mathbb{H}^3 \times [r,1] =: \mathcal{C}_r$$

and $\mathsf{m}(\mathcal{C}_E) \geq r'\mathsf{m}(\mathcal{C}_r)$.

For every $\xi \in \partial\mathbb{H}^3$, set

$$I_\xi := \big\{t \in [r,1] : \mathsf{m}_{C_t(\xi)}(C_t(\xi) \cap D) < (1 - \varepsilon^{32})\mathsf{m}_{C_t(\xi)}(C_t(\xi))\big\}$$

where $C_t(\xi)$ is the circle with radius t centered at ξ.

For every $\xi \in \partial\mathbb{H}^3$, $\{C_t(\xi) : t \in [r, 1]\}$ swipe a positive portion of $\partial\mathbb{H}^3$; hence, by Fubini's theorem, $|I_\xi| \ll_r \varepsilon^{32}$ for every $\xi \in \partial\mathbb{H}^3$. Therefore,

$$\sigma\big(\{C \in \mathcal{C}_r : \mathsf{m}_C(C \cap D) < (1 - \varepsilon^{32})\mathsf{m}_C(C)\}\big) \ll_r \varepsilon^{32}.$$

Let $\hat{E}_\varepsilon = \{g \in E : \mathsf{m}_{C_g}(C_g \cap D) \geq (1 - \varepsilon^{32})\mathsf{m}_{C_g}(C_g)\}$ where C_g is the circle so that the convex hull of C_g corresponds to Hg. In view of the above estimate and since $\mathcal{C}_E \subset \mathcal{C}_r$, we have $\mathrm{vol}(E \setminus \hat{E}_\varepsilon) \ll_{r,r'} \varepsilon^{32}\mathrm{vol}(E)$.

Let $\mathcal{C}' \subset \mathcal{C}$ be the conull subset where Lemma 2.7(2) holds true. Let E_ε be a compact subset of

$$\hat{E}_\varepsilon \cap \{g \in G : C_g \in \mathcal{C}'\}$$

so that $\mathrm{vol}(E \setminus E_\varepsilon) \ll_{r,r'} \varepsilon^{32}\mathrm{vol}(E)$. In particular, E_ε satisfies the first claim in the lemma. We now verify that it also satisfies the second claim.

Assume contrary to the claim that for every n there is some $p_n \in \mathbf{V}(\mathfrak{l})$ and some $C_n \in \mathcal{C}_{E_\varepsilon}$ so that

$$\mathsf{m}_{C_n}\big(\{\xi \in C_n : \Psi(\xi) \in \mathsf{N}_{1/n}(p_n)\}\big) > \varepsilon\mathsf{m}_{C_n}(C_n). \tag{2.11}$$

Passing to a subsequence, if necessary, we assume that $C_n \to C \in \mathcal{C}_{E_\varepsilon}$ and $p_n \to p \in \mathbf{V}(\mathfrak{l})$ —recall that $\mathbf{V}(\mathfrak{l})$ is compact.

For each n, let $C'_n := \{\xi \in C_n : \Psi(\xi) \in \mathsf{N}_{1/n}(p_n)\} \cap D$. In view of the fact that $E_\varepsilon \subset \hat{E}_\varepsilon$ and using (2.11), we get that

$$\mathsf{m}_{C_n}(C'_n) \geq \varepsilon\mathsf{m}_{C_n}(C_n)/2.$$

Let $C' := \limsup C'_n$; then $C' \subset C \cap D$ and $\mathsf{m}_C(C') \geq \varepsilon\mathsf{m}_C(C)/2$. Moreover, for every $\xi \in C'$ there exist some $\xi_n \to \xi$ with $\xi_n \in C'_n$. Since Ψ is continuous on D, we get that $\Psi(\xi) = p$, i.e., $\Psi(C') = p$. This contradicts the fact that $C' \in \mathcal{C}_{E_\varepsilon} \subset \mathcal{C}'$ and finishes the proof. $\square$

3. A Γ-equivariant circle preserving map

In this section we state one of the main results of this paper, Proposition 3.1. We then complete the proofs of Theorem 1.2 and Theorem 1.1 using Proposition 3.1.

Let the notation be as in §2.1. In particular, L is a number field and $\mathbf{G}$ is an L-group. For every $v \in \mathcal{S}$ and $\sigma \in \Sigma_v$, let $\mathfrak{l}_v$ denote $\mathbb{C}$ if v is an Archimedean place, and an extension of degree at most 2 of L_v so that ${}^\sigma\mathbf{G}$ is $\mathfrak{l}_v$-split if v is a non-Archimedean place. Recall from (2.2) that

$${}^\sigma\mathbf{G} \text{ is } \mathfrak{l}_v\text{-isomorphic to } \mathrm{PGL}_2 \times \mathrm{PGL}_2.$$

For every g, let $\mathfrak{C}_g \subset \mathbb{P}\mathfrak{l}_v \times \mathbb{P}\mathfrak{l}_v$ be the graph of the linear fractional[1] transformation $g : \mathbb{P}\mathfrak{l}_v \to \mathbb{P}\mathfrak{l}_v$.

[1]If $[r, 1] \in \mathbb{P}\mathfrak{l}_v$, then $g([r, 1]) = [\frac{ar+b}{cr+d}, 1]$ where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with the usual convention, see also §6.

Let B_v denote the group of upper triangular matrices in $\mathrm{PGL}_2(\mathfrak{l}_v)$. We may define $\mathfrak{C}_g \subset B_v\backslash\mathrm{PGL}_2(\mathfrak{l}_v) \times B_v\backslash\mathrm{PGL}_2(\mathfrak{l}_v) = \mathbb{P}\mathfrak{l}_v \times \mathbb{P}\mathfrak{l}_v$ alternatively as follows. Let $g \in \mathrm{PGL}_2(\mathfrak{l}_v)$, the stabilizer of $\mathfrak{C}_g$ in $\mathrm{PGL}_2(\mathfrak{l}_v) \times \mathrm{PGL}_2(\mathfrak{l}_v)$ is

$$\mathrm{Stab}(\mathfrak{C}_g) = \{(h, g^{-1}hg) : h \in \mathrm{PGL}_2(\mathfrak{l}_v)\}. \tag{3.1}$$

Recall from §2.6 that $\mathcal{C}$ denotes the space of circles in $\mathbb{S}^2 = \partial\mathbb{H}^3$; the space $\mathcal{C}$ is equipped with a natural $\mathrm{PGL}_2(\mathbb{C})$-invariant measure σ. Recall also that m denotes the Lebesgue measure on $\mathbb{S}^2$, and that for every $C \in \mathcal{C}$, m_C denotes the Lebesgue measure on C.

3.1. **Proposition.** *Assume $X = G/\Gamma$ contains infinitely many closed H-orbits. Let $\mathfrak{l}_v$ and $\mathbf{G}$ be as above and assume that*

$$\sigma(\Gamma) \subset \mathrm{PGL}_2(\mathfrak{l}_v) \times \mathrm{PGL}_2(\mathfrak{l}_v) \text{ is unbounded.}$$

There exists a σ-equivariant measurable map

$$\Psi : \mathbb{S}^2 \to \mathbb{P}\mathfrak{l}_v \times \mathbb{P}\mathfrak{l}_v$$

with the following properties.

(1) *For a.e. $C \in \mathcal{C}$ we have $(\Psi|_C)_*\mathsf{m}_C$ is non-atomic. In particular, the essential range of $\Psi|_C$ is infinite for a.e. $C \in \mathcal{C}$.*
(2) *For a.e. $C \in \mathcal{C}$ there exists some $g_C \in \mathrm{PGL}_2(\mathfrak{l}_v)$ so that the essential image of $\Psi|_C$ is contained in $\mathfrak{C}_{g_C}$.*

This proposition will be proved in §6. Our goal in this section is to complete the proofs of Theorem 1.2 and Theorem 1.1 using Proposition 3.1.

The argument in this section are measurable versions of well-known arguments, see e.g., [19, 7]. In the sequel by an *inversion* of a circle C, we mean a linear fractional transformation on C of order 2 where C is identified with $\mathbb{P}\mathbb{R}$; similarly we define an inversion of $\mathfrak{C}_g$. The group of inversions on $\mathfrak{C}_g$ is identified with $\mathrm{Stab}(\mathfrak{C}_g)$, see (3.1).

3.2. **Lemma.** *Let $\varphi : \mathbb{P}\mathbb{R} \to \mathfrak{C}_g$ be a Borel measurable map so that the essential image of φ has at least three points. Let $\mathcal{I}$ be a subset of inversions on $\mathbb{P}\mathbb{R}$ so that the group generated by $\mathcal{I}$ contains $\mathrm{PSL}_2(\mathbb{R})$. Assume further that there exists a Borel map $f : \mathcal{I} \to \mathrm{Stab}(\mathfrak{C}_g)$ which satisfies the following:*

$$\text{for any } \iota \in \mathcal{I},\ \varphi \circ \iota = f(\iota) \circ \varphi \quad \mathsf{m}_{\mathbb{P}\mathbb{R}}\text{–a.e. on } \mathbb{P}\mathbb{R}. \tag{3.2}$$

Then f extends to a continuous homomorphism from $\mathrm{PSL}_2(\mathbb{R})$ into $\mathrm{Stab}(\mathfrak{C}_g)$.

Proof. First note that since the essential image of φ has at least three points, any linear fractional transformation on $\mathfrak{C}_g$ is uniquely determined by its restriction to the essential image of φ.

In view of this and (3.2) the map $\iota_1 \circ \cdots \circ \iota_n \mapsto f(\iota_1) \circ \cdots \circ f(\iota_n)$ is a well-defined measurable homomorphism from the group generated by $\mathcal{I}$ into $\mathrm{Stab}(\mathfrak{C}_g)$.

The claim follows from this as the group generated by $\mathcal{I}$ contains $\mathrm{PSL}_2(\mathbb{R})$ and any measurable homomorphism is continuous, see e.g. [29, Ch. VII, Lemma 1.4]. $\square$

Let $(\xi, \xi') \in \mathbb{S}^2 \times \mathbb{S}^2$, with $\xi \neq \xi'$. We parametrize the family of circles passing through $\{\xi, \xi'\}$ as follows: using the stereographic projection with a pole different from $\{\xi, \xi'\}$, we assume $\{\xi, \xi'\} \subset \mathbb{R}^2$. A circle passing through $\{\xi, \xi'\}$ is uniquely determined by a point on the orthogonal bisector of the line segment $\overline{\xi\xi'}$ in $\mathbb{R}^2$, or it is centered at ∞ (which yields the straight line through $\{\xi, \xi'\} \subset \mathbb{R}^2$).

In view of the above, we let $\{C_t(\xi, \xi') : t \in \mathbb{S}^1\}$ denote the one parameter family of circles in $\mathbb{S}^2$ passing through ξ and ξ'.

Given a triple $(\xi, \xi', C) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathcal{C}$ we say $\{\xi, \xi'\}$ and C are *linked* if ξ and ξ' belong to different connected components of $\mathbb{S}^2 \setminus C$. For every circle $C \in \mathcal{C}$, let

$$E_C \subset \mathbb{S}^2 \times \mathbb{S}^2 \tag{3.3}$$

denote the set of $(\xi, \xi') \in \mathbb{S}^2 \times \mathbb{S}^2$ so that $\{\xi, \xi'\}$ and C are linked.

If $(\xi, \xi') \in E_C$, we may also parametrize the circles passing through $\{\xi, \xi'\}$ using points on C. Indeed, for every $\theta \in C$, there is a unique circle $C(\xi, \xi', \theta)$ passing through them —this is a two-to-one map: $C(\xi, \xi', \theta) \cap C = \{\theta, \theta'\}$.

3.3. **Lemma.** *Let $C \in \mathcal{C}$ and $(\xi, \xi') \in \mathbb{S}^2 \times \mathbb{S}^2$; assume that $\{\xi, \xi'\}$ and C are linked. Define*

$$\iota_{\xi,\xi'} : C \to C$$

as follows. Let $p \in C$, there exists a unique circle C' which passes through $\{\xi, \xi', p\}$. Then $C' \cap C = \{p, q\}$. Define $\iota_{\xi,\xi'}(p) = q$.

The map $\iota_{\xi,\xi'}$ is an inversion on C.

Proof. First note that $\iota_{\xi,\xi'}$ has order two. The fact that $\iota_{\xi,\xi'}$ is an inversion on C could be seen, e.g., as follows: we may assume C is the unit circle in the plane, $\xi' = \infty$ and $\xi = (0, b)$ for some $0 \leq b < 1$. Let pr denote the stereographic projection of C onto the line $\{y = b\} \cup \{\infty\}$. Then $\mathrm{pr}(0, 1) = \infty$ and $\iota_{\xi,\xi'}(0, 1) = (0, -1)$. Moreover, if $\mathrm{pr} \circ \iota_{\xi,\xi'}(p) = (a, b)$ with $a \neq 0$, then $\mathrm{pr} \circ \iota_{\xi,\xi'}(q) = (\frac{b^2-1}{a}, b)$.

Since $a \cdot \frac{b^2-1}{a} = b^2 - 1$ is a constant, $\iota_{\xi,\xi'}$ is an inversion as we claimed. □

3.3.1. **Remark.** We also need an analogue of Lemma 3.3 in the target space, i.e., for $\mathfrak{C}_g$. This can be seen by a direct computation which involves solving a quadratic equation. As was done in the proof of Lemma 3.3, one may also simplify this computation as follows: We may reduce to the case where the graph is given by $zw = 1$, $\xi = (\infty, \infty)$, $\xi' = (r, s)$ with r, s both finite and every line $az + b$ through ξ' intersects $zw = 1$. Then, on the finite points of the graph, the inversion is given by $z \mapsto -z - \frac{b}{a}$.

3.4. **Lemma.** *Let $E' \subset E_C$ be a subset with positive measure. For every $(\xi, \xi') \in E'$, let $\iota_{\xi,\xi'}$ be constructed as in Lemma 3.3. Then the group generated by the $\mathcal{I}' := \{\iota_{\xi,\xi'} : (\xi, \xi') \in E'\}$ contains $\mathrm{PSL}_2(\mathbb{R})$, where we identify C with $\mathbb{PR}$.*

Proof. Let us write

$$\mathrm{Inv} := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{R}, a^2 + bc = -1 \right\}.$$

We equip Inv with the Lebesgue measure, then $\mathcal{I}' \subset \mathrm{Inv}$ has positive measure. Moreover, any set of positive measure in Inv generates a subgroup which contains $\mathrm{PSL}_2(\mathbb{R})$.

To see this, note that for a.e. $(g, g') \in \mathrm{Inv}$ we have $g\mathrm{Inv} \cap g'\mathrm{Inv}$ is one dimensional. Therefore, there are $g, g' \in \mathcal{I}'$ so that $Q := (g\mathcal{I}')(g'\mathcal{I}')$ has positive measure in $\mathrm{PSL}_2(\mathbb{R})$. Now $Q^{-1}Q$ contains an open neighborhood of the identity and $\mathrm{PSL}_2(\mathbb{R})$ is connected, hence, $Q^{-1}Q$ generates a subgroup which contains $\mathrm{PSL}_2(\mathbb{R})$. $\square$

Let the notation be as in Proposition 3.1. In particular,

$$\Psi : \mathbb{S}^2 \to \mathbb{P}\mathfrak{l}_v \times \mathbb{P}\mathfrak{l}_v$$

is a σ-equivariant measurable map in Proposition 3.1.

3.5. **Lemma.** *Let $\mathcal{V} = \{(C, \theta, \theta') \in \mathcal{C} \times \mathbb{S}^2 \times \mathbb{S}^2 : \theta, \theta' \in C\}$. Then $\mathcal{V}$ is a subvariety of $\mathcal{C} \times \mathbb{S}^2 \times \mathbb{S}^2$; equip $\mathcal{V}$ with the natural measure. There exists a conull Borel measurable subset*

$$\mathcal{A} \subset \mathcal{V}$$

so that $\Psi(\theta), \Psi(\theta')$ belong to the essential image of $\Psi|_C$ for all $(C, \theta, \theta') \in \mathcal{A}$.

Proof. Since Ψ is measurable, for any $n \in \mathbb{N}$ there exists a compact subset $D_n \subset \mathbb{S}^2$ with $\mathsf{m}(\mathbb{S}^2 \setminus D_n) < 1/n$ so that $\Psi|_{D_n}$ is continuous; we may also assume that $D_1 \subset D_2 \subset \cdots$.

For every $m \in \mathbb{N}$, let $\mathcal{C}_m$ denote the set of circles with radius $\geq 1/m$. Then $\mathcal{C}_m$ is a compact subset of $\mathcal{C}$ and $\mathcal{C} = \cup \mathcal{C}_m$. For every $m, n \in \mathbb{N}$ let $\mathcal{C}_{m,n} = \{C \in \mathcal{C}_m : \mathsf{m}_C(C \cap D_n) > 0\}$. Then $\cup_n \mathcal{C}_{m,n}$ is a conull Borel subset of $\mathcal{C}_m$ for every m.

Let $C \in \mathcal{C}_{m,n}$ and let $\theta \in C \cap D_n$ be a density point for $C \cap D_n$. Then by continuity of $\Psi|_{D_n}$, for every $r > 0$, we have $\Psi^{-1}(\mathsf{N}_r(\Psi(\theta)) \cap D_n \cap C$ is an open subset of $D_n \cap C$; since θ is a density point of $C \cap D_n$, we get that $\mathsf{m}_C(\Psi^{-1}(\mathsf{N}_r(\Psi(\theta)) \cap D_n \cap C) > 0$. In particular, $\Psi(\theta)$ belongs to the essential image of $\Psi|_C$.

Define $\mathcal{A}'_{m,n} = \{(C, \theta, \theta') : C \in \mathcal{C}_{m,n}, \theta, \theta' \in C \cap D_n\}$. Then $\mathcal{A}'_{m,n}$ is a Borel subset of $\mathcal{V}$. Using a countable basis of open subsets, we see that

$$\mathcal{A}_{m,n} = \{(C, \theta, \theta') \in \mathcal{A}'_{m,n} : \theta, \theta' \text{ are density points of } C \cap D_n\}$$

is a conull subset of $\mathcal{A}'_{m,n}$.

This in view of the above argument implies that $\mathcal{A} = \cup_{m,n} \mathcal{A}_{m,n}$ satisfies the claim in the lemma. $\square$

3.6. **Lemma.** *Let the notation be as in Proposition 3.1. For a.e. $C \in \mathcal{C}$, there is a conull subset $E'_C \subset E_C$, see* (3.3)*, so that for all $(\xi, \xi') \in E'_C$ the following hold.*

(1) $\Psi(\xi) \neq \Psi(\xi')$.
(2) *For a.e.* $t \in \mathbb{S}^1$, *there exists* $g_t \in \mathrm{PGL}_2(\mathfrak{l}_v)$ *so that the essential image of* $\Psi|_{C_t(\xi,\xi')}$ *is contained in* $\mathfrak{C}_{g_t}$.
(3) *For a.e.* $\theta \in \mathbb{S}^1$ *the following holds. Let* $C \cap C(\xi,\xi',\theta) = \{\theta,\theta'\}$ *where* $C(\xi,\xi',\theta)$ *is the circle passing through* $\{\xi,\xi',\theta\}$. *Then* $\Psi(\theta) \neq \Psi(\theta')$ *and they both belong to the essential image of* $\Psi|_C$.

Proof. Let $\mathcal{C}' \subset \mathcal{C}$ be a conull subset where Proposition 3.1(1) and (2) hold. Then, for any $C \in \mathcal{C}'$, we have the essential image of $\Psi|_C$ is a subset of $\mathfrak{C}_{g_C}$ and the essential image is an infinite subset.

Applying Lemma 2.7(1) with Ψ and $\Psi(\xi)$ (for every $\xi \in \mathbb{S}^2$ so that $\Psi(\xi)$ is defined), the set of $\xi' \in \mathbb{S}^2$ such that $\Psi(\xi') = \Psi(\xi)$ is a null set. That is: for a.e. $(\xi,\xi') \in \mathbb{S}^2 \times \mathbb{S}^2$ we have $\Psi(\xi) \neq \Psi(\xi')$.

Note also that since $\mathcal{C}'$ is conull, for a.e. $(\xi,\xi') \in \mathbb{S}^2 \times \mathbb{S}^2$ we have

$$C_t(\xi,\xi') \in \mathcal{C}' \quad \text{ for a.e. } t \in \mathbb{S}^1.$$

In consequence, for a.e. $(\xi,\xi') \in E_C$ parts (1) and (2) hold true. We now show that (3) also holds for a.e. $C \in \mathcal{C}'$ and a.e. $(\xi,\xi') \in E_C$.

Let $\mathcal{A}$ be as in Lemma 3.5. That is:

$$\mathcal{A} \subset \{(C,\theta,\theta') \in \mathcal{C} \times \mathbb{S}^2 \times \mathbb{S}^2 : \theta,\theta' \in C\} = \mathcal{V}$$

is a conull Borel measurable subset so that $\Psi(\theta), \Psi(\theta')$ belong to the essential image of $\Psi|_C$ for all $(C,\theta,\theta') \in \mathcal{A}$.

Define

$$\mathcal{A}' := \mathcal{A} \cap \mathrm{pr}^{-1}(\mathcal{C}')$$

where $\mathrm{pr} : \mathcal{V} \to \mathcal{C}$ is the projection map.

Let $\mathcal{B}' \subset \mathcal{A}' \times \mathbb{S}^2 \times \mathbb{S}^2$ be the set of points $(C,\theta,\theta',\xi,\xi')$ where $(C,\theta,\theta') \in \mathcal{A}'$, $(\xi,\xi') \in E_C$, then $\mathcal{B}'$ is Borel subset. Let

$$\mathcal{B} := \big\{(C,\theta,\theta',\xi,\xi') \in \mathcal{B}' : C \cap C(\xi,\xi',\theta) = \{\theta,\theta'\}\big\};$$

note that $\mathcal{B}$ is also a Borel set. Indeed, $\mathcal{B}$ is the inverse image of the diagonal in $\big((\mathbb{S}^1 \times \mathbb{S}^1)/(\mathbb{Z}/2)\big) \times \big((\mathbb{S}^1 \times \mathbb{S}^1)/(\mathbb{Z}/2)\big)$ under the map $(C,\theta,\theta',\xi,\xi') \mapsto (\{\theta,\theta'\}, C \cap C(\xi,\xi',\theta))$ where we identified C with $\mathbb{S}^1$.

By the definition of $\mathcal{B}$ and Lemma 3.5, for any $(C,\theta,\theta',\xi,\xi') \in \mathcal{B}$ we have $\Psi(\theta)$ and $\Psi(\theta')$ belong to the essential image of $\Psi|_C$.

We now show that, after possibly removing a null subset from $\mathcal{B}$, the first claim in (3) also holds. Let $\check{\mathcal{B}} \subset \mathcal{B}$ be the set of $(C,\theta,\theta',\xi,\xi') \in \mathcal{B}$ so that $\Psi(\theta) = \Psi(\theta')$. We claim that $\check{\mathcal{B}}$ is a null set.

To see this, observe that if we fix (C,θ) and vary $(\xi,\xi') \in E_C$, there is a unique $\theta' \in C$ so that $(C,\theta,\theta',\xi,\xi') \in \mathcal{B}$; indeed $\{\theta,\theta'\} = C \cap C(\xi,\xi,\theta)$. Assume now contrary to the claim that $\check{\mathcal{B}}$ has positive measure. By Fubini's theorem, thus, there exists some (C,θ) so that

$$\check{\mathcal{B}}_{(C,\theta)} := \{(\xi,\xi',\theta') : (C,\theta,\theta',\xi,\xi') \in \check{\mathcal{B}}\}$$

has positive measure —this fiber is the graph of the analytic map $(\xi,\xi') \mapsto \theta'$, as we remarked above.

Moreover, by varying $(\xi,\xi') \in E_C$, we cover every $\theta \neq \theta' \in C$; they appear as the intersection of $C \cap C(\xi,\xi,\theta)$. Using the implicit function theorem, thus there is a subset $J \subset C$ with $\mathsf{m}_C(J) > 0$ so that $\Psi(\theta') = \Psi(\theta)$ for all $\theta' \in J$. This contradicts the fact that $C \in \mathcal{C}'$ —recall that Proposition 3.1(1) holds for all $C \in \mathcal{C}'$.

The proof is complete. $\square$

3.7. **Lemma.** *Let the notation be as in Proposition 3.1. In particular,*

$$\Psi : \mathbb{S}^2 \to \mathbb{P}\mathfrak{l}_v \times \mathbb{P}\mathfrak{l}_v$$

is a Γ-equivariant measurable map which satisfies parts (1) and (2) in Proposition 3.1. Then $\mathfrak{l}_v = \mathbb{C}$ and Ψ agrees with a rational map from $\mathbb{S}^2$ into $\mathbb{PC} \times \mathbb{PC}$ almost everywhere.

Proof. Let $C \in \mathcal{C}$ be so that Lemma 3.6 and Proposition 3.1 holds true and let $E'_C \subset E_C$ be as in Lemma 3.6. In particular the essential image of $\Psi|_C$ belongs to $\mathfrak{C} := \mathfrak{C}_{g_C}$ for some $g_C \in \mathrm{PGL}_2(\mathfrak{l}_v)$. Let $\mathcal{I}$ be the collection of inversions of C obtained by $(\xi,\xi') \in E'_C$ as defined in Lemma 3.3. Then by Lemma 3.4, $\mathcal{I}$ generates a subgroup which contains the connected component of the identity $\mathrm{Stab}_G(C)^\circ \simeq \mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{Stab}_G(C)$.

Recall that the essential image of $\Psi|_C$ in $\mathfrak{C}$ is infinite, therefore, an inversion is uniquely determined by its restriction to the essential image of Ψ. By Lemma 3.6 and Remark 3.3.1, Ψ induces a map f from $\mathcal{I}$ into the set of inversions on $\mathfrak{C}$.

Since the essential image of $\Psi|_C$ in $\mathfrak{C}$ is infinite, we get from Lemma 3.2 that f extends to a continuous homomorphism from $\mathrm{Stab}_G(C)^\circ \simeq \mathrm{PSL}_2(\mathbb{R})$ into $\mathrm{Stab}_G(\mathfrak{C}) \simeq \mathrm{PGL}_2(\mathfrak{l}_v)$. Such a homomorphism can only arise from algebraic constructions as follows: There exists a continuous homomorphism of fields $\vartheta : \mathbb{R} \to \mathfrak{l}_v$ and an isomorphism of algebraic groups $\varphi : {}^{\vartheta}\mathrm{PGL}_2 \to \mathrm{PGL}_2$ so that $f(g) = \varphi(\vartheta^0(g))$ for all $g \in \mathrm{PSL}_2(\mathbb{R})$ where $\vartheta^0 : \mathrm{PGL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\vartheta(\mathbb{R}))$ is the isomorphism induced by ϑ, see [29, Ch. I, §1.8].

Since there are no monomorphism from $\mathbb{R}$ into non-Archimedean local fields, continuous or not, we get that $\mathfrak{l}_v = \mathbb{C}$.

We now show that Ψ agrees with a rational map[2] almost surely. Let C and C' be two circles which intersect at two points and both satisfy Lemma 3.6 —the set of intersecting circles has positive measure in $\mathcal{C}$, hence, two such circles exist. Let $C \cap C' = \{\xi,\xi'\}$ where $\xi \neq \xi'$. Using the stereographic projection of $\mathbb{S}^2$ with ξ as the pole, we get coordinates on $\mathbb{R}^2$ induced by two lines ℓ and ℓ' corresponding C and C', respectively, which intersect at the image of ξ'. Thus Ψ induces a measurable map in two variables which is rational in each variable. In view of [28, Lemma 17], we thus get that Ψ agrees with a rational map almost surely. $\square$

[2]In the case at hand this assertion can be proved using more elementary arguments, e.g. by choosing three parallel circles.

3.8. **Proofs of the main theorems.** In this section we complete the proofs of Theorem 1.2 and Theorem 1.1 assuming Proposition 3.1.

Proof of Theorem 1.2. In view of Lemma 3.7 we may assume $\mathfrak{l}_v = \mathbb{C}$ and Ψ agrees with a rational map almost surely. Theorem 1.2 follows from this by [28, §1.3] as the action of $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ on its boundary is strictly effective. $\square$

Proof of Theorem 1.1. We recall the argument from [28, Proof of Thm. 1, p. 97]. Let the notation be as in §2.1; in particular, v_0 and $\sigma = \mathrm{id}$ are the place and the embedding which give rise to the lattice Γ in $\mathrm{PGL}_2(\mathbb{C})$. By Theorem 1.2 for any $(v, \sigma) \neq (v_0, \mathrm{id})$ we have $\sigma(\Gamma)$ is bounded in $\mathbf{G}(\mathsf{L}_v)$.

Let $\mathbf{G}' = \mathrm{R}_{\mathsf{L}/\mathbb{Q}}(\mathbf{G})$ where $\mathrm{R}_{\mathsf{L}/\mathbb{Q}}$ is the restriction of scalars. Then $\mathbf{G}'(\mathbb{R})$ is naturally identified with $\prod \mathbf{G}(\mathsf{L}_v)$ where the product is taken over all the Archimedean places. Let $\varphi(\Gamma)$ denote the image of Γ in $\mathbf{G}'$ — note that $\varphi(\Gamma)$ is isomorphic to Γ and $\varphi(\Gamma) \subset \mathbf{G}'(\mathbb{Q})$.

Let $\mathbf{F}$ be the Zariski closure of $\varphi(\Gamma)$ in $\mathbf{G}'$. Then the natural map $\varrho : \mathbf{F} \to \mathbf{G}$ is an $\mathbb{R}$-epimorphism. Let $\mathbf{K} = \ker(\varrho)$. Recall that any compact subgroup of a real algebraic group is itself algebraic. In view of this and since $\varphi(\Gamma)$ has bounded image in $\mathbf{K}(\mathbb{R})$, we get that $\mathbf{K}(\mathbb{R})$ is compact.

Moreover, $\sigma(\Gamma)$ is bounded in $\mathbf{G}(\mathsf{L}_v)$ for all non-Archimedean places and Γ is finitely generated, hence, $\varphi(\Gamma) \cap \mathbf{F}(\mathbb{Z})$ in finite index in $\varphi(\Gamma)$. Further, since $\mathbf{K}(\mathbb{R})$ is compact, $\varrho(\mathbf{F}(\mathbb{Z}))$ is discrete in $\mathbf{G}(\mathbb{R})$; we get that Γ and $\varrho(\mathbf{F}(\mathbb{Z}))$ are commensurable. The proof is complete. $\square$

4. The cocycle and equivariant measurable maps

In this section we recall a construction due to Margulis which produces an equivariant measurable map between certain projective spaces.

Let $\Gamma \subset G$ be a uniform lattice and let $\mathfrak{l}$ be a local field of characteristic zero. We assume fixed a homomorphism

$$\rho : \Gamma \to \mathrm{PGL}_2(\mathfrak{l})$$

whose image is *unbounded and Zariski dense* — in our application, ρ will actually be a monomorphism.

In the sequel we will consider measurable maps which are ρ-equivariant. Since ρ is assumed fixed throughout, we abuse the notation, and often refer to these maps as Γ-equivariant maps.

4.1. **Characteristic maps and cocylces.** Let B_0 (resp. B) denote the group of upper triangular matrices in G (resp. $\mathrm{PGL}_2(\mathfrak{l})$). We now recall from [29, Ch. V] the construction of a measurable map from $\mathbb{S}^2 \cong B_0\backslash G$ to $B\backslash\mathrm{PGL}_2(\mathfrak{l})$ which is associated to the representation ρ. This approach relies on the multiplicative ergodic theorem.

We fix a fundamental domain F for Γ in G using a Dirichlet fundamental domain as follows, see e.g. [34]. Let d be the bi-K-invariant metric on G

induced from the hyperbolic metric on $\mathbb{H}^3$. Let

$$\tilde{F} = \{g \in G : d(e,g) \le d(e, g\gamma) \text{ for all } \gamma \in \Gamma\}$$

where e denotes the identity element in G.

Then $\tilde{F}\Gamma = G$, and $\tilde{F}^\circ \cap \tilde{F}^\circ\gamma = \emptyset$ for all $e \ne \gamma \in \Gamma$ where $\tilde{F}^\circ = \{g \in G : d(e,g) < d(e,g\gamma) \text{ for all } e \ne \gamma \in \Gamma\}$ is the interior of $\tilde{F}$. Note that the closure of $\tilde{F}$ is compact, and $\partial\tilde{F}$ is a union of finitely many manifolds with lower dimension. Let $\{e, \gamma_1, \dots, \gamma_n\} = \{\gamma \in \Gamma : \tilde{F} \cap \tilde{F}\gamma \ne \emptyset\}$.

We now define F inductively. Let $F_0 = \tilde{F}$. Let $0 < i \le n$, and suppose $F_0, \dots, F_{i-1}$ are defined; put $F_i = F_{i-1} \setminus F_{i-1}\gamma_i$. Let $F = F_n$. Then

$$G = F\Gamma \quad \text{and} \quad F\gamma \cap F = \emptyset \text{ for all } \gamma \ne e. \tag{4.1}$$

In view of (4.1), for any $g \in G$ there exists a unique $\gamma_g \in \Gamma$ so that $g \in F\gamma_g$. Set

$$\omega(g) := \rho(\gamma_g).$$

Then $\omega : G \to \mathrm{PGL}_2(\mathfrak{l})$ is a Borel map and

$$\omega(g\gamma) = \omega(g)\rho(\gamma) \text{ for all } g \in G \text{ and } \gamma \in \Gamma. \tag{4.2}$$

Define $b'_\omega(g,y) = \omega(gy)\omega(y)^{-1}$ for all $g \in G$ and $y \in G$. Note that by (4.2) we have $b'_\omega(g,y) = b'_\omega(g, y\gamma)$ for all $\gamma \in \Gamma$. Define

$$b_\omega(g,x) = b'_\omega(g, \pi^{-1}(x)) \text{ for all } g \in G \text{ and } x \in X.$$

Then $b_\omega(g_1g_2, x) = b_\omega(g_1, g_2x)b_\omega(g_2, x)$. That is: b_ω is a cocycle.

Define the cocycle

$$u(n,x) = b_\omega(a_n, x) \text{ for all } x \in X \text{ and all } n \in \mathbb{Z}; \tag{4.3}$$

where a_t is defined in (2.1).

4.2. **Theorem** (Cf. [29], Ch. V and VI). *Let the notation be as above.*

(1) *There exist some $\lambda_1 > 0$ so that the following holds. Let $w \in \mathfrak{l}^2 \setminus \{0\}$. Then for* vol_X*-a.e. $x \in X$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \log(\|u(n,x)w\|/\|w\|) = \lambda_1$$

(2) *There is a unique ρ-equivariant Borel measurable map*

$$\psi : G \to B\backslash \mathrm{PGL}_2(\mathfrak{l}) \cong \mathbb{P}\mathfrak{l}$$

defined as follows: $\psi(g) = [w]$, $0 \ne w \in \mathfrak{l}^2$, if and only if

$$\lim_{n\to-\infty} \frac{1}{n} \log(\|\omega(a_n g)w\|/\|w\|) = \lambda_1.$$

Moreover, $\psi(bg) = \psi(g)$ for all $b \in B_0$. In particular, ψ induces a ρ-equivariant Borel map from $\mathbb{S}^2 \cong B_0\backslash G$ to $B\backslash\mathrm{PGL}_2(\mathfrak{l})$ which we continue to denote by ψ.

Proof. This theorem is proved in [29, Ch. V] using the multiplicative ergodic theorem. Indeed [29, Ch. V, Thm. 5.15] shows $\lambda_1 > 0$. The desired ρ-equivariant map is constructed in [29, Ch. V, Thm. 3.2] from G to $B\backslash\mathrm{PGL}_2(\mathfrak{l})$, and it is shown in [29, Ch. V, Thm. 3.3(ii)] that this map factors through a map from $B_0\backslash G \cong \mathbb{S}^2$.

An alternative approach to the existence of a ρ-equivariant Borel map from $\mathbb{S}^2$ to $B\backslash\mathrm{PGL}_2(\mathfrak{l})$ is due to Furstenberg, see [29, Ch. VI, Thm. 4.3] —this approach is based on the amenability of B_0, see also [17, 20]. □

4.3. **A set of uniform convergence.** Let $\varepsilon > 0$. Let F be as in (4.1). Let $F'' \subset F$ be a compact subset $\mathrm{vol}_G(F'') \geq (1-\varepsilon^{64})\mathrm{vol}_G(F)$. Let $F''_\varepsilon \subset F$ be as in Lemma 2.8.

Let $e_1 = (1,0) \in \mathfrak{l}^2$. Assuming ε is small enough, there exists a compact subset $F'_\varepsilon \subset F''_\varepsilon$ with $\mathrm{vol}_X(F'_\varepsilon) > 1-\varepsilon^8$ so that the convergence in Theorem 4.2(1) is uniform. That is for every $\eta > 0$ there exists some n_η so that for all $n > n_\eta$ and any $x \in F'_\varepsilon$ we have

$$\Big|\big(\frac{1}{n}\log\|u(n,x)e_1\|\big) - \lambda_1\Big| < \eta \quad \text{for all } n > n_\eta. \tag{4.4}$$

Since the measure vol_X is G-invariant and in particular, $\mathrm{SO}(2)$-invariant, the following holds. There exists a compact subset $F_\varepsilon \subset F'_\varepsilon$ with

$$\mathrm{vol}_X(F_\varepsilon) > 1-\varepsilon^4$$

so that for every $g \in F_\varepsilon$ we have $|\{\theta \in [0,2\pi] : r_\theta g \in F'_\varepsilon\}| > 1-\varepsilon^4$ where r_θ denotes the rotation matrix with angle θ.

Let $\tau > 0$ be fixed. For every $\alpha > 0$, let $F_\varepsilon(\tau,\alpha) \subset F_\varepsilon$ be the subset with the property that for every $g \in F_\varepsilon(\tau,\alpha)$ we have

$$|\{\theta \in [0,2\pi] : \mathsf{B}(a_\tau r_\theta g, \alpha) \subset F\omega(a_\tau r_\theta g)\}| > 2(1-2\varepsilon^4)2\pi \tag{4.5}$$

where for every $h \in G$ we have $\omega(h) \in \Gamma$ is the element so that $h \in F\omega(h)$ and $\mathsf{B}(h,\alpha)$ denote the ball of radius α centered at h.

Let τ and $\varepsilon > 0$ be fixed. There exists some $\alpha_0 = \alpha_0(\varepsilon)$ so that

$$\mathrm{vol}(F_\varepsilon(\tau,\alpha_0)) > 1-2\varepsilon^4. \tag{4.6}$$

Random walks and distances. Recall that $\mathcal{T}$ denotes the Bruhat-Tits tree (if $\mathfrak{l}$ is non-Archimedean) or $\mathbb{H}^3$ if $\mathfrak{l} = \mathbb{C}$. Let $\mathrm{d}_\mathcal{T}$ denote the right $\mathrm{PGL}_2(\mathfrak{l})$-invariant metric on $\mathcal{T}$ —that is: if $\mathfrak{l} = \mathbb{C}$, then $\mathrm{d}_\mathcal{T}$ is the hyperbolic metric and if $\mathfrak{l}$ is non-Archimedean, then $\mathrm{d}_\mathcal{T}$ is the path metric on a tree. We fix a base point $o \in \mathcal{T}$ which we assume to be the image of the identity element of $\mathrm{PGL}_2(\mathfrak{l})$ in $\mathcal{T}$. Abusing the notation, given an element $g \in \mathrm{PGL}_2(\mathfrak{l})$ we sometimes write $\mathrm{d}_\mathcal{T}(g,o)$ for $\mathrm{d}_\mathcal{T}(o.g,o)$.

Recall that Γ is uniform. Therefore, F is bounded and we have

$$\mathrm{d}_\mathcal{T}(o.u(n,x),o) \leq L_1 n \quad \text{for all } x \in X \text{ and } n \geq 1 \tag{4.7}$$

where L_1 depends on the representation ρ.

4.4. **Lemma.** *Let $0 < \varepsilon < 1/2$ be so that $L_1\varepsilon < 0.01\lambda_1$. Let F_ε be defined as in §4.3. There exists some N_0 so that for any $n > N_0$ we have the following.*

(1) *For every geodesic $\xi = \{\xi_t\} \subset \mathcal{T}$ with $\xi_0 = o$ and every $g \in F_\varepsilon$, there exists a subset $R_{g,\xi} \subset [0, 2\pi]$ with $|R_{g,\xi}| > 2(1-\varepsilon)\pi$ so that for all $\theta \in R_{g,\xi}$ we have*
$$\mathrm{d}_{\mathcal{T}}(o.u(n, r_\theta g), \xi_t) > t + \lambda_1 n/3 \qquad \textit{for all } t \geq 0$$
(2) *For every geodesic $\xi = \{\xi_t\} \subset \mathcal{T}$ with $\xi_0 = o$ and every $g \in F_\varepsilon$ we have*
$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}_{\mathcal{T}}(o.u(n, r_\theta g), \xi_t)\,\mathrm{d}\theta > t + \lambda_1 n/5$$
for all $t \geq 0$.

Proof. The proof is a special case of the argument presented in the proof of Lemma 2.5; we repeat parts of the argument for the convenience of the reader.

Apply (4.4) with $\eta = 1/2$ and assume $n > n_{1/2}$ for the rest of the discussion.

For any $\theta \in [0, 2\pi]$, let $q_\theta = o.u(n, r_\theta g)$. By the hyperbolic law of cosines, see (2.3), we have
$$\cosh(\overline{q_\theta\xi_t}) = \cosh(\overline{o\xi_t})\cosh(\overline{oq_\theta}) - \cos(\alpha)\sinh(\overline{o\xi_t})\sinh(\overline{oq_\theta})$$
where α is the angle between $\overline{o\xi_t}$ and $\overline{oq_\theta}$ —if $\mathcal{T}$ is a tree, the above gets the trivial form (2.4).

Then in view of Lemma 2.8 and the fact that $g \in F_\varepsilon \subset \mathcal{C}_{F''_\varepsilon}$, there exists some $\delta > 0$ so that
$$|\{\theta \in [0, 2\pi] : \psi(r_\theta g) \notin \mathsf{N}_\delta(\xi_\infty)\}| > 2(1-\varepsilon)\pi \tag{4.8}$$
where $\xi_\infty = \lim_{t\to\infty} \xi_t \in B\backslash \mathrm{PGL}_2(\mathfrak{l})$.

Therefore, we have
$$\mathrm{d}_{\mathcal{T}}(o.u(n, r_\theta g), \xi_t) > t + \tfrac{\lambda_1 n}{2} - O_\delta(1) \tag{4.9}$$
for all θ which lies in the set appearing in (4.8). Assuming n is large enough, depending on δ, we get part (1).

Recall that $\mathrm{d}_{\mathcal{T}}(o.u(n, h), o) \leq L_1 n$ for all $h \in F$. In particular we have
$$\mathrm{d}_{\mathcal{T}}(o.u(n, h), \xi_t) \geq t - L_1 n \quad \text{for all } n. \tag{4.10}$$

Part (2) now follows from (4.9) and (4.10) as Lemma 2.5 was proved using (2.9) and (2.10). $\square$

4.5. **Lemma.** *Let $0 < \varepsilon < 1/2$ with $L_1\varepsilon < 0.01\lambda_1$ and let $\tau > N_0$ be a fixed parameter, where N_0 is as in Lemma 4.4. Suppose $x \in X$ is so that there exists some $g_x \in F_\varepsilon(\tau, \alpha_0)$ with*
$$\mathrm{dist}(a_\tau r_\theta g_x, a_\tau r_\theta x) < \alpha_0/2 \quad \textit{for all } \theta \in [0, 2\pi], \tag{4.11}$$
see (4.6). Then the following hold.

(1) *Let $\xi = \{\xi_t\} \subset \mathcal{T}$ with $\xi_0 = o$ be a geodesic. There exists a subset $\hat{R}_{x,\xi} \subset [0, 2\pi]$ with $|\hat{R}_{x,\xi}| > 2(1-2\varepsilon)\pi$ so that for all $\theta \in \hat{R}_{x,\xi}$ we have*

$$\mathrm{d}_{\mathcal{T}}(o.u(\tau, r_\theta x), \xi_t) > t + \tau\lambda_1/3.$$

(2) *Let $\xi = \{\xi_t\} \subset \mathcal{T}$ with $\xi_0 = o$ be a geodesic. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}_{\mathcal{T}}(o.u(\tau, r_\theta x), \xi_t)\,\mathrm{d}\theta > t + \tau\lambda_1/5.$$

Proof. We first prove part (1). Let

$$R_{g_x} = \{\theta \in [0, 2\pi] : \mathsf{B}(a_\tau r_\theta g_x, \alpha_0) \subset F\}.$$

Then by the definition of $F_\varepsilon(\tau, \alpha_0)$, see (4.5), we have $|R_{g_x}| > 2(1-2\varepsilon^4)\pi$.

Let $R_{g_x,\xi}$ be as in Lemma 4.4(1) applied to g_x and ξ and put

$$\hat{R}_{x,\xi} := R_{g_x} \cap R_{g_x,\xi}.$$

Note that $|\hat{R}_{x,\xi}| > 2(1-2\varepsilon)\pi$.

Let now $\theta \in \hat{R}_{x,k}$. By (4.11), we have

$$\mathrm{dist}(a_\tau r_\theta g_x, a_\tau r_\theta x) < \alpha_0/2.$$

Since $\theta \in R_{x,g_x}$, we have $u(\tau, r_\theta x) = u(\tau, r_\theta g_x)$. Moreover, since $\theta \in R_{g_x,\xi}$ and $\tau > N_0$, we get from Lemma 4.4(1) that

$$\mathrm{d}_{\mathcal{T}}(o.u(\tau, r_\theta x), \xi_t) = \mathrm{d}_{\mathcal{T}}(o.u(\tau, r_\theta g_x), \xi_t) > t + \lambda_1\tau/3,$$

as was claimed in part (1).

The proof of part (2) is similar to the proof Lemma 2.5 as we now explicate. Recall from (4.7) that

$$\mathrm{d}_{\mathcal{T}}(o.u(\tau, r_\theta x), \xi_t) \geq t - L_1\tau \tag{4.12}$$

for any $\theta \in [0, 2\pi]$.

Using part (1) and (4.12) we obtain the following.

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}_{\mathcal{T}}(o.u(\tau, r_\theta x), \xi_t)\,\mathrm{d}\theta &> (1-2\varepsilon)(t + \tfrac{\lambda_1\tau}{3}) + 2(t - L_1\tau)\varepsilon \\ &> t + \lambda_1\tau/4 - 2L_1\lambda_1\varepsilon\tau \\ {\scriptstyle L_1\varepsilon < 0.01\lambda_1 \rightsquigarrow} &> t + \lambda_1\tau/5.\end{aligned}$$

The proof is complete. $\square$

5. The main lemma

The following lemma is one of the pivotal ingredients in the proof of Proposition 3.1, and is one of the main technical tools in this paper.

5.1. **Lemma** (Main Lemma). *Let the notation be as in §4. Further, assume that there are infinitely many closed H-orbits $\{Hx_i : i \in \mathbb{N}\}$ in $X = G/\Gamma$. There exists some $\lambda_0 = \lambda_0(\rho) > 0$, and for every $\varepsilon > 0$, there exist positive* *integers* i_0, τ, *and* N *with the following properties. For all* $i > i_0$, *there exists a subset* $Z_i \subset Hx_i$ *with* $\mu_{Hx_i}(Z_i) > 1 - \varepsilon$ *so that*

$$\mathrm{d}_{\mathcal{T}}(o.u(n\tau, z), o) > \lambda_0 \tau n \quad \textit{for all } z \in Z_i \textit{ and all } n > N$$

where μ_{Hx_i} *denote the* H*-invariant probability measure on* Hx_i *for every* i.

The proof of this lemma relies on results in §4, equidistribution theorems in homogeneous dynamics, and certain maximal inequalities. The proof will occupy the rest of this section.

We begin with the following theorem which is a special case of a theorem of Mozes and Shah [31] —the proof in [31] builds on seminal works on unipotent dynamics by Dani, Margulis, and Ratner.

5.2. **Theorem.** *Let* $\Gamma \subset G$ *be a lattice. Assume that there are infinitely many closed* H*-orbits* $\{Hx_i : i \in \mathbb{N}\}$ *in* X. *For every* i *let* μ_{Hx_i} *denote the* H*-invariant probability measure on* Hx_i. *Then*

$$\int f \, \mathrm{d}\mu_{Hx_i} \to \int f \, \mathrm{dvol} \qquad \textit{for any } f \in C_c(G/\Gamma).$$

This theorem plays an important role in the sequel. We record a corollary of this theorem here which will be used in §6.

5.2.1. **Corollary.** *Let the notation be as in Theorem 5.2. Let* $0 < \varepsilon < 1/2$ *and for each* i, *let* $Z_i \subset Hx_i$ *be a subset with* $\mu_{Hx_i}(Z_i) > 1 - \varepsilon$. *Let* $\delta > 0$ *and for each* i, *let* $\mathcal{N}_{i,\delta}$ *be the open* δ*-neighborhood of* Z_i. *Then there exists some* i_1 *so that*

$$\mathrm{vol}_X(\mathcal{N}_{i,\delta}) > 1 - 2\varepsilon \quad \textit{for all } i > i_1.$$

Proof. This follows from Theorem 5.2. However one needs to practice caution as the geometry of the sets Z_i is not controlled. We remedy this issue using the fact that δ is fixed. Throughout, we assume that δ is less than the injectivity radius of every point $x \in X$, and write μ_i for μ_{Hx_i}.

For each i, fix a covering $\{B_{i,j}\}$ of $X \setminus \mathcal{N}_{i,\delta}$ with balls of radius $\delta/4$ with multiplicity depending only on X. For all i and j, let $B^+_{i,j}$ denote the ball with the same center as $B_{i,j}$ and radius $\delta/3$. Set $\mathcal{O}_i = \cup_j B_{i,j}$ and $\mathcal{O}^+_i = \cup_j B^+_{i,j}$. Then $Z_i \cap \mathcal{O}^+_i = \emptyset$ for all i; hence, $\mu_i(\mathcal{O}^+_i) < \varepsilon$.

For each i, let f_i be a continuous function so that $\mathbb{1}_{\mathcal{O}_i} \le f_i \le \mathbb{1}_{\mathcal{O}^+_i}$. Since δ is fixed and the multiplicity of $\{B_{i,j}\}$ is bounded by a constant depending on X, the number of balls $\{B_{i,j}\}$ is uniformly bounded (independent of i). Thus we may choose such f_i so that furthermore $\mathcal{F} = \{f_i : i \ge 1\}$ is precompact.

It suffices to show that $\mathrm{vol}(f_i) \le 2\varepsilon$ for all large enough i. Indeed if this is established, then $\mathrm{vol}(\mathcal{O}_i) \le 2\varepsilon$ which implies the result as $X \setminus \mathcal{N}_{i,\delta} \subset \mathcal{O}_i$.

Assume contrary to the claim that there is a subsequence $i_n \to \infty$ so that $\mathrm{vol}(f_{i_n}) > 2\varepsilon$. Passing to a subsequence, we assume $\{f_{i_n}\}$ converges uniformly to f for some continuous function f. Thus

$$|\mathrm{vol}(f_{i_n})-\mu_{i_n}(f_{i_n})|\leq|\mathrm{vol}(f_{i_n})-\mathrm{vol}(f)| \\ +|\mathrm{vol}(f)-\mu_{i_n}(f)|+|\mu_{i_n}(f)-\mu_{i_n}(f_{i_n})|$$

Note that $\mu_i(f_i)\leq\mu_i(\mathcal{O}_i^+)<\varepsilon$. Therefore, if n is large enough, Theorem 5.2, $f_{i_n}\to f$, and the above imply that $\mathrm{vol}(f_{i_n})<2\varepsilon$. This contradiction finishes the proof. □

5.3. **Maximal inequalities.** Let $Y=Hx\subset X$ be a closed H-orbit and let μ be the probability H-invariant measure on Y. For any $\tau>0$, define an averaging operator A_τ on the space of Borel functions on Y by

$$A_\tau\varphi(y)=\frac{1}{2\pi}\int_0^{2\pi}\varphi(a_\tau r_\theta y)\,\mathrm{d}\theta.$$

Let $\mathcal{R}=[0,2\pi]^{\mathbb{Z}}$ be equipped with $\mathrm{d}\nu:=\big(\frac{\mathrm{d}\,\theta}{2\pi}\big)^{\otimes\mathbb{Z}}$.

Let $\tau>0$ and define $\eta_Y:\mathcal{R}\times Y\to\mathcal{R}\times Y$ by

$$\eta_Y((\theta_n),y)=(\eta(\theta_n),a_\tau r_{\theta_1}y)$$

where $\eta:\mathcal{R}\to\mathcal{R}$ is the shift map $\eta\big((\theta_n)\big)=\big((\theta_{n+1})\big)$.

Then the measure $\nu\times\mu$ is η_Y-invariant and ergodic. We refer the reader to [2, Ch. 2], see in particular, [2, Prop. 2.9, 2.14, 2.23, and §2.6].

For any $f\in L^1(Y,\mu)$, we have $1\otimes f\in L^1(\mathcal{R}\times Y,\nu\times\mu)$. Therefore, in view of the maximal inequality for η_Y, there exists an absolute constant $D>0$ so that the following holds. Let $f\in L^1(Y,\mu)$; for any $c>0$ we have

$$\nu\times\mu\Big\{(\theta,y)\in\mathcal{R}\times Y:\sup_n\frac{1}{n}\sum_{\ell=1}^{n}1\otimes f(\eta_Y^\ell(\theta,y))\geq c\Big\}\leq\frac{D\|f\|_1}{c}. \tag{5.1}$$

We also need a maximal inequality similar to and more general than Kolmogorov's inequality in the context of the law of large numbers —see also [3, §3] and [2, Ch. 3].

Consider the space $\mathcal{W}=Y^{\mathbb{N}}$ and let ω_y be the Markov measure associated to A_τ and y. That is: for bounded Borel function $\phi_0,\ldots,\phi_m$ on Y we have

$$\int\phi_1(w_1)\cdots\phi_m(w_m)\,\mathrm{d}\omega_y(w)=(\phi_1A_\tau(\cdots(\phi_{m-1}A_\tau(\phi_m))\cdots))(y)$$

where $w=(\cdots,w_{-1},w_1,w_2,\ldots)$.

The main case of interest to us is the trajectories obtained using the operator A_τ. That is: trajectories of the form

$$\big((w_j)_{j\in\mathbb{Z}}\big)=\big((w_j)_{j\leq0},(a_\tau r_{\theta_j}\cdots a_\tau r_{\theta_1}w_0)_{j\geq1}\big) \tag{5.2}$$

for a random $(\theta_j)\in\mathcal{R}$. Let $z\in Y$ we let $\mathcal{W}_z$ be the space of all paths as in (5.2) with $w_0=z$. In this case, the measure ω_z is obtained by pushing forward ν to $\mathcal{W}_z$.

Fix some $z \in Y$. Let $\rho : \Gamma \to \mathrm{PGL}_2(\mathfrak{l})$ be a representation as in Lemma 5.1. For every $n \geq 1$ define $u_{n,z} : \mathcal{R} \to \mathrm{PGL}_2(\mathfrak{l})$ by

$$u_{z,n}((\theta_j)) = u(\tau, r_{\theta_n} w_{n-1}) \cdots u(\tau, r_{\theta_2} w_1) u(\tau, r_{\theta_1} z); \tag{5.3}$$

where $w_1 = a_\tau r_{\theta_1} z$ and for all $j > 1$ we have $w_j = a_\tau r_{\theta_j} w_{j-1}$, see (5.2).

Fix some $z \in Y$. For all $(\theta_j) \in \mathcal{R}$ and all $n \geq 1$, define $\phi_{\theta,n} : Y \to \mathbb{R}$ by

$$\phi_{\theta,n}(y) = \mathrm{d}_{\mathcal{T}}\big(o.u(\tau, y) u_{z,n-1}((\theta_j)), o\big) - \mathrm{d}_{\mathcal{T}}\big(o.u_{z,n-1}((\theta_j)), o\big). \tag{5.4}$$

In view of (4.7), there is some $L = L(\rho, \tau)$, *but independent of* Y, so that

$$|\phi_{\theta,n}| \leq L \quad \text{for all } \theta \text{ and } n. \tag{5.5}$$

Put $\varphi_n\big((\theta_j)\big) := \phi_{\theta,n}(r_{\theta_n} w_{n-1}) - \frac{1}{2\pi}\int_0^{2\pi} \phi_{\theta,n}(r_\theta w_{n-1})\,\mathrm{d}\theta$.

5.4. **Lemma.** *For every $c > 0$ and $\delta > 0$, there exists some $N_1 = N_1(c, \delta, L)$ with the following property. Let $z \in Y$ and define φ_ℓ as above. Then*

$$\nu\Big(\{\theta = (\theta_j) \in \mathcal{R} : \max_{n \geq N_1} \frac{1}{n} |\textstyle\sum_{\ell=1}^n \varphi_\ell(\theta)| > c\}\Big) \leq \delta.$$

Proof. This lemma is proved using the following maximal inequality which follows, e.g. by combining [25, p. 386] with [13, Thm 1.1], see also [23, 8].

Let $(\Omega, \mathcal{B}, \beta)$ be a standard probability space and let $\{\zeta_n\}$ be a sequence of bounded Borel functions on Ω so that $\mathbb{E}_\beta(\zeta_n | \zeta_{n-1}, \ldots, \zeta_1) = 0$ for every n. Then for every $N \geq N_1 \geq 1$ and every $c > 0$ we have

$$\beta\big(\{\omega \in \Omega : \max_{N_1 \leq n \leq N} \frac{1}{n} |\textstyle\sum_{\ell=1}^n \zeta_\ell(\omega)| > c\}\big) \leq \\ \frac{1}{c^2}\Big(\textstyle\sum_{n=N_1}^N \frac{\int \zeta_n^2}{n^2} + \frac{1}{N_1^2}\sum_{n=1}^{N_1} \int \zeta_n^2\Big). \tag{5.6}$$

Returning to our setup, we now observe that

$$\mathbb{E}_\nu(\varphi_n | \varphi_{n-1}, \ldots, \varphi_1) = \mathbb{E}_\nu\big(\mathbb{E}_\nu(\varphi_n | \theta_{n-1}, \ldots, \theta_1) | \varphi_{n-1}, \ldots, \varphi_1\big).$$

Moreover, we have $\mathbb{E}_\nu(\varphi_n | \theta_{n-1}, \ldots, \theta_1) = 0$. Hence

$$\mathbb{E}_\nu(\varphi_n | \varphi_{n-1}, \ldots, \varphi_1) = 0.$$

Therefore, we may apply (5.6) with the space $(\mathcal{R}, \mathcal{B}^{\otimes \mathbb{Z}}, \nu)$ and the sequence $\{\varphi_n\}$ of functions. Since $\int \varphi_n^2 \leq 2L^2$, see (5.5), and $\sum \frac{1}{n^2}$ is a convergent series, the lemma follows. $\square$

5.5. **Conclusion of the proof of Lemma 5.1.** Let $0 < \varepsilon < 1/2$ with $L_1 \varepsilon < 0.01\lambda_1$, see (4.7), and let $\tau = N_0 + 1$ where N_0 is as in Lemma 4.4.

Let $\delta > 0$ be small enough so that for any $g, g' \in G$ with $g = hg'$ and $\|h - I\| \leq \delta$, we have

$$\mathrm{dist}(a_\ell r_\theta g, a_\ell r_\theta g') < \alpha_0/2 \tag{5.7}$$

for all $\theta \in [0, 2\pi]$ and all $0 < \ell < 2\tau$, see (4.6).

Let $\mathcal{N}$ denote the δ-neighborhood of $F_\varepsilon(\tau,\alpha_0)$ in X. In view of (5.7) and Lemma 4.5(2) we have the following. Let $\xi=\{\xi_t\}\subset\mathcal{T}$ with $\xi_0=o$ be a geodesic. Then

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}_{\mathcal{T}}(o.u(\tau,r_\theta x),\xi_t)\,\mathrm{d}\theta>t+\tau\lambda_1/5 \tag{5.8}$$

for all $x\in\mathcal{N}$. Let us write $\lambda_2:=\lambda_1/5$.

By Theorem 5.2, there exists some i_0 so that for all $i>i_0$ we have

$$\mu_{Hx_i}(\mathcal{N}\cap Y_i)\geq 1-2\varepsilon^4.$$

Fix some $i>i_0$ and let $Y=Y_i$ and $\mu=\mu_{Hx_i}$.

By Fubini's theorem, there exists a compact subset $Y'\subset\mathcal{N}\cap Y$ with $\mu(Y')\geq 1-\varepsilon^3$ so that for all $y\in Y'$, we have

$$|\{\theta\in[0,2\pi]:r_\theta y\in\mathcal{N}\cap Y\}|\geq 1-\varepsilon.$$

Apply the maximal inequality in (5.1) with Y, $f=\mathbb{1}_{Y\setminus Y'}$, and τ as above. In consequence, there exists an absolute constant $D>0$ so that

$$\nu\times\mu\Big\{(\theta,y)\in\mathcal{R}\times Y:\sup_n\frac{1}{n}\sum_{\ell=1}^n 1\otimes f(\eta^\ell(\theta,y))\geq\varepsilon\Big\}\leq D\varepsilon^2. \tag{5.9}$$

Therefore, if ε is small enough, Fubini's theorem implies the following: there exists some $Z'\subset Y$ with $\mu(Z')>1-2\varepsilon$ so that for every $z\in Z'$ we have

$$\nu\Big(\{\theta\in\mathcal{R}:\sup_n\frac{1}{n}\sum_{\ell=1}^n 1\otimes f(\eta_Y^\ell(\theta,z))\geq\varepsilon\}\Big)\leq\varepsilon. \tag{5.10}$$

Let $\theta=(\theta_j)$ be in the complement of the set on the left side of (5.10) and let $(w_j)\in\mathcal{W}_z$ be the path obtain from this θ, see (5.2). Then

$$\frac{1}{n}\sum_{\ell=1}^n\mathbb{1}_{Y'}(w_\ell)\geq 1-\varepsilon. \tag{5.11}$$

Put $I_n(\theta):=\{1\leq\ell\leq n:w_\ell\in Y'\}$ and $I'_n(\theta):=\{1\leq\ell\leq n\}\setminus I_n(\theta)$. Then, by (5.11), we have

$$\#I_n(\theta)\geq(1-\varepsilon)n. \tag{5.12}$$

Apply now Lemma 5.4 with this $z\in Z'$ and with $c=0.1\lambda_2$ and $\delta=\varepsilon$. Let N_1 be as in Lemma 5.4 for these choices. Then

$$\nu\Big(\{\theta\in\mathcal{R}:\max_{n\geq N_1}\frac{1}{n}|\textstyle\sum_{\ell=1}^n\varphi_\ell(\theta)|>0.1\lambda_2\}\Big)\leq\varepsilon. \tag{5.13}$$

Let $\mathcal{R}_z\subset\mathcal{R}$ be the compliment of the union of sets appearing on the left sides of (5.10) and (5.13). Let $\theta\in\mathcal{R}_z$ and write I_n and I'_n for $I_n(\theta)$ and $I'_n(\theta)$, respectively. Let $\ell\in I_n$. Then

$$w_\ell=a_\tau r_{\theta_{\ell-1}}\cdots a_\tau r_{\theta_1}z\in Y',$$

and by the definition of Y', $r_\beta w_\ell\in Y\cap\mathcal{N}$ for some $\beta\in[0,2\pi]$.

Apply (5.8) with $g = r_\beta w_\ell$ and the geodesic segment $\{\xi_t\} \subset \mathcal{T}$ connecting o to $q := o.u_\ell((\theta_j, w_j))$, see (5.3). Let us put $t = \mathrm{d}_\mathcal{T}(q, o)$ and parametrize so that $\xi_t = o$. In consequence, we have the following:

$$\begin{aligned} t + \lambda_2\tau &< \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}_\mathcal{T}(o.u(\tau, r_\theta r_\beta w_\ell)u_\ell((\theta_j, w_j)), \xi_t)\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}_\mathcal{T}(o.u(\tau, r_\theta r_\beta w_\ell)u_\ell((\theta_j, w_j)), o)\,\mathrm{d}\theta. \end{aligned} \tag{5.14}$$

Moreover, by (5.4) we have

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_{\theta,\ell+1}(r_\theta w_\ell)\,\mathrm{d}\theta = \\ \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}_\mathcal{T}(o.u(\tau, r_\theta w_\ell)u_\ell((\theta_j, w_j)), o) - \mathrm{d}_\mathcal{T}(q, o)\,\mathrm{d}\theta.$$

Recall that $\mathrm{d}_\mathcal{T}(q, o) = t$, therefore, using the above and (5.14) we get that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_{\theta,\ell+1}(r_\theta w_\ell)\,\mathrm{d}\theta \ge \lambda_2\tau \quad \text{if } \ell \in I_n. \tag{5.15}$$

Recall further from (4.7) that $\mathrm{d}_\mathcal{T}(u(\tau, g), e) \le L_1\tau$ for all $g \in F$; thus, we may use the triangle inequality and get also the trivial estimate

$$\begin{aligned} &\Big|\frac{1}{2\pi}\int_0^{2\pi} \phi_{\theta,\ell+1}(r_\theta w_\ell)\,\mathrm{d}\theta\Big| \\ &\qquad\le \frac{1}{2\pi}\int_0^{2\pi} \big|\mathrm{d}_\mathcal{T}(o.u(\tau, r_\theta w_\ell)u_\ell((\theta_j, w_j)), o) - \mathrm{d}_\mathcal{T}(q, o)\big|\,\mathrm{d}\theta \\ &\qquad\le L_1\tau \end{aligned} \tag{5.16}$$

for all $0 \le \ell \le n-1$.

In view of (5.15) and (5.16), for every $n \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{\ell=1}^n \frac{1}{2\pi}\int_0^{2\pi} \phi_{w,\ell}(r_\theta w_{\ell-1})\,\mathrm{d}\theta &\ge (\#I_n)\lambda_2\tau - (n - \#I_n)L_1\tau \\ {\scriptstyle (5.12)\rightsquigarrow} &\ge (1-\varepsilon)\lambda_2\tau n - \varepsilon L_1\tau n \\ {\scriptstyle \varepsilon<0.1\ \&\ \varepsilon L_1<0.1\lambda_2\rightsquigarrow} &\ge \lambda_2\tau n/2 \end{aligned} \tag{5.17}$$

Let now $n > N_1$. Therefore, since $((\theta_j, w_j)) \in \mathcal{R}_z$, we conclude from (5.13) that

$$\frac{1}{n}\Big|\sum_{\ell=1}^n \varphi_\ell(\theta)\Big| \le 0.1\lambda_2. \tag{5.18}$$

Recall again from (5.4) the definition of $\phi_{\theta,n}(r_{\theta_n} w_{n-1})$, also recall that $\varphi_n = \phi_{\theta,n} - \frac{1}{2\pi}\int_0^{2\pi}\phi_{\theta,n}$. We thus obtain

$$\sum_{\ell=1}^n \varphi_\ell(\theta) = \mathrm{d}_\mathcal{T}\big(o.u_{z,n}(\theta), o\big) - \sum_{\ell=1}^n \frac{1}{2\pi}\int_0^{2\pi} \phi_{\theta,\ell}(r_\theta w_{\ell-1})\,\mathrm{d}\theta.$$

This, together with (5.18) and (5.17), implies that for all $n \geq N_1$ we have

$$\mathrm{d}_{\mathcal{T}}\big(o.u_{z,n}(\theta), o\big) \geq (\tau/2 - 1/10)\lambda_2 n \geq \lambda_2 \tau n/3 = \lambda_1 \tau n/15. \tag{5.19}$$

To get Lemma 5.1 from (5.19) it remains to note that trajectories

$$\{o_{\mathbb{H}^3}.a_\tau r_{\theta_n} \cdots a_\tau r_{\theta_1} g_z : (\theta_j) \in \mathcal{R}\}$$

give rise to the rotation invariant distribution on the boundary circle corresponding to Hg_z, recall that $g_z \in F$. Moreover, for ν-a.e. $\theta = (\theta_j) \in \mathcal{R}$ there exists a unique geodesic $\{\xi_{\theta,t}\}$ with $\xi_{\theta,0} = g_z$ so that the trajectory $o_{\mathbb{H}^3}.a_\tau r_{\theta_n} \cdots a_\tau r_{\theta_1} g_z$ is at a sublinear distance from $\{\xi_{\theta,t}\}$, see e.g. [24, Thm. 2.1]. Indeed even a central limit theorem holds for these trajectories [16, 15, 22, 4]. $\square$

6. Proof of Proposition 3.1

In this section we complete the proof of Proposition 3.1; the proof uses Lemma 5.1. We begin with some preliminary statements.

Let the notation be as in Proposition 3.1. In particular, recall that for any $g \in \mathrm{PGL}_2(\mathfrak{l}_v)$, $\mathfrak{C}_g$ denotes the graph of the linear fractional transformation

$$g : \mathbb{P}\mathfrak{l}_v \to \mathbb{P}\mathfrak{l}_v.$$

We use the projective coordinates for $\mathbb{P}\mathfrak{l}_v$, in these coordinates we have: $\mathbb{P}\mathfrak{l}_v = \{[r,s] : r, s \in \mathfrak{l}_v\}$ and

$$\mathfrak{C}_g = \big\{\big([r,s], [ar + bs, cr + ds]\big)\big\}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let $\alpha, \beta \in \mathfrak{l}_v$, and define

$$\mathrm{Crs}(\alpha, \beta) = \{([\alpha, 1], [s, 1]) \in \mathbb{P}\mathfrak{l}_v \times \mathbb{P}\mathfrak{l}_v\} \cup \{([r, 1], [\beta, 1]) \in \mathbb{P}\mathfrak{l}_v \times \mathbb{P}\mathfrak{l}_v\}.$$

We refer to $\mathrm{Crs}(\alpha, \beta)$ as *crosses*.

6.1. **Lemma.** *Let $\{g_i\}$ be a squence of elements in $\mathrm{PGL}_2(\mathfrak{l}_v)$. Then at least one of the following holds.*

1. *There exists a subsequence $\{g_{i_m}\}$ and some $g \in \mathrm{PGL}_2(\mathfrak{l}_v)$ so that $\mathfrak{C}_{g_{i_m}} \to \mathfrak{C}_g$, or*
2. *there exists a subsequence $\{g_{i_m}\}$ so that $\{\mathfrak{C}_{g_{i_m}}\}$ converges to a cross or a union of a line and a point.*

Proof. If there exists a subsequence $\{g_{i_m}\}$ and some $g \in \mathrm{PGL}_2(\mathfrak{l}_v)$ so that $g_{i_m} \to g$, then $\mathfrak{C}_{g_{i_m}} \to \mathfrak{C}_g$. That is: part (1) holds.

Therefore, we may assume that $g_i \to \infty$ and will show that part (2) holds in this case. Passing to a subsequence we may assume all g_i's are in one $\mathrm{PSL}_2(\mathfrak{l}_v)$ coset, hence we assume $\det(g_i) = k$ for all i.

Recall that $\mathbb{P}\mathfrak{l}_v = \{[r,s] : r, s \in \mathfrak{l}_v\}$, and

$$\mathfrak{C}_{g_i} = \big\{\big([r,s], [a_i r + b_i s, c_i r + d_i s]\big)\big\}$$

where $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$.

First let us assume that for all but finally many g_i's we have $c_i \neq 0$. Omitting these finitely many terms, we assume $c_i \neq 0$. Using the non-homogeneous coordinates, we have

$$\mathfrak{C}_{g_i} = \big\{([r,1],[\tfrac{a_i r+b_i}{c_i r+d_i},1]) : -d_i/c_i \neq r \in \mathsf{l}_v\big\} \quad \cup \\ \{([1,0],[a_i/c_i,1])\} \cup \{([-d_i/c_i,1],[1,0])\}$$

Alternatively, except for two points, the graph $\mathfrak{C}_{g_i}$ is given by the equation

$$(s - a_i/c_i)(r + d_i/c_i) = -k/c_i^2 \tag{6.1}$$

—recall that $\det(g_i) = k$. The missing two points can be obtained by taking limit as $r \to \infty$ or $s \to \infty$.

Now if the sequence $\{c_i\}$ is bounded away from ∞ and 0, then since $g_i \to \infty$ we get from (6.1) that $\mathfrak{C}_{g_i}$ converges to $\{([1,0],[-d_i/c_i,1])\}$ or $\{([a_i/c_i,1],[1,0])\}$. Therefore, we may assume that passing to a subsequence either $c_i \to 0$ or $c_i \to \infty$. In either case we get the conclusion in part (2).

It remains to consider the case where $c_{i_m} = 0$ along a subsequence. In this case, $\{\mathfrak{C}_{g_{i_m}}\}$ converges to the union of a line and a point. □

For $j = 1,2$, let

$$\mathfrak{p}_j : \mathrm{PGL}_2 \times \mathrm{PGL}_2 \to \mathrm{PGL}_2$$

be the projection onto the j-th component; put $\sigma_j = \mathfrak{p}_j \circ \sigma : \Gamma \to \mathrm{PGL}_2(\mathsf{l}_v)$. For $j = 1,2$, let u_j denote the cocycle corresponding to σ_j, see §4.1.

If $\sigma(\Gamma) \subset {}^\sigma\mathbf{G}(\mathsf{L}_v)$ is unbounded, then either $\sigma_1(\Gamma)$ or $\sigma_2(\Gamma)$ is unbounded. Using Lemma 5.1, we can show that indeed both these projections are unbounded.

6.2. **Lemma.** *Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold which contains infinitely many totally geodesic surfaces, $\{\mathsf{S}_i : i \in \mathbb{N}\}$. For every i, let $g_i \in F$ (F is our fixed fundamental domain for Γ in G) be so that S_i corresponds to $Hg_i\Gamma$, see §2. For every i, let $\Delta_i = g_i^{-1}Hg_i \cap \Gamma$.*

Suppose $\sigma_j(\Gamma)$ is unbounded for some $j = 1,2$. Then both $\sigma_1(\Delta_i)$ and $\sigma_2(\Delta_i)$ are unbounded for all large enough i. In particular, both $\sigma_1(\Gamma)$ and $\sigma_2(\Gamma)$ are unbounded.

Proof. For every i, let ${}^\sigma\mathbf{H}_i$ denote the Zariski closure of $\sigma(\Delta_i)$. Let j be so that $\sigma_j(\Gamma)$ is unbounded. Then it follows from Lemma 5.1 that $\sigma_j(\Delta_i)$ is unbounded for all large enough i. Indeed by loc. cit. there exists N and τ so that for all large enough i we have the following. There is a subset $Z_i \subset Hg_i\Gamma$ with $\mu_{Hg_i\Gamma}(Z_i) > 1/2$ such that

$$\mathrm{d}_{\mathcal{T}}(o.u_j(n\tau,z),o) > \lambda_0\tau n \tag{6.2}$$

for all $z \in Z_i$, all $n > N$, and $j = 1,2$. Let $z \in Z_i$. Since $Hg_i\Gamma$ is closed and $a_{n\tau}z \in Hg_i\Gamma$ for all n, there is a sequence $k \to \infty$ so that $u_j(n_k\tau,z) = \sigma_j(\delta_k)$

where $\delta_k \in \Delta_i$ is so that $a_{n_k\tau} g_z \in F\delta_k$ and $g_z \in F$ is the lift of z, see §4.1. This and (6.2) imply that $\sigma_j(\Delta_i)$ is unbounded for $j = 1, 2$.

Therefore, by Lemma 2.2 we have: for all large enough i, there exists some $g_{v,i} \in \mathrm{PGL}_2(\mathfrak{l}_v)$ so that

$$^{\sigma}\mathbf{H}_i(\mathfrak{l}_v) \cap \{(h, g_{v,i}^{-1} h g_{v,i}) : h \in \mathrm{PGL}_2(\mathfrak{l}_v)\}$$

is a subgroup of index at most 8 in $^{\sigma}\mathbf{H}_i(\mathfrak{l}_v)$.

The claims hold for all such i. □

6.3. **The definition of Ψ.** Recall that $M = \mathbb{H}^3/\Gamma$ is a closed hyperbolic 3-manifold containing infinitely many totally geodesic surfaces, $\{\mathsf{S}_i : i \in \mathbb{N}\}$.

For each $i \in \mathbb{N}$, we let $\Delta_i \subset \Gamma$ be defined as in Lemma 6.2. Let $^{\sigma}\mathbf{H}_i \subset \mathrm{PGL}_2 \times \mathrm{PGL}_2$ denote the Zariski closure of $\sigma(\Delta_i)$ for every i.

Let i be large enough so that Lemma 6.2 holds true. In particular, there exists some $g_{v,i} \in \mathrm{PGL}_2(\mathfrak{l}_v)$ so that

$$\{(h, g_{v,i}^{-1} h g_{v,i}) : h \in \mathrm{PSL}_2(\mathfrak{l}_v)\}$$

has index at most 8 in $^{\sigma}\mathbf{H}_i(\mathfrak{l}_v)$.

Let $\mathfrak{C}_i = \mathfrak{C}_{g_{v,i}} \subset \mathbb{P}\mathfrak{l}_v \times \mathbb{P}\mathfrak{l}_v$ denote the graph of $g_{v,i} : \mathbb{P}\mathfrak{l}_v \to \mathbb{P}\mathfrak{l}_v$. Then $^{\sigma}\mathbf{H}_i(\mathfrak{l}_v)$ is the stabilizer of $\mathfrak{C}_i$ in $\mathrm{PGL}_2(\mathfrak{l}_v) \times \mathrm{PGL}_2(\mathfrak{l}_v)$.

Recall that each S_i gives rise to a periodic H-orbits, $Hx_i = Hg_i\Gamma$. For every i, the orbit Hx_i corresponds to a closed Γ-orbit

$$\mathcal{C}_i = \{C_i\Gamma\} \subset \mathcal{C}.$$

Identify X with F, then Hx_i is identified with a subset $Y_i \subset F$; note that Y_i has only finitely many connected components. For every $y \in Hx_i$ we let $g_y \in F$ be the point corresponding to y. The orbit Hg_y gives rise to a plane P_y in $\mathbb{H}^3$ and a circle

$$C_y := \partial P_y = C_i\gamma_y \in \mathcal{C}_i \text{ for some } \gamma \in \Gamma. \tag{6.3}$$

By Lemma 6.2, $\sigma_j(\Gamma)$ is unbounded for $j = 1, 2$. For $j = 1, 2$, let u_j be the cocycle and ψ_j the equivariant map constructed in §4.1 using the representation σ_j. Define

$$\Psi := (\psi_1, \psi_2). \tag{6.4}$$

We will show that Ψ satisfies the claims in Proposition 3.1.

6.4. **Lemma.** *Let $\varepsilon > 0$. There exist*

- *natural numbers $N = N(\varepsilon)$ and $\tau = \tau(\varepsilon)$,*
- *for every $i > N$, a subset $Z'_i \subset Hx_i$ with $\mu_{Hx_i}(Z'_i) > 1 - \varepsilon$, and*
- *for every $z \in Z'_i$, a subset $R_z \subset [0, 2\pi]$ with $|R_z| > 2(1 - \varepsilon)\pi$*

so that the following hold.

(1) *For every $z \in Z'_i$, $\theta \in R_z$, $n > N_0$, and $j = 1, 2$ we have*

$$\mathrm{d}_{\mathcal{T}}(o.u_j(n\tau, r_\theta z), o) > \lambda_0 \tau n. \tag{6.5}$$

(2) *For every* $z \in Z'_i$ *and* $\theta \in R_z$ *put*

$$\beta_{g_z,\theta} := \lim_n a_{n\tau} r_\theta g_z \in C_z = C_i\gamma_z$$

where g_z, C_z, *and* γ_z *are as in* (6.3). *Then*

$$\Psi(\beta_{g_z,\theta}) \in \mathfrak{C}_i\sigma(\gamma_z) = \mathfrak{C}_{g_v,i}\sigma(\gamma_z). \tag{6.6}$$

Proof. By Lemma 6.2, $\sigma_j(\Gamma)$ is unbounded for $j = 1, 2$. Let

$$i_0 = i_0(\varepsilon^2/2),\ \tau = \tau(\varepsilon^2/2), \text{ and } N = N(\varepsilon^2/2)$$

be so that the conclusion of Lemma 5.1 holds with $\varepsilon^2/2$ and both σ_1 and σ_2. Moreover, assume that Lemma 6.2 holds true for all $i > i_0$.

Let $Z_i \subset Y_i$ be a subset with $\mu_{Hx_i}(Z_i) > 1 - \varepsilon^2$ so that the conclusion of Lemma 5.1 holds for all $z \in Z_i$ and for both σ_1 and σ_2.

For every $z \in Z_i$, let

$$R_z = \{\theta \in [0, 2\pi] : r_\theta z \in Z_i\}.$$

By Fubini's theorem there is a subset $Z'_i \subset Z_i$ with $\mu_{Hx_i}(Z'_i) > 1-\varepsilon$ so that for all $z \in Z'_i$ we have $|R_z| > 2(1 - \varepsilon)\pi$.

We will show that the lemma holds with this N, Z'_i, and R_z. Let $z \in Z'_i$. Then by Lemma 5.1 we have

$$\mathrm{d}_{\mathcal{T}}(o.u_j(n\tau, r_\theta z), o) > \lambda_0\tau n, \qquad \theta \in R_z,\ n > N, \text{ and } j = 1, 2. \tag{6.7}$$

This establishes the part (1).

Let $\theta \in R_z$. Then by (6.5) and Lemma 2.4 we have the following. There exists a unique geodesic $\{\xi_t : t \in \mathbb{R}\} \subset \mathcal{T}$ with $\xi_0 = o$ so that

$$o.u_j(n\tau, r_\theta z) \to \xi_\infty \in \partial\mathcal{T}. \tag{6.8}$$

Note also that $a_t r_\theta \in H$ for all t and θ. Therefore, $\{p_{g_z}(a_{n\tau_0} r_\theta g_z) : n \in \mathbb{Z}\} \subset P_z$ where $p_{g_z} : Hg_z \to P_z$ is the projection map. Moreover, since $Hg_z\Gamma$ is closed, we get the following:

$$o.u_j(n_m\tau, r_\theta z) \in \mathcal{T}' \text{ for a sequence } n_m \to \infty \tag{6.9}$$

where $\mathcal{T}'$ is the subtree corresponding to $\sigma(\gamma)^{-1}\mathbf{H}_i\sigma(\gamma)$.

Since by (6.8) we have $o.u_j(n_m\tau, r_\theta z) \to \xi_\infty$, the definition of Ψ, (6.4), and (6.9) imply that $\Psi(\beta_{g_z,\theta}) \in \partial\mathcal{T}' = \mathfrak{C}_i\sigma(\gamma_z)$.

This finishes the proof of part (2) and the lemma. □

Recall that for a circle C we denote the length measure on C by m_C. The following is a crucial step in the proof Proposition 3.1.

6.5. **Lemma.** *For every* $\varepsilon > 0$, *there exists a subset* $\hat{F}_\varepsilon \subset F$ *with*

$$\mathrm{vol}_X(\hat{F}_\varepsilon) > 1 - 4\varepsilon$$

and for every $g \in \hat{F}_\varepsilon$ *a subset* $\hat{C}_{g,\varepsilon} \subset C_g$ *with* $\mathsf{m}_{C_g}(\hat{C}_{g,\varepsilon}) \geq (1 - 2\varepsilon)\mathsf{m}_{C_g}(C_g)$ *so that one of the the following holds.*

(1) *There exists some* $h_g \in \mathrm{PGL}_2(\mathfrak{l}_v)$ *so that* $\Psi(\hat{C}_{g,\varepsilon}) \subset \mathfrak{C}_{h_g}$, *or*
(2) $\Psi(\hat{C}_{g,\varepsilon})$ *is contained in a cross or the union of a line and a point.*

Proof. Fix some $\varepsilon > 0$. Let i_0, N, τ, Z'_i, and R_z be as in Lemma 6.4 applied with this ε.

Let F_ε and N_0 be as in Lemma 4.4. Fix a geodesic $\{\xi_t\}$ emanating from o for the rest of the proof. Let $R_g \subset [0, 2\pi]$ be the set where Lemma 4.4 holds for $\{\xi_t\}$, g, u_1, and u_2. In particular, $|R_g| > 2(1-\varepsilon)\pi$ and for $j = 1, 2$ we have

$$\mathrm{d}_{\mathcal{T}}(o.u_j(n, r_\theta g), o) > \lambda_1 n/3 \quad \text{for all } \theta \in R_g,\, n > N_0. \tag{6.10}$$

For any $g \in F_\varepsilon$, let P_g be the plane corresponding to Hg; put $C_g = \partial P_g$. For any $\theta \in R_g$, define

$$\beta_{g,\theta} := \lim_n p_g(a_{n\tau} r_\theta g) \in C_g.$$

where $p_g : Hg \to P_g$ is the natural projection. Then $\Psi(\beta_{g,\theta}) \in \Psi(C_g)$.

Given i and $\delta > 0$, let $\mathcal{N}_{i,\delta}$ be the δ-neighborhood of Z'_i. By Corollary 5.2.1, there exists some $i_1(\delta)$ so that $\mathrm{vol}_X(\mathcal{N}_{i,\delta}) > 1 - 3\varepsilon$ for all $i > i_1(\delta)$.

Claim. *For every $\eta > 0$, there exists some $\delta > 0$ so that the following holds. Let $i > i_\eta := \max\{i_0, i_1(\delta)\}$ and let $g \in F_\varepsilon \cap \mathcal{N}_{i,\delta}$. Then there exists some $C'_{g,\eta} \subset C_g$ with*

$$\mathsf{m}_{C_g}(C'_{g,\eta}) > (1 - 3\varepsilon)\mathsf{m}_{C_g}(C_g)$$

so that $\Psi(C'_{g,\eta})$ lies in the η-neighborhood of $\mathfrak{C}_h$ for some $h \in \mathrm{PGL}_2(\mathfrak{l}_v)$.

Let us first assume the claim and finish the proof of the lemma. For every $m \in \mathbb{N}$, let $\eta_m = 1/m$ and let δ_m and i_m be given by applying the claim with η_m. Set

$$\hat{F}_\varepsilon := F_\varepsilon \cap \Big(\bigcap_{\ell \geq 1} \bigcup_{m \geq \ell} \mathcal{N}_{i_m, \delta_m} \Big);$$

note that $\mathrm{vol}_X(\hat{F}_\varepsilon) \geq 1 - 4\varepsilon$.

Let $g \in \hat{F}_\varepsilon$ and let C_g be the corresponding circle. Then there exists a subsequence $\{m_k\}$ so that $g \in \mathcal{N}_{i_{m_k}, \delta_{m_k}}$. In view of the claim, for every k, there exists some $C'_{g,m_k} \subset C_g$ and some $h_k \in \mathrm{PGL}_2(\mathfrak{l}_v)$ with $\mathsf{m}_{C_g}(C'_{g,m_k}) > (1 - 2\varepsilon)\mathsf{m}_{C_g}(C_g)$ so that

$$\Psi(C'_{g,m_k}) \text{ lies in the } 1/m_k\text{-neighborhood of } \mathfrak{C}_{h_k}. \tag{6.11}$$

Apply Lemma 6.1 with the sequence $\{\mathfrak{C}_{h_k} : k \in \mathbb{N}\}$. Then there exists a subsequence $\{k_i\}$ so that one of the following holds.

(1) $\lim_i \mathfrak{C}_{h_{k_i}} = \mathfrak{C}_h$ for some $h \in \mathrm{PGL}_2(\mathfrak{l}_v)$, or
(2) $\lim_i \mathfrak{C}_{h_{k_i}}$ is contained in a cross or the union of a line and a point.

Let

$$\hat{C}_{g,\varepsilon} := \bigcap_{\ell \geq 1} \bigcup_{i \geq \ell} C'_{g_{k_i}, m_{k_i}},$$

then $\mathsf{m}_C(\hat{C}_{g,\varepsilon}) > (1 - 2\varepsilon)\mathsf{m}(C_g)$. Moreover, for every $\beta \in \hat{C}_{g,\varepsilon}$ we have $\Psi(\beta) \in \lim_i \mathfrak{C}_{h_{k_i}}$. The lemma follows. □

Proof of the claim. Let $g \in F_\varepsilon$. Recall that we fixed a geodesic $\{\xi_t\}$ emanating from o, and let $R_g \subset [0, 2\pi]$ be the set where Lemma 4.4 holds for $\{\xi_t\}$, g, u_1, and u_2. For $g \in F_\varepsilon$ and $\theta \in R_g$, define

$$u(n\tau, r_\theta g) = (u_1(n\tau, r_\theta g), u_2(n\tau, r_\theta g)).$$

For any $z \in Z'_i$, let R_z be as in Lemma 6.4. Similarly, for all $i > i_0$, $z \in Z'_i$, and $\theta \in R_z$ define

$$u(n\tau, r_\theta g_z) = \big(u_1(n\tau, r_\theta g_z), u_2(n\tau, r_\theta g_z)\big).$$

Thanks to (6.5), (6.10), and Lemma 2.4, there exists some n_η so that for all $n \geq n_\eta$ we have $(o_1, o_2).u(n\tau, r_\theta g_z)$ and $(o_1, o_2).u(n\tau, r_\theta g)$ approximate $\Psi(\beta_{g_z,\theta})$ and $\Psi(\beta_{g,\theta})$, respectively, within $\eta/4$.

Let $n = n_\eta$ and let $\delta > 0$ be so that if $\mathrm{d}(h_1, h_2) < \delta$ for $h_1, h_2 \in G$, then $\mathrm{d}(a_{n\tau}h_1, a_{n\tau}h_2) \leq \eta/4$ where d denotes the right invariant Riemannian metric on G.

Apply Corollary 5.2.1 with the sets $\mathcal{N}_{i,\delta}$ and let $i_1(\delta)$ be as in that Corollary. Let $i > \max\{i_0, i_1(\delta)\}$ and let $g \in F_\varepsilon \cap \mathcal{N}_{i,\delta}$. Then there exists some $g_z \in Z'_i$ so that $\mathrm{d}(g, g_z) < \delta$.

The claim thus holds with $C'_{g,\eta} = \{\beta_{g,\theta} : \theta \in R_z \cap R_g\}$. □

Proof of Proposition 3.1. First note that in view of Lemma 2.7, Ψ satisfies part (1) in the proposition —recall that $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts transitively on $\mathbb{P}\mathfrak{l}_v \times \mathbb{P}\mathfrak{l}_v$,

We now show that Ψ also satisfies part (2) in the proposition. We claim that there exists a full measure subset $\hat{F} \subset F$, and for every $g \in \hat{F}$ a full measure subset $\hat{C}_g \subset C_g$ so that one of the following holds.

(1) There exists some $h_g \in \mathrm{PGL}_2(\mathfrak{l}_v)$ so that $\Psi(\hat{C}_g) \subset \mathfrak{C}_{h_g}$, or
(2) $\Psi(\hat{C}_g)$ is contained in a cross or the union of a line and a point.

Apply Lemma 6.5 with $\varepsilon = 1/m$ for all $m \in \mathbb{N}$. Let $\hat{F}_m = \hat{F}_{1/m}$ and for every $g \in \hat{F}_m$ let $\hat{C}_{m,g} = \hat{C}_{1/m,g}$ denote the sets obtained by that lemma. Define

$$\hat{F} = \bigcap_{\ell \geq 1} \bigcup_{m \geq \ell} \hat{F}_m.$$

Then $\hat{F}$ is conull in F.

Moreover, for every $g \in \hat{F}$ there exists a subsequence m_k so that $g \in \hat{F}_{m_k}$ for all k. Let $\hat{C}_g = \bigcap_{\ell \geq 1} \bigcup_{k \geq \ell} \hat{C}_{m_k,g}$. Then $\hat{C}_g \subset C_g$ is conull in C_g. Moreover, $\hat{C}_g$ satisfies (1) or (2) in the claim —recall that the same property holds for $\hat{C}_{m_k,g}$ by Lemma 6.5.

We now show that (1) above holds almost surely. To see this, set $\mathcal{L}_1 := \{\mathfrak{C}_h : h \in \mathrm{PGL}_2(\mathfrak{l}_v)\}$ and $\mathcal{L}_2 := \{\text{union of a line and a point or crosses}\}$. Note that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are Γ-invariant. Moreover, Ψ is Γ-equivariant and Γ acts ergodically on $\mathcal{C}$. Therefore, either the essential image of $\Psi|_C$ belongs to $\mathcal{L}_1$ a.e. $C \in \mathcal{C}$ or the essential image of $\Psi|_C$ belongs to $\mathcal{L}_2$ a.e. $C \in \mathcal{C}$.

Let $\xi \neq \xi' \in \mathbb{S}^2$ be so that $\Psi(\xi) = ([r, 1], [s, 1])$ and $\Psi(\xi') = ([r', 1], [s', 1])$ with $r \neq r'$ and $s \neq s'$ —in view of Lemma 2.7 we may find such points. Then there are exactly two crosses passing through both of $\Psi(\xi)$ and $\Psi(\xi')$; similarly for union of a line and a point. However, the set of circles in $\mathbb{S}^2$ passing through $\{\xi, \xi'\}$ covers the entire $\mathbb{S}^2$. Therefore, the essential image of $\Psi|_C$ belongs to $\mathcal{L}_2$ a.e. $C \in \mathcal{C}$ would contradict Lemma 2.7.

In consequence, we have: for a.e. $g \in \hat{F}$ there exists some $h_g \in \mathrm{PGL}_2(\mathfrak{l}_v)$ so that $\Psi(\hat{C}_g) \subset \mathfrak{C}_{h_g}$. Since Ψ is Γ-equivariant, this concludes the proof of the proposition. □

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