EFFECTIVE COUNTING OF SIMPLE CLOSED GEODESICS ON HYPERBOLIC SURFACES

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Abstract. We prove a quantitative estimate, with a power saving error term, for the number of simple closed geodesics of length at most L on a compact surface equipped with a Riemannian metric of negative curvature. The proof relies on the exponential mixing rate for the Teichmüller geodesic flow.

1. Introduction

Let $g \geq 2$, and let S be a compact Riemann surface of genus g. Let $\mathcal{T}(S)$ be the Teichmüller space of complete hyperbolic metrics on S, and let

$$\mathcal{M}(S) = \mathcal{T}(S)/\operatorname{Mod}_g$$

be the corresponding moduli space, where Mod_g is the mapping class group of S.

Let $M \in \mathcal{M}(S)$. Problems related to the asymptotic growth rate of the number of closed geodesics on M have been long studied. In particular, thanks to works of Delsart, Huber, and Selberg we have the following: There exists some $\delta = \delta(M) > 0$ so that the number of closed geodesics of length at most L on M equals

$$\operatorname{Li}(e^L) + O_M(e^{L-\delta}), \tag{1}$$

where $\operatorname{Li}(x) = \int_2^x \frac{dt}{\log t}$; see [Bus] and references there.

More generally, the growth rate of the number of closed geodesics on a negatively curved compact manifold was studied by Margulis, [Mar]. His proof, which is different from the above mentioned works, is based on the mixing property of the Margulis measure for the geodesic flow. In the constant negative curvature case, Margulis' method combined with an exponential mixing rate for the geodesic flow, also provides an estimate like (1) — albeit with a weaker power saving δ, see e.g. [MMO].

1.1. Simple closed geodesics.

Simple closed geodesics. The aforementioned fundamental results do not provide any estimates for the number of simple closed geodesics on M. Indeed, very few closed geodesics on M are simple, [BS2], and it is hard to discern them in $\pi_1(M)$, [BS1]. More explicitly, it was shown in [Ri] that the number of *simple* closed geodesics of length at most L on M is bounded above and below by $O_M(L^{6g-6})$.

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In her PhD thesis, [Mir1] and [Mir2], Mirzakhani proved an asymptotic growth rate for the number of simple closed geodesics of a given topological type on a hyperbolic surface M — recall that two simple closed geodesics γ and γ' on M are of the same topological type if there exists some $\mathbf{g} \in \mathrm{Mod}_g$ so that $\gamma' = \mathbf{g}\gamma$.

Let X be a compact surface equipped with a Riemannian metric of negative curvature. We emphasize that the curvature is not assumed to be constant; indeed, elements in $\mathcal{M}(S)$ will be denoted by M to minimize the confusion. By a multi-geodesic γ on X we mean $\gamma = \sum_{i=1}^{d} a_i\gamma_i$ where γ_i's are disjoint, essential, simple closed geodesics, and $a_i > 0$ for all $1 \le i \le d$. In this case, we define $\ell_X(\gamma) := \sum a_i \ell_X(\gamma)$, where ℓ_X denotes the length function on X. The multi-geodesic γ will be called integral (resp. rational) if $a_i \in \mathbb{Z}$ (resp. $a_i \in \mathbb{Q}$).

Given a rational multi-geodesic γ_0 on X, define

$$s_X(\gamma_0, L) := \#\{\gamma \in \mathrm{Mod}_g .\gamma_0 : \ell_X(\gamma) \le L\}.$$

Mirzakhani, [Mir2, Thm. 1.1], proved the following estimate when M is a hyperbolic surface:

$$s_M(\gamma_0, L) \sim n_{\gamma_0}(M) L^{6g-6}, \tag{2}$$

where $n_{\gamma_0} : \mathcal{M}(S) \to \mathbb{R}^+$ (the *Mirzakhani* function) is a continuous proper function; geometric informations carried by n_{γ_0} are also studied in [Mir2].

In this paper we obtain a quatitative version of (2); moreover, our approach allows us to prove such a result in the more general setting of *variable* negative curvature.

Theorem 1.1. *There exists some $\kappa = \kappa(g) > 0$ so that the following holds. Let X be a compact surface of genus g equipped with a Riemannian metric of negative curvature. Let γ_0 be a rational multi-geodesic on X. Then*

$$s_X(\gamma_0, L) = n_{\gamma_0}(X) L^{6g-6} + O_{\gamma_0,X}(L^{6g-6-\kappa})$$

where $n_{\gamma_0}(X)$ is a positive constant which depends on γ_0 and X.

The proof of Theorem 1.1 is based on the study of a related counting problem in the space of geodesic measured laminations on S, à la Mirzakhani. The space of measured laminations on S, which we denote by $\mathcal{ML}(S)$, is a piecewise linear integral manifold homeomorphic to $\mathbb{R}^{6g-6}$; but it does not have a natural differentiable structure, [Th1]. Train tracks were introduced by Thurston as a powerful technical device for understanding measured laminations. Roughly speaking, train tracks are induced by squeezing almost parallel strands of a very long simple closed geodesic to simple arcs on a surface; they provide linear charts for $\mathcal{ML}(S)$.

The mapping class group Mod_g of S acts naturally on $\mathcal{ML}(S)$. Moreover, there is a natural Mod_g-invariant locally finite measure on $\mathcal{ML}(S)$, the Thurston measure μ_{Th}, given by the piecewise linear integral structure on $\mathcal{ML}(S)$, [Th1]. For any open subset $U \subset \mathcal{ML}(S)$ and any $t > 0$, we have

$$\mu_{\mathrm{Th}}(tU) = t^{6g-6}\mu_{\mathrm{Th}}(U).$$

On the other hand, any metric of negative curvature X on S induces the length function $\lambda \mapsto \ell_X(\lambda)$ on $\mathcal{ML}(S)$, which satisfies $\ell_X(t\lambda) = t\ell_X(\lambda)$ for all $t > 0$. It is proved in [Mir1, App. A] that ℓ_M is a convex function on $\mathcal{ML}(S)$ when M is a hyperbolic surface. This fact remains valid in the more general setting of variable negative curvature, see §5.5.

The source of the polynomially effective error term in Theorem 1.1 is the exponential mixing property of the Teichmüller geodesic flow proved by Avila, Gouëzel, and Yoccoz, [AGY, AR, AG]. We combine this estimate with ideas developed by Margulis in his PhD thesis, [Mar], to prove the following theorem which is of independent interest — see Theorem 7.1 for a more general statement.

Let τ be a train track and let $U(\tau)$ be the corresponding train track chart. For every $\lambda \in U(\tau)$ we let $\|\lambda\|_\tau$ denote the sum of the weights of λ in $U(\tau)$, see §5.

Theorem 1.2. *There exists some $\kappa_1 = \kappa_1(g) > 0$ so the following holds. Let τ be a maximal train track. Let $L \geq 1$ and let γ_0 be a simple close curve on M. There exists a constant $c_{\gamma_0} > 0$ so that*

$$\#\{\gamma \in U(\tau) \cap \mathrm{Mod}_g . \gamma_0 : \|\gamma\|_\tau \leq L\} = c_{\gamma_0} \mathrm{vol}_\tau L^{6g-6} + O_{\tau,\gamma_0}(L^{6g-6-\kappa_1})$$

where $\mathrm{vol}_\tau = \mu_{\mathrm{Th}}\{\lambda \in U(\tau) : \|\lambda\|_\tau \leq 1\}$.

It is worth noting that in view of Theorem 1.2, the asymptotic behavior of the number of points in one Mod_g-orbit in the cone $\{\lambda : \|\lambda\|_\tau \leq L\}$ and that of the number of integral points in this cone agree up to multiplicative constant.

Theorem 1.2, in the more general form Theorem 7.1, plays a crucial role in our analysis. Indeed, using the aforementioned convexity of the length function, we will prove Theorem 1.1 using Theorem 7.1 in §8.

It is an intriguing problem to investigate the asymptotic behavior of functions similar to and different from $s_X(\gamma_0, L)$ or the complexity considered in Theorem 1.2. For instance, for a suitable formulation of a combinatorial length — using intersection numbers — the count is exactly a polynomial, see [FLP]. We also refer the reader to [CMP] where a related problem is studied for thrice punctured sphere.

1.2. **Outline of the paper.** In §2 we collect some preliminary results. In §3 we prove an equidistribution result with an error term, Proposition 3.2, which may be of independent interest; see, e.g. [KM, LMir]. The proof of this proposition is based on the exponential mixing rate for the Teichmüller geodesic flow, [AGY], and the so called *thickening* technique, see [Mar, EMc]. In §4 we prove Proposition 4.1; this proposition is one of the main ingredients in the proof, and could be compared to arguments in [Mar, Chap. 6]. We will recall some basic facts about $\mathcal{ML}(S)$, and study the relation between the linear structures on $\mathcal{ML}(S)$ and the space of quadratic differentials in §5 and §6. The orbital counting in sectors of $\mathcal{ML}(S)$ is studied in §7; the main result here is Theorem 7.1. We prove Theorem 1.1 in §8.

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2. Preliminaries and notation

Let $\mathcal{Q}(S)$ denote the moduli space of quadratic differentials on S, and let $\mathcal{Q}_1(S)$ be the moduli space of quadratic differentials with area one on S. For any $\alpha = (\alpha_1, \dots, \alpha_k, \varsigma)$ with $\sum \alpha_i = 4g-4$ and $\varsigma \in \{\pm 1\}$, define $\mathcal{Q}_1(\alpha)$ to be (a connected component) of the stratum of quadratic differentials consisting of pairs (M, q) where $M \in \mathcal{M}(S)$ and q is a unit area quadratic differential on M whose zeros have multiplicities $\alpha_1, \dots, \alpha_k$ and $\varsigma = 1$ if q is the quare of an abelian differential and -1 otherwise. Then

$$\mathcal{Q}_1(S) = \bigsqcup_{\alpha} \mathcal{Q}_1(\alpha).$$

Put $\mathcal{Q}(\alpha) := \{tq : t \in \mathbb{R}, q \in \mathcal{Q}_1(\alpha)\}$. Let $\Sigma \subset S$ be a set of k distinct marked points. Let $\mathcal{Q}^1\mathcal{T}(\alpha)$ denote the space of quadratic differentials (M, q) equipped with an equivalence class of homeomorphisms $f : S \to M$ that send the marked points to the zeros of q. The equivalence relation is isotopy rel marked points. Let $\pi : \mathcal{Q}^1\mathcal{T}(\alpha) \to \mathcal{Q}_1(\alpha)$ be the forgetful map which forgets the marking f; this is an infinite degree branched covering.

Similarly, let $\Omega(S)$ denote the moduli space of Abelian differentials on S, and let $\Omega_1(S)$ be the moduli space of area one Abelian differentials. For any $\alpha = (\alpha_1, \dots, \alpha_k)$, we let $\mathcal{H}(\alpha)$ denote the corresponding stratum, and let $\mathcal{H}_1(\alpha)$ denote the area one abelian differentials.

Note that passing to a branched double cover $\hat{M}$ of M, we may realize $\mathcal{Q}_1(\alpha)$ as an *affine invariant submanifold* in $\mathcal{H}_1(\hat{\alpha})$ corresponding to odd cohomology classes on $\hat{M}$, see §2.1. However, even if q belongs to a compact subset of $\mathcal{Q}_1(S)$, the complex structure on $\hat{M}$ may have very short closed curves in the hyperbolic metric, e.g. a short saddle connection between two distinct zeros on (M, q) could lift to a short loop in $\hat{M}$. Note however that if $(\hat{M}, \omega)$ is the aforementioned double cover of (M, q), then the length of the shortest saddle connection in ω is bounded by the length of the shortest saddle connection in q, i.e. compact subsets of $\mathcal{Q}_1(\alpha)$ lift to compact subsets of $\mathcal{H}_1(\hat{\alpha})$.

2.1. **Period coordinates.**

Let $x = (M, \omega) \in \mathcal{H}(\alpha)$, and let $\Sigma \subset M$ be the set of zeros of ω. Passing to a finite cover, which we continue to denote by $\mathcal{H}(\alpha)$, we assume there are no orbifold points in $\mathcal{H}(\alpha)$. Define the period map

$$\Phi : \mathcal{H}(\alpha) \to H^1(M, \Sigma, \mathbb{C}).$$

Let us recall that Φ can be defined as follows. Let $\#\Sigma = k$. Fix a triangulation T of the surface by saddle connections of x, that is: $2g + k - 1$ directed edges $\delta_1, \dots, \delta_{2g+k-1}$ which form a basis for $H_1(M, \Sigma, \mathbb{Z})$. Define

$$\Phi(x) = \Big(\int_{\delta_i} \omega \Big)_{i=1}^{2g+k-1}.$$

Note that this map depends on the triangulation T. If T' is any other triangulation, and Φ' is the corresponding period map, then $\Phi' \circ \Phi^{-1}$ is linear. For any $x \in \mathcal{H}(\alpha)$, there is a neighborhood $\mathsf{B}(x)$ of x so that the restriction of Φ to $\mathsf{B}(x)$ is a homeomorphism onto $\Phi(\mathsf{B}(x))$, see §2.9. We always choose $\mathsf{B}(x)$ small enough so that, using the Gauss-Manin connection, the triangulation at $y \in \mathsf{B}(x)$ can be identified with the triangulation at x.

We define the period coordinates at $x = (M, q) \in \mathcal{Q}(\alpha)$ as follows. If $\varsigma = 1$, then q is a square of an abelian differential, and we may define period coordinates as above. If $\varsigma = -1$, we use the orienting double cover $\mathcal{H}(\hat{\alpha})$ to define the period coordinates: in this case there is a canonical injection from $\mathcal{Q}(\alpha)$ into $\mathcal{H}(\hat{\alpha})$. Any Riemann surface in the image of this map is equipped with an involution. This way we get the period map from $\mathcal{Q}(\alpha)$ to $H^1_{\mathrm{odd}}(M, \Sigma, \mathbb{C})$ — the anti-invariant subspace of the cohomology for the involution.

Put $h := 2g + k - 2$ if $\varsigma = 1$ and $h := 2g + k - 3$ if $\varsigma = -1$; the number h is the topological entropy of the Teichmüller geodesic flow on $\mathcal{Q}_1(\alpha)$.

2.2. $\mathrm{SL}(2,\mathbb{R})$**-action on** $\mathcal{H}_1(\alpha)$**.** Let $x \in \mathcal{H}_1(\alpha)$, we write $\Phi(x)$ as a $2\times n$ matrix. The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ in these coordinates is linear. We choose a fundamental domain for the action of the mapping class group and think of the dynamics on the fundamental domain. Then, the $\mathrm{SL}(2,\mathbb{R})$-action becomes

$$\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} A(g,x),$$

where $A(g,x) \in \mathrm{Sp}(2g,\mathbb{Z}) \ltimes \mathbb{Z}^{k-1}$ is the *Kontsevich-Zorich cocycle*. That is: $A(g,x)$ is the change of basis one needs to perform to return the point gx to the fundamental domain. It can be interpreted as the monodromy of the Gauss-Manin connection restricted to the orbit of $\mathrm{SL}(2,\mathbb{R})$.

In the sequel, we let $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, and $\bar{u}_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$.

We have the following.

Theorem 2.1 (Veech-Masur)**.** *The space* $\mathcal{H}_1(\alpha)$ *carries a natural measure* μ *in the Lebesgue measure class such that*

(1) $\mathcal{H}_1(\alpha)$ *has finite measure,*
(2) μ *is* $\mathrm{SL}(2,\mathbb{R})$*-invariant and ergodic.*

More generally, for any affine invariant manifold, $\mathcal{M} \subset \mathcal{H}_1(\alpha)$, we let μ denote the $\mathrm{SL}(2,\mathbb{R})$-invariant affine measure on $\mathcal{M}$. In particular, all the strata in $\mathcal{Q}_1(S)$ are equipped with such invariant measures.

2.3. **Mapping class group action.** We denote elements in Mod_g using bold letters, e.g., $\mathbf{g}$ denotes an element in Mod_g. The action of Mod_g on $\mathcal{Q}^1\mathcal{T}(\alpha)$ commutes with the action of $\mathrm{SL}(2,\mathbb{R})$, we will however denote both these actions as left action and write, e.g. $\mathbf{g} \cdot \tilde{x}$ or simply denoted by $\mathbf{g}\tilde{x}$.

2.4. **The constants.** In the sequel we will use $\kappa_\bullet$ and $N_\bullet$, $\bullet = 1, 2, \dots$ to denote various constants. Unless it is explicitly mentioned otherwise, these constants are allowed only to depend on the genus. The constants $\kappa_\bullet$ are meant to indicate small positive numbers while $N_\bullet$ are used for constants which are expected to be > 1.

We will also use the notation $A \ll B$. This expression means: there exists a constant $c > 0$ so that $A \leq cB$; the implicit constant c is permitted to depend on the genus, but (unless otherwise noted) not on anything else. We write $A \asymp B$ if $A \ll B \ll A$. If a constant (implicit or explicit) depends on another parameter others than the genus, we will make this clear by writing, e.g. $\ll_\epsilon$, $C(x)$, etc.

We also adopt the following $\star$-notation. We write $B = A^{\pm\star}$ if $B = A^{\pm c}$ where $c > 0$ depends only on the genus. Similarly, one defines $B \ll A^\star$, $B \gg A^\star$. Finally, we also write $A \asymp B^\star$ if $A^\star \ll B \ll A^\star$ (possibly with different exponents).

2.5. **Modified Hodge norm.** Let M be a Riemann surface. By definition, M has a complex structure. Let $\mathcal{H}_M$ denote the set of holomorphic 1-forms on M. One can define the *Hodge inner product* on $\mathcal{H}_M$ by

$$\langle \omega, \eta \rangle = \frac{\mathbf{i}}{2} \int_M \omega \wedge \bar{\eta}.$$

We have a natural map $r : H^1(M, \mathbb{R}) \to \mathcal{H}_M$ which sends a cohomology class $c \in H^1(M, \mathbb{R})$ to the holomorphic 1-form $r(c) \in \mathcal{H}_M$ such that the real part of $r(c)$ (which is a harmonic 1-form) represents c. We can thus define the Hodge inner product on $H^1(M, \mathbb{R})$ by $\langle c_1, c_2 \rangle = \langle r(c_1), r(c_2) \rangle$. Then

$$\langle c_1, c_2 \rangle = \int_M c_1 \wedge *c_2,$$

where $*$ denotes the Hodge star operator and we choose harmonic representatives of c_1 and $*c_2$ to evaluate the integral. We denote the associated norm by $\| \cdot \|_M$. This is the *Hodge norm*, see [FK].

If $x = (M, \omega) \in \mathcal{H}_1(\alpha)$, we will often write $\| \cdot \|_{\mathrm{H},x}$ to denote the Hodge norm $\| \cdot \|_M$ on $H^1(M, \mathbb{R})$. Since $\| \cdot \|_{\mathrm{H},x}$ depends only on M, we have $\|c\|_{\mathrm{H},kx} = \|c\|_{\mathrm{H},x}$ for all $c \in H^1(M, \mathbb{R})$ and all $k \in \mathrm{SO}(2)$.

Let $E(x) = \mathrm{span}\{[\mathrm{Re}(\omega)], [\mathrm{Im}(\omega)]\}$ — the space $E(x)$ is often referred to as the *standard space*. We let

$$p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R}) \tag{3}$$

denote the natural projection; p defines an isomorphism between $E(x)$ and $p(E(x)) \subset H^1(M, \mathbb{R})$.

For our applications in the sequel (and in order to account for the *loss* of hyperbolicity in the thin part of the moduli space) we need to consider a modification of the Hodge norm.

The classes c_α and $*c_\alpha$. Let α be a homology class in $H_1(M,\mathbb{R})$. We let $*c_\alpha \in H^1(M,\mathbb{R})$ be the cohomology class so that

$$\int_\alpha \omega = \int_M \omega \wedge *c_\alpha$$

for all $\omega \in H^1(M,\mathbb{R})$. Then,

$$\int_M *c_\alpha \wedge *c_\beta = i(\alpha,\beta),$$

where $i(\cdot,\cdot)$ denotes the algebraic intersection number. Let $*$ denote the Hodge star operator, and let

$$c_\alpha = *^{-1}(*c_\alpha).$$

Then, for any $\omega \in H^1(M,\mathbb{R})$ we have

$$\langle \omega, c_\alpha \rangle = \int_M \omega \wedge *c_\alpha = \int_\alpha \omega,$$

where $\langle \cdot,\cdot \rangle$ is the Hodge inner product. We note that $*c_\alpha$ is a purely topological construction which depends only on α, but c_α depends also on the complex structure of M.

Fix $\epsilon_* > 0$ (the *Margulis constant*) so that any two geodesics of hyperbolic length less than ϵ_* must be disjoint.

Let σ denote the hyperbolic metric in the conformal class of M. For any closed curve α on M, let $\ell_M(\alpha)$ denote the length of the geodesic representative of α in the metric σ.

We recall the following.

Theorem 2.2. [ABEM, Thm. 3.1] *For any constant $L > 1$ there exists a constant $c > 1$, such that for any simple closed curve α with $\ell_M(\alpha) < L$, we have*

$$\frac{1}{c}\ell_M(\alpha)^{1/2} \le \|c_\alpha\|_M < c\,\ell_M(\alpha)^{1/2}. \tag{4}$$

Furthermore, if $\ell_M(\alpha) < \epsilon_*$ and β is the shortest simple closed curve crossing α, then

$$\frac{1}{c}\ell_M(\alpha)^{-1/2} \le \|c_\beta\|_M < c\,\ell_M(\alpha)^{-1/2}.$$

Short bases. Suppose $(M,\omega) \in \mathcal{H}_1(\alpha)$. Fix $\epsilon_1 < \epsilon_*$ and let $\alpha_1,\dots,\alpha_k$ be the curves with hyperbolic length less than ϵ_1 on M. For every $1 \le i \le k$, let β_i be the shortest curve in the flat metric defined by ω with $i(\alpha_i,\beta_i) = 1$. We can pick simple closed curves γ_r, $1 \le r \le 2g-2k$ on M so that the hyperbolic length of each γ_r is bounded by a constant L depending only on the genus, and so that the α_j, β_j and γ_j form a symplectic basis $\mathcal{S}$ for $H_1(M,\mathbb{R})$. We will call such a basis *short.* A short basis is not unique, and in the following we fix some measurable choice of a short basis at each point of $\mathcal{H}_1(\alpha)$.

We recall the definition of a modified Hodge norm from [EMM]; this is similar (but not the same) to the one defined in [ABEM]. The modified norm is defined on the tangent space to the space of pairs (M,ω) where M is a Riemann surface and ω is a holomorphic 1-form on M. Unlike the Hodge norm, the modified Hodge norm will depend not only on the complex structure on M but also on the choice of a holomorphic 1-form ω on M. Let $\{\alpha_i,\beta_i,\gamma_r\}_{1\le i\le k, 1\le r\le 2g-2k}$ be a short basis for $x = (M,\omega)$.

We can write any $\theta \in H^1(M,\mathbb{R})$ as

$$\theta = \sum_{i=1}^{k} a_i(*c_{\alpha_i}) + \sum_{i=1}^{k} b_i \ell_{\alpha_i}(\sigma)^{1/2}(*c_{\beta_i}) + \sum_{r=1}^{2g-2k} u_i(*c_{\gamma_r}), \tag{5}$$

We then define

$$\|\theta\|''_x = \|\theta\|_{\mathrm{H},x} + \left(\sum_{i=1}^{k} |a_i| + \sum_{i=1}^{k} |b_i| + \sum_{r=1}^{2g-2k} |u_r|\right). \tag{6}$$

Note that $\|\cdot\|''$ depends on the choice of a short basis; however, switching to a different short basis can change $\|\cdot\|''$ by at most a fixed multiplicative constant depending only on the genus.

From (6) we have: for $1 \le i \le k$,

$$\| *c_{\alpha_i}\|''_x \asymp 1, \tag{7}$$

see §2.4 for the notation $\asymp$. Similarly, we have

$$\| *c_{\beta_i}\|''_x \asymp \| *c_{\beta_i}\|_{\mathrm{H},x} \asymp \frac{1}{\ell_M(\alpha_i)^{1/2}}. \tag{8}$$

In addition, in view of Theorem 2.2, if γ is any other moderate length curve on M, $\|*c_\gamma\|''_x \asymp \| *c_\gamma\|_{\mathrm{H},x} = O(1)$. Thus, if $\mathcal{B}$ is a short basis at $x = (M,\omega)$, then for any $\gamma \in \mathcal{B}$,

$$\mathrm{Ext}_\gamma(x)^{1/2} \asymp \|*c_\gamma\|_{\mathrm{H},x} \le \|*c_\gamma\|'' \tag{9}$$

By $\mathrm{Ext}_\gamma(x)$ we mean the extremal length of γ in M, where $x = (M,\omega)$.

Remark. From the construction, we see that the modified Hodge norm is greater than the Hodge norm. Also, if the flat length of shortest curve in the flat metric defined by ω is greater than ϵ_1, then for any cohomology class c, for some N depending on ϵ_1 and the genus,

$$\|c\|'' \le N\|c\|_{\mathrm{H},x}; \tag{10}$$

i.e., the modified Hodge norm is within a multiplicative constant of the Hodge norm.

Note however that for a fixed absolute cohomology class c, $\|c\|''_x$ is not a continuous function of x, as x varies in a Teichmüller disk; this is due to the dependence on the choice of a short basis. To remedy this, we pick a positive, continuous, SO(2)-bi-invariant function ϕ on $SL(2,\mathbb{R})$ which is supported on a neighborhood of the identity with $\int_{SL(2,\mathbb{R})} \phi(g)\,dg = 1$, and define

$$\|c\|'_x = \|c\|_{\mathrm{H},x} + \int_{SL(2,\mathbb{R})} \|c\|''_{gx}\,\phi(g)\,dg.$$

It follows from [EMM, Lemma 7.4] that for a fixed c, $\log \|c\|'_x$ is uniformly continuous as x varies in a Teichmüller disk. In fact, there is a constant m_0 such that for all $x \in \mathcal{H}_1(\alpha)$, all $c \in H^1(M,\mathbb{R})$ and all $t > 0$,

$$e^{-m_0 t}\|c\|'_x \le \|c\|'_{a_t x} \le e^{m_0 t}\|c\|'_x. \tag{11}$$

Remark 2.3. Even though $\|\cdot\|'_x$ is uniformly continuous as long as x varies in a Teichmüller disk, it may be only measurable in general (because of the choice of short basis).

2.6. **Relative cohomology.** For $c \in H^1(M,\Sigma,\mathbb{R})$ and $x = (M,\omega) \in \mathcal{H}_1(\alpha)$, let $\mathfrak{p}_x(c)$ denote the harmonic representative of $p(c)$, where $p : H^1(M,\Sigma,\mathbb{R}) \to H^1(M,\mathbb{R})$ is the natural map. We view $\mathfrak{p}_x(c)$ as an element of $H^1(M,\Sigma,\mathbb{R})$. Then, (similarly to [EMM, §7], see also [ABEM] and [EMR]) we define the modified Hodge norm $\| \ \|'$ on $H^1(M,\Sigma,\mathbb{R})$ as follows.

$$\|c\|'_x = \|p(c)\|'_x + \sum_{(z,z')\in\Sigma\times\Sigma} \left| \int_{\gamma_{z,z'}} (c - \mathfrak{p}_x(c)) \right|,$$

where $\gamma_{z,z'}$ is any path connecting the zeroes z and z' of ω. Since $c - \mathfrak{p}_x(c)$ represents the zero class in absolute cohomology, the integral does not depend on the choice of $\gamma_{z,z'}$. Note that the $\|\cdot\|'$ norm on $H^1(M,\Sigma,\mathbb{R})$ is invariant under the action of $\mathrm{SO}(2)$.

As above, we pick a positive continuous $\mathrm{SO}(2)$-bi-invariant function ϕ on $\mathrm{SL}(2,\mathbb{R})$ supported on a neighborhood of the identity such that $\int_{\mathrm{SL}(2,\mathbb{R})} \phi(g)\,dg = 1$, and define

$$\|c\|_x = \int_{\mathrm{SL}(2,\mathbb{R})} \|c\|'_{gx}\,\phi(g)\,dg. \tag{12}$$

Then, the $\|\cdot\|_x$ norm on $H^1(M,\Sigma,\mathbb{R})$ is also invariant under the action of $\mathrm{SO}(2)$.

By [EMM, Lemma 7.5] there exists some N_1 so that

$$e^{-N_1 t}\|c\|_x \le \|(a_t)_* c\|_{a_t x} \le e^{N_1 t}\|c\|_x. \tag{13}$$

2.7. **The AGY-norm.** Let $\|\cdot\|_{\mathrm{AGY},x}$ denote the norm defined in [AGY, §2.2.2]. We recall the definition: let $x = (M,\omega) \in \mathcal{H}_1(\alpha)$. For any $c \in H^1(M,\Sigma,\mathbb{C})$, define

$$\|c\|_{\mathrm{AGY},x} = \sup_\gamma \frac{|c(\gamma)|}{|\Phi(x)(\gamma)|} \tag{14}$$

where the supremum is taken over all saddle connections of ω. This defines a norm and the corresponding Finsler metric is complete, see [AGY].

We note that $\| \ \|_x$ and $\| \ \|_{\mathrm{AGY},x}$ are commensurable to each other on compact subsets of $\mathcal{H}_1(\alpha)$.

For every $x = (M,q) \in \mathcal{Q}_1(\alpha)$, we define the norms $\| \ \|_x$ and $\| \ \|_{\mathrm{AGY},x}$ using the branched double cover $\hat{M}$.

Lemma 2.4. *Let $c \in H^1(M,\Sigma,\mathbb{C})$, $t \ge 0$ and $s \in [0,1]$. Then*

$$e^{-2-2t}\|c\|_{\mathrm{AGY},x} \le \|(a_t u_s)_* c\|_{\mathrm{AGY},a_t u_s x} \le e^{2+2t}\|c\|_{\mathrm{AGY},x}. \tag{15}$$

Proof. This is proved in [AG, Lemma 5.2], see also [AGY, eq. (2.13)], we recall the argument. Write $c = a' + \mathbf{i}b'$ and $\Phi(x) = a + \mathbf{i}b$. Then the definition (14), implies that for all $t \ge 0$ and $|s| \le 1$ we have

$$\begin{aligned} \|(a_t u_s)_* c\|_{\mathrm{AGY},a_t u_s x} &= \sup_\gamma \tfrac{|e^t(a'(\gamma)+sb'(\gamma))+\mathbf{i}e^{-t}b'(\gamma)|}{|e^t(a(\gamma)+sb(\gamma))+\mathbf{i}e^{-t}b(\gamma)|} \\ &\le e^{2t} \sup_\gamma \tfrac{|a'(\gamma)+sb'(\gamma)+\mathbf{i}b'(\gamma)|}{|a(\gamma)+sb(\gamma)+\mathbf{i}b(\gamma)|}. \end{aligned} \tag{16}$$

By the triangle inequality, for every $|s| \leq 1$ and every $z = z_1 + \mathbf{i}z_2 \in \mathbb{C}$ we have

$$|u_s z| = |z_1 + sz_2 + \mathbf{i}z_2| \leq |z_1 + \mathbf{i}z_2| + |z_2| \leq 2|z|;$$

since $z = u_{-s}u_s z$, we also get that $|u_s z| \geq |z|/2$.

This observation and (16) imply that

$$\|(a_t u_s)_* c\|_{\mathrm{AGY}, a_t u_s x} \leq 4e^{2t} \sup_\gamma \tfrac{|a'(\gamma)+\mathbf{i}b'(\gamma)|}{|a(\gamma)+\mathbf{i}b(\gamma)|}.$$

The lower bound follows similarly. □

2.8. **Non-divergence results.** Recall that $\mathcal{Q}_1(\alpha)$ is realized as an affine invariant submanifold in $\mathcal{H}_1(\hat{\alpha})$, moreover, compact subsets of $\mathcal{Q}_1(\alpha)$ lift to compact subsets of $\mathcal{H}_1(\hat{\alpha})$. Let $u : \mathcal{H}_1(\hat{\alpha}) \to [2, \infty]$ be the function constructed in [EMas] and [Ath].

Theorem 2.5. *There exists a compact subset $K'_\alpha \subset \mathcal{Q}_1(\alpha)$ and some $N_2 > 0$ with the following property. For every t_0 and every $x \in \mathcal{Q}_1(\alpha)$, there exists*

$$s \in [0, 1/2] \text{ and } t_0 \leq t \leq \max\{2t_0, N_2 \log u(x)\}$$

such that $x' = a_t u_s x \in K'_\alpha$.

Proof. The stratum $\mathcal{Q}_1(\alpha)$ is an affine invariant submanifold in $\mathcal{H}_1(\hat{\alpha})$. The claim thus follows from [Ath, Thm. 2.2] and [AG, Lemma 6.3] applied with $\delta = 1/2$. □

2.9. **Period box.** Let $\tilde{x} = (M, q) \in \mathcal{Q}^1\mathcal{T}(\alpha)$. For every $r > 0$ define

$$\mathsf{R}_r(\tilde{x}) := \big\{\Phi(\tilde{x}) + a' + \mathbf{i}b' : a', b' \in H^1(M, \Sigma, \mathbb{R}), \|a' + \mathbf{i}b'\|_{\mathrm{AGY}, \tilde{x}} \leq r\big\}.$$

Let now $r > 0$ be so that Φ^{-1} is a homeomorphism on $\mathsf{R}_r(\tilde{x}) \cap \Phi(\mathcal{Q}^1\mathcal{T}(\alpha))$. Put

$$\mathsf{B}_r(\tilde{x}) = \Phi^{-1}\Big(\mathsf{R}_r(\tilde{x})\Big).$$

The open subset $\mathsf{B}_r(\tilde{x})$ will be called a *period box* of radius r centered at $\tilde{x}$. Thanks to [AG, Prop. 5.3], $\mathsf{B}_r(\tilde{x})$ is well defined for all $0 < r \leq 1/2$ and *all* $\tilde{x} \in \mathcal{Q}^1\mathcal{T}(\alpha)$. We also have the following.

Lemma 2.6. *There exists some N_3 so that for all $x \in \mathcal{Q}_1(\alpha)$ and every $0 < r \leq u(x)^{-N_3}$ the following hold. Let $\tilde{x} \in \mathcal{Q}^1\mathcal{T}(\alpha)$ be a lift of x. Then*

(1) *The restriction of the covering map π to $\mathsf{B}_r(\tilde{x})$ is injective.*
(2) *For all $\tilde{x}_1, \tilde{x}_2 \in \mathsf{B}_r(\tilde{x})$, the Teichmüller distance between $\tilde{x}_1$ and $\tilde{x}_2$ is at most 1.*

Proof. The argument is similar to the one used in the proof of [EMM, Lemma 8.2].

For part (2) we will need the following two facts: $d_{\mathcal{T}}((a_t u_s)^{\pm 1} z, (a_t u_s)^{\pm 1} z') \leq 16e^{2t}$ for all $t \geq 0$ and $s \in [-1, 1]$ where $d_{\mathcal{T}}$ denotes the Tichmüller distance. Moreover, there exist a constant $C \geq 1$ so that

$$C^{-1} d_{\mathrm{AGY}}(z, z') \leq d_{\mathcal{T}}(z, z') \leq C d_{\mathrm{AGY}}(z, z') \quad \text{for all } z, z' \in K'_\alpha,$$

where $K'_\alpha \subset \mathcal{Q}_1(\alpha)$ is the compact set introduced in Theorem 2.5.

We now turn to the proof of the lemma. For every $x \in K'_\alpha$, there exists $0 < r(x) \le 1/2$ so that $\mathsf{B}_{r(x)}(x)$ is embedded in the sense that the projection from the Teichmüller space $\mathcal{Q}^1\mathcal{T}(\alpha)$ to the Moduli space $\mathcal{Q}_1(\alpha)$ restricted to $\mathsf{B}_{r(x)}(\tilde{x})$ is injective. Let $r_0 = \inf_{x\in K'_\alpha} r(x)$. By compactness of K'_α, $r_0 > 0$. Decreasing r_0 if necessary, we assume that for all $x \in K'_\alpha$ and all $\tilde{x}_1, \tilde{x}_2 \in \mathsf{B}_{r_0}(\tilde{x})$, the Teichmüller distance between $\tilde{x}_1$ and $\tilde{x}_2$ is at most 1.

Let $N \ge 1$ be so that

$$C2^{4N_2-N+16} < r_0 \le 1/2. \tag{17}$$

where N_2 is as in Theorem 2.5.

We will show that $N_3 = N$ satisfies the claims in the lemma. First note that in view of [AG, Prop. 5.3], $\mathsf{B}(\tilde{x}) := \mathsf{B}_{u(x)^{-N}}(\tilde{x})$ is well defined for all $x \in \mathcal{Q}_1(\alpha)$ and all the lifts $\tilde{x} \in \mathcal{Q}^1\mathcal{T}(\alpha)$. Suppose now that there exists $x \in \mathcal{Q}_1(\alpha)$ and $\tilde{x}_1, \tilde{x}_2 \in \mathsf{B}(\tilde{x})$ such that $\tilde{x}_2 = \mathbf{g}\tilde{x}_1$ for some $\mathbf{g}$ in the mapping class group. Write

$$\tilde{x}_i = \tilde{x} + v_i, \qquad \text{where } \|v_i\|_{\mathrm{AGY},x} \le u(x)^{-N}$$

By Theorem 2.5, there exists $s \in [0, 1/2]$ and $\tau \le N_2 \log u(x)$ such that $x' \equiv a_\tau u_s x \in K'_\alpha$.

Let $x'_i = a_\tau u_s x_i$, and put $\tilde{x}'_i = a_\tau u_s \tilde{x}_i$; also put $\tilde{x}' = a_\tau u_s \tilde{x}$. Then, in view of (15) we have

$$\|v_i\|_{\mathrm{AGY},x'_i} \le e^{2+2\tau} u(x)^{-N} \le 8u(x)^{2N_2-N+2} \le 2^{2N_2-N+5} \le r_0 \tag{18}$$

where for the last estimate we used (17) and the fact that $u(x) \ge 2$. However, $\tilde{x}'_2 = \mathbf{g}\tilde{x}'_1$, thus, both x'_1 and x'_2 belong to the projection of $\mathsf{B}_{r_0}(\tilde{x}')$; this contradicts the fact that $\mathsf{B}_{r_0}(x')$ is embedded.

This contradiction shows that $\mathsf{B}_{u(x)^{-N}}(x)$ is embedded, establishing part (1).

We now turn to part (2). We use the above notation. Let $\tilde{x}_1, \tilde{x}_2 \in \mathsf{B}_{u(x)^{-N}}(x)$, and define $x'_i = a_\tau u_s x_i \in K'_\alpha$ and $\tilde{x}'_i = a_\tau u_s \tilde{x}_i$ as above. Then (18) implies that

$$d_{\mathrm{AGY}}(\tilde{x}'_1, \tilde{x}'_2) \le 16u(x)^{2N_2-N+2}$$

Hence, $d_{\mathcal{T}}(\tilde{x}'_1, \tilde{x}'_2) \le 16Cu(x)^{2N_2-N+2}$. Since $\tilde{x}_i = (a_\tau u_s)^{-1}\tilde{x}'_i$, we conclude that

$$d_{\mathcal{T}}(\tilde{x}_1, \tilde{x}_2) \le Cu(x)^{4N_2-N+16} < 1$$

where we used (17) and $u(x) \ge 2$ in the last inequality. The proof is complete. □

For every $x \in \mathcal{Q}_1(\alpha)$ we put

$$r(x) = u(x)^{-N_3}; \tag{19}$$

for every compact subset $K \subset \mathcal{Q}_1(\alpha)$, let $r(K) = \inf\{r(x) : x \in K\}$.

For every $0 < r \le r(x)$, we let $\mathsf{B}_r(x)$ denotes $\pi(\mathsf{B}_r(\tilde{x}))$ where $\tilde{x} \in \mathcal{Q}^1\mathcal{T}(\alpha)$ is an arbitrary lift of x. We refer to $\mathsf{B}_r(x)$ as the ball of radius r centered at x.

2.10. **Horospherical foliation.** Given a point $x = (M, q) \in \mathcal{Q}_1(\alpha)$, the tangent space $\mathrm{T}_x\mathcal{Q}_1(\alpha)$ decomposes as

$$\mathrm{T}_x\mathcal{Q}_1(\alpha) = \mathbb{R}\mathbf{v}(x) \oplus E^{\mathrm{u}}(x) \oplus E^{\mathrm{s}}(x)$$

where $\mathbf{v}(x)$ with $\|\mathbf{v}(x)\|_{\mathrm{AGY},x} = 1$ determines the direction of the Teichmüller geodesic flow,

$$E^{\mathrm{u}}(x) = \mathrm{T}_x\mathcal{Q}_1(\alpha) \cap \mathrm{D}\Phi_x^{-1}\big(H^1(\dagger, \ddagger, \mathbb{R})\big), \text{ and}$$
$$E^{\mathrm{s}}(x) = \mathrm{T}_x\mathcal{Q}_1(\alpha) \cap \mathrm{D}\Phi_x^{-1}\big(\mathbf{i}H^1(\dagger, \ddagger, \mathbb{R})\big).$$

where $(\dagger, \ddagger) = (M, \Sigma)$ if $\varsigma = 1$ and $(\dagger, \ddagger) = (\hat{M}, \hat{\Sigma})$ if $\varsigma = -1$ — recall that $\hat{M}$ is the orienting double cover of M and we use Φ to locally identify $\mathbb{R}\mathcal{Q}_1(\alpha)$ with $H^1(M, \Sigma, \mathbb{C})$ if $\varsigma = 1$ and with the $H^1_{\mathrm{odd}}(\hat{M}, \hat{\Sigma}, \mathbb{C})$ if $\varsigma = -1$.

If $\Phi(x) = a + \mathbf{i}b$ for some $x \in \mathcal{Q}_1(\alpha)$, then

$$E^{\mathrm{u}}(x) = \{a' \in H^1(M, \Sigma, \mathbb{R}) : i(a', b) = 0\}, \tag{20}$$

and $E^{\mathrm{s}}(x) = \{\mathbf{i}b' \in \mathbf{i}H^1(M, \Sigma, \mathbb{R}) : i(a, b') = 0\}$ when $\varsigma = 1$. Similarly, one can define $E^{\mathrm{u,s}}$ in the case $\varsigma = -1$.

The subspaces $E^{\mathrm{u,s}}(x)$ depend smoothly on x, moreover, they are integrable. We denote the corresponding leaves by $W^{\mathrm{u}}(x)$ and $W^{\mathrm{s}}(x)$, respectively. Also put

$$W^{\mathrm{cu}}(x) := \{a_t W^{\mathrm{u}}(x) : t \in \mathbb{R}\}$$

and $W^{\mathrm{cs}}(x) := \{a_t W^{\mathrm{s}}(x) : t \in \mathbb{R}\}$.

Let μ_x^{u} and μ_x^{s} denote the leafwise measures of the natural measure μ along $W^{\mathrm{u}}(x)$ and $W^{\mathrm{s}}(x)$, respectively. Then $y \mapsto \mu_y^{\mathrm{u,s}}$ is constant along $W^{\mathrm{u,s}}(x)$, respectively, and we have

$$(a_t)_*\mu_x^{\mathrm{u}} = e^{-ht}\mu_{a_t x}^{\mathrm{u}} \quad \textit{and} \quad (a_t)_*\mu_x^{\mathrm{s}} = e^{ht}\mu_{a_t x}^{\mathrm{s}}; \tag{21}$$

see also [AG, §4] where these measures are defined using volume forms.

If $\mathsf{B}_r(x)$ is a period box centered at x, then $\mu|_{\mathsf{B}_r(x)}$ has a product structure as $\mathrm{dLeb} \times \mathrm{d}\mu^{\mathrm{s}} \times \mathrm{d}\mu^{\mathrm{u}}$, see e.g. [AG, Prop. 4.1].

Given $x \in \mathcal{Q}_1(\alpha)$ and a period box $\mathsf{B}_r(x)$ with center x and $0 \le r \le r(x)$, we let

$$\mathsf{B}_r^{\mathrm{u,s}}(x) = \text{the connected component of } x \text{ in } \mathsf{B}_r(x) \cap W^{\mathrm{u,s}}(x).$$

Define $\mathsf{B}_r^{\bullet}(x)$ for $\bullet = \mathrm{cu}, \mathrm{cs}$ similarly.

We also denote functions which are supported on the leaves W^{u}, W^{cu}, etc. using the same superscript, e.g., ϕ^{u} denotes a function which is supported on a leaf $W^{\mathrm{u}}(x)$.

We use the norm $\|\cdot\|_{\mathrm{AGY},x}$ to induce a metric $d_{W^{\mathrm{u,s}}(x)}$ on $\mathsf{B}_r^{\mathrm{u,s}}(x)$ for $0 < r < r(x)$. Hence notions such as diam etc. refer to this metric.

Let $\tilde{W}^{\bullet}(\tilde{x})$ denote the foliation $\bullet$ in $\mathcal{Q}^1\mathcal{T}(\alpha)$, and define $\mathsf{B}^{\bullet}(\tilde{x})$ accordingly.

Let $w^{\mathrm{u,s}} \in E^{\mathrm{u,s}}(x)$. Then

$$\|(a_t)_* w^{\mathrm{u}}\|_{\mathrm{AGY},a_t x} \ge \|w^{\mathrm{u}}\|_{\mathrm{AGY},x} \quad \text{and} \quad \|(a_t)_* w^{\mathrm{s}}\|_{\mathrm{AGY},a_t x} \le \|w^{\mathrm{s}}\|_{\mathrm{AGY},x}, \tag{22}$$

see [AG, Lemma 5.2]. Moreover, we have the following uniform hyperbolicity estimate.

Proposition 2.7. *Let $K \subset \mathcal{Q}_1(\alpha)$ be a compact subset. There exist some $\kappa_2(K)$ and some $t_0 = t_0(K)$ with the following property. Let $t \geq t_0$; suppose that $x, a_t x \in K$, moreover, assume that*

$$|\{\tau \in [0,t] : a_\tau x \in K\}| \geq t/3.$$

Then

$$\|(a_t)_* w\|_{\mathrm{AGY}, a_t x} \leq e^{-\kappa_2(K)t}\|w\|_{\mathrm{AGY},x} \quad \text{and} \quad \|(a_t)_* w\|_{a_t x} \leq e^{-\kappa_2(K)t}\|w\|_x$$

for all $w \in E^{\mathrm{s}}(x)$ and all $t \geq t_0$.

Proof. Let $\| \ \|_{\mathrm{ABEM},x}$ denote the modified Hodge norm defined in [ABEM, §3]. Let C be a constant so that

$$C^{-1}\|v\|_{\mathrm{ABEM},y} \leq \|v\|_{\mathrm{AGY},y} \leq C\|v\|_{\mathrm{ABEM},y} \tag{23}$$

for all $y \in K$.

In view of [ABEM, Thm. 3.15], there exists some $\kappa_3(K)$ so that under our assumptions in this proposition we have

$$\|(a_t)_* w\|_{\mathrm{ABEM}, a_t x} \leq e^{-\kappa_3 t}\|w\|_{\mathrm{ABEM},x}. \tag{24}$$

We now compute

$$\begin{aligned} \|(a_t)_* w\|_{\mathrm{AGY}, a_t x} &\leq C\|(a_t)_* w\|_{\mathrm{ABEM}, a_t x} && \text{since } a_t x \in K \\ &\leq C e^{-\kappa_3 t}\|w\|_{\mathrm{ABEM},x} && \text{by (24)} \\ &\leq C^2 e^{-\kappa_3 t}\|w\|_{\mathrm{AGY},x} && \text{since } x \in K. \end{aligned}$$

The claim thus holds with $\kappa_2 = \kappa_3/2$ and $t_0 = \frac{4\log C}{\kappa_3}$. □

Lemma 2.8. *Let K'_α be as in Theorem 2.5. There is a positive constant N_4 and for every $0 < \theta < 1$ there exists $\kappa_4(\theta)$, and a compact subset $K_\alpha(\theta) \supset K'_\alpha$ with the following properties. Let $x \in \mathcal{Q}_1(\alpha)$, $0 < r \leq r(x)$, and let $\mathsf{B}_r(x)$ be a period box centered at x. Put*

$$\mathsf{H}^{\mathrm{u}}_t(x,\theta) := \big\{y \in \mathsf{B}^{\mathrm{u}}_r(x) : |\{\tau \in [0,t] : a_\tau y \in K_\alpha(\theta)\}| \geq \theta t\big\}.$$

Then for every $t \geq N_4 \log u(x)$, we have

$$\mu^{\mathrm{u}}_x\left(\mathsf{B}^{\mathrm{u}}_r(x) - \mathsf{H}^{\mathrm{u}}_t(x,\theta)\right) \leq e^{-\kappa_4(\theta)t}\mu^{\mathrm{u}}_x(\mathsf{B}^{\mathrm{u}}_r(x)).$$

Proof. See [AG, Prop. 6.1]. □

We apply the above with $\theta = 0.5$, and put

$$K_\alpha = K_\alpha(0.5), \quad \kappa_4 := \kappa_4(0.5), \quad \text{and} \quad \mathsf{H}^{\mathrm{u}}_t(x) := \mathsf{H}^{\mathrm{u}}_t(x, 0.5) \tag{25}$$

for the rest of the paper.

We have the following corollary

Corollary 2.9. *Let $x \in \mathcal{Q}_1(\alpha)$, and let $t \geq N_4 \log u(x)$. For every $y \in \mathsf{H}^{\mathrm{u}}_t(x)$ and every $w \in E^{\mathrm{u}}(x)$ we have*

$$\|(a_{-t})_* w\|_{\mathrm{AGY}, a_{-t}y} \leq e^{-0.5\kappa_2(K_\alpha)t}\|w\|_{\mathrm{AGY},y}.$$

Proof. Let $\tau_0 < \tau_1$ be the first and the last time so that $a_\tau y \in K_\alpha$. Then in view of Lemma 2.8, $\tau_1 - \tau_0 \geq 0.5t \geq (\tau_1 - \tau_0)/3$. Therefore, by (22) and Proposition 2.7, we have

$$\begin{aligned}\|(a_{-t})_* w\|_{\mathrm{AGY}, a_{-t}y} &\leq \|(a_{-\tau_1})_* w\|_{\mathrm{AGY}, a_{-\tau_1}y} \\ &\leq e^{-0.5\kappa_2(K_\alpha)t}\|(a_{-\tau_0})_* w\|_{\mathrm{AGY}, a_{-\tau_0}y} \\ &\leq e^{-0.5\kappa_2(K_\alpha)t}\|w\|_{\mathrm{AGY},y}\end{aligned}$$

as we claimed. □

2.11. **Smooth structure on affine manifolds.** As it is done in [AG, §5.2], we use the affine structure to define a smooth structure on $\mathcal{Q}^1\mathcal{T}(\alpha)$ and $\mathcal{Q}_1(\alpha)$. Let us recall the definition of a C^k-norm from [AG], see also [AGY].

Let $W \subset \mathcal{Q}_1(\alpha)$ be an affine submanifold. For a function φ on W define

$$c_k(\varphi) = \sup |\,\mathrm{D}^k \varphi(x, v_1, \ldots, v_k)|,$$

where the supremum is taken over x in the domain of φ and $v_1, \ldots, v_k \in \mathrm{T}_x W$ with AGY-norm at most 1. Define the C^k-norm of φ as $\|\varphi\|_{C^k} = \sum_{j=0}^k c_j(\varphi)$.

By a C^k function we mean a function whose C^k-norm is finite. The space of compactly supported C^k functions on W will be denoted by $C_c^k(W)$, similarly, we define $C_c^\infty(W)$.

In the sequel we will only need C^1-norm of functions. To avoid confusion between this norm and other relevant norms which will be used, and also since we often use the letter C to denote various constants, define

$$\mathcal{C}^1(\varphi) := \|\varphi\|_{C^1}.$$

for any C^1 function φ.

In the sequel we will need to replace the characteristic functions of certain sets with their smooth approximations. The following lemmas will provide such approximations.

Lemma 2.10 (Cf. [AG], Prop. 5.8)**.** *There exists N_5 so that the following holds. Let $x \in \mathcal{Q}_1(\alpha)$. Let $D \subset W^{\mathrm{u}}(x)$ be a compact set, and let $\epsilon \leq 0.1r(D)$, see* (19). *There exists a finite collection $\{\varphi_i\}$ of C^∞ functions on $W^{\mathrm{u}}(x)$ with the following properties:*

1. $0 \leq \varphi_i \leq 1$ *for all* i.
2. $\mathcal{C}^1(\varphi_i) \leq N_5 \epsilon^{-N_5}$.
3. *For every i, φ_i is supported on $\mathsf{B}^{\mathrm{u}}_\epsilon(y_i)$ for some $y_i \in D$.*
4. *The covering $\{\mathsf{B}^{\mathrm{u}}_\epsilon(y_i)\}$ of D has multiplicity at most N_5.*
5. $\sum \varphi_i \leq 1$, *and the equality holds on a neighborhood of D.*

Proof. This is proved in [AG, Prop. 5.8]. It is worth mentioning that [AG, Prop. 5.8] is stated for balls of size $\asymp 1$, to get our claim here, one needs to apply the argument there not to the AGY norm, but to the AGY norm scaled by $1/\epsilon$. □

Let W be one of the following: $\mathcal{Q}_1(\alpha)$, $W^{\mathrm{u,s}}(x)$, or $W^{\mathrm{cu,cs}}(x)$, for some $x \in \mathcal{Q}_1(\alpha)$. Let $E \subset W$ be a compact subset. For any $0 < \epsilon < 0.1r(E)$ define

$$E^W_{+,\epsilon} = \{y \in W : r(y) \geq \epsilon \text{ and } \mathsf{B}_\epsilon(y) \cap E \neq \emptyset\};$$

note that $E^W_{+,\epsilon}$ is an open subset of W which contains E.

Let $r > 0$ and $L > 1$. Let $\mathcal{S}_W(E, r, L)$ denote the class of Borel functions $0 \le f \le 1$ supported and defined everywhere in E with the following properties: for all $\epsilon \le r/10L$ there exist $\varphi_{+,\epsilon}, \varphi_{-,\epsilon} \in C^\infty_c(E^W_{+,\epsilon})$ so that

($\mathcal{S}$-1) $\varphi_{-,\epsilon} \le f \le \varphi_{+,\epsilon}$,
($\mathcal{S}$-2) $\mathcal{C}^1(\varphi_{\pm,\epsilon}) \le \epsilon^{-L}$, and
($\mathcal{S}$-3) $\|\varphi_{+,\epsilon} - \varphi_{-,\epsilon}\|_2 \le \epsilon^{1/2}\|f\|_2$.

If W is clear from the context, we denote $\mathcal{S}_W(E, r, L)$ and $E^W_{+,\epsilon}$ simply by $\mathcal{S}(E, r, L)$ and $E_{+,\epsilon}$, respectively.

Lemma 2.11. *There exists some L depending only on α so that for all $0 < r \le r(x)$,*

$$1_{\mathsf{B}^{\mathrm{u,s}}_r(x)} \in \mathcal{S}_{W^{\mathrm{u,s}}(x)}(\mathsf{B}^{\mathrm{u,s}}_r(x), r, L).$$

Similarly, $1_{\mathsf{B}_r(x)} \in \mathcal{S}(\mathsf{B}_r(x), r, L)$ for all $0 < r \le r(x)$.

Proof. We will show the claims hold if we choose $L > 2N_5$, see Lemma 2.10, large enough. Apply Lemma 2.10 with ϵ and $D = \mathsf{B}^{\mathrm{u}}_{r-2\epsilon}$, and denote by $\{\varphi_{i,-}\}$ the functions obtained from that lemma. For a second time, apply Lemma 2.10 with ϵ and $D = \mathsf{B}^{\mathrm{u}}_r(x)$, and denote by $\{\varphi_{i,+}\}$ the functions thus obtained. Put

$$\varphi_{\epsilon,-} = \sum \varphi_{i,-} \quad \text{and} \quad \varphi_{\epsilon,+} = \sum \varphi_{i,+}.$$

These functions satisfy ($\mathcal{S}$-1) thanks to Lemma 2.10(1) and (5). Moreover, they satisfy ($\mathcal{S}$-2) thanks to Lemma 2.10(1)—(4) and the fact that $L > 2N_5$.

To see ($\mathcal{S}$-3), first note that $\mu^{\mathrm{u}}_x\big(\mathsf{B}^{\mathrm{u}}_r(x) - \mathsf{B}^{\mathrm{u}}_{r-2\epsilon}\big) \ll \epsilon$ where the implied constant depends only on α. The claim in ($\mathcal{S}$-3) thus holds true in view of Lemma 2.10(5) if we choose L large enough, depending on α.

The second claim follows from the first claim, using the product structure of $\mathsf{B}_r(x)$ and of the measure μ. □

We fix once and for all some L so that Lemma 2.11 holds true and drop L from the notation. In particular, $\mathcal{S}(E, r, L)$ will be denoted by $\mathcal{S}(E, r)$.

Abusing the notation we will write $\mathcal{S}(x, r)$ for $\mathcal{S}(E, r)$ if the compact subset E is not relevant except for the fact that it is a compact subset containing the point x.

3. Translates of horospheres

In this section we will use a fundamental result of Avila, Gouëzel, and Yoccoz, [AGY, AG] together with Margulis' thickening technique, [Mar, EMc, KM], to study translations of pieces of the horospherical foliations along the geodesic flow.

Theorem 3.1 (Exponential Mixing, [AGY, AR, AG])**.** *Let $(\mathcal{M},\mu)$ be an affine invariant manifold. There exists a positive constant $\kappa=\kappa(\mathcal{M},\mu)$ so that the following holds. Let $\Psi_1,\Psi_2\in C_c^\infty(\mathcal{M})$, then*

$$\left|\int \Psi_1(a_tx)\Psi_2(x)\,\mathrm{d}\mu(x)-\mu(\Psi_1)\mu(\Psi_2)\right|\ll \mathcal{C}^1(\Psi_1)\mathcal{C}^1(\Psi_2)e^{-\kappa t}$$

where the implied constant depends on $(\mathcal{M},\mu)$.

We remark that combining [AGY, AR, AG] and [Rn], the $\mathcal{C}^1$ norm in Theorem 3.1 may be replaced by the p-Hölder norm for any $p>0$. However, if we use the p-Hölder norm, the constant κ will, in general, depend on p; in particular, κ tends to 0 as p tends to 0, see [Rn, Thm. 1] and [AGY, Thm. 2.14].

It is also worth mentioning that the $\mathcal{C}^1$ norm in Theorem 3.1 may be taken to include derivatives only in the direction of $\mathrm{SO}(2)\subset\mathrm{SL}(2,\mathbb{R})$, see [CHH] and [Rn, Thm. 1] and references there. Our choice, $\mathcal{C}^1$, is more restrictive; this is tailored to our applications later, e.g., we will use the estimate that $\|\phi\|_\infty\le\mathcal{C}^1(\phi)$ for any $\phi\in C_c^\infty(\mathcal{M})$.

Proposition 3.2. *There exists some κ_5, depending on α, with the following property. Let $x\in\mathcal{Q}_1(\alpha)$, $0<r\le r(x)$, and let $\mathsf{B}_r(x)$ be a period box centered at x. Let $\psi^{\mathrm{u}}\in C_c^\infty(\mathsf{B}_r^{\mathrm{u}}(x))$, then for any $\phi\in C_c^\infty(\mathcal{Q}_1(\alpha))$ we have*

$$\left|\int_{W^{\mathrm{u}}(x)}\phi(a_ty)\psi^{\mathrm{u}}(y)\,\mathrm{d}\mu_x^{\mathrm{u}}(y)-\int_{\mathcal{Q}_1(\alpha)}\phi\,\mathrm{d}\mu\int_{W^{\mathrm{u}}(x)}\psi^{\mathrm{u}}\,\mathrm{d}\mu_x^{\mathrm{u}}\right|\le\mathcal{C}^1(\phi)\mathcal{C}^1(\psi^{\mathrm{u}})e^{-\kappa_5t}.$$

We need some notation; we discuss the case $\varsigma=1$, the case $\varsigma=-1$ is similar. Let $\Phi(x)=a+\mathbf{i}b$; recall from (20) that

$$E^{\mathrm{u}}(x)=\{a'\in H^1(M,\Sigma,\mathbb{R}):i(a',b)=0\}.$$

Similarly $E^{\mathrm{s}}(x)=\{\mathbf{i}b'\in\mathbf{i}H^1(M,\Sigma,\mathbb{R}):i(a,b')=0\}$.

These spaces can alternatively be described as follows. Recall that subspace $E(x)=\mathrm{span}\{a,b\}$, then $E_{\mathbb{C}}(x)$ is $\mathrm{SL}_2(\mathbb{R})$ equivariant. Let

$$H^1_{\mathbb{C}}(x)^\perp:=\{c\in H^1(M,\Sigma,\mathbb{C}):p(c)\wedge p(E(x)_{\mathbb{C}})=0\},$$

similarly define $H^1_{\mathbb{R}}(x)^\perp$.

The unstable leaf $W^{\mathrm{u}}(x)$ is locally identified with $\Phi(x)+sb+w$ for $s\in\mathbb{R}$ and $w\in H^1_{\mathbb{R}}(x)^\perp$. Similarly the center stable leaf $W^{\mathrm{cs}}(x)$ is locally identified with $\Phi(x)+\tau\mathbf{v}(x)+s'\mathbf{i}a+\mathbf{i}w'$ where $\tau,s'\in\mathbb{R}$ and $w'\in H^1_{\mathbb{R}}(x)^\perp$.

Let $0<r\le 0.1r(x)$ and let $y\in\mathsf{B}_r(x)$. Write $\Phi(y)=a_y+\mathbf{i}b_y$. We define the stable projection $y^{\mathrm{u}}\in B^{\mathrm{u}}_{2r}(x)$ as the unique point so that $\Phi(y)=\Phi(y^{\mathrm{u}})+\tau\mathbf{v}(y)+sa_y+w$ where $\tau,s\in\mathbb{R}$ with $|\tau|,|s|\le 2r$ and $w\in H^1_{\mathbb{R}}(y)^\perp$ with $\|w\|_{\mathrm{AGY},x}\le 2r$. Put

$$\mathsf{FB}_r(x)=\{y\in\mathsf{B}_r(x):y^{\mathrm{u}}\in\mathsf{B}^{\mathrm{u}}_r(x)\}.$$

Then $\mathsf{B}^{\mathrm{u}}_r(x)\subset\mathsf{FB}_r(x)$.

For every $0 < \delta < 0.1r$ and every $y \in \mathsf{B}_r(x)$, let

$$\mathsf{D}^{\rm cs}_\delta(y) = \{a_\tau z : |\tau| \leq \delta, z \in W^{\rm s}(y), \Phi(z) = \Phi(y) + w, \|w\|_{{\rm AGY},x} \leq \delta\}.$$

For every $y \in \mathsf{B}^{\rm u}_r(x)$, let $p^{\rm cs}_y : W^{\rm cs}(y) \cap \mathsf{B}_r(x) \to W^{\rm cs}(x)$ be the projection along unstable leaves. Then $0.5 \leq {\rm Jac}(p^{\rm cs}_y) \leq 2$, moreover we have

$$W^{\rm cs}(x) \cap \mathsf{B}_{0.1\delta}(x) \subset p^{\rm cs}_y\big(\mathsf{D}^{\rm cs}_\delta(y)\big) \subset W^{\rm cs}(x) \cap \mathsf{B}_{10\delta}(x).$$

We now begin the proof of the Proposition 3.2.

Proof. The idea is to relate the integral $\int_{W^{\rm u}(x)} \phi(a_t y)\psi^{\rm u}(y)\,{\rm d}\mu^{\rm u}_x(y)$ to correlations of the function $a_{-t}\phi$ with a *thickening* of $\psi^{\rm u}$ in the direction of $W^{\rm cs}(x)$. Then we may use Theorem 3.1 to conclude the proof.

To that end, let $0 < \epsilon < 0.01r(x)$ be a parameter which will be fixed later. In particular, it will be taken to be of the form $e^{-\kappa t}$. Let $\psi^{\rm cs}$ be a smooth function supported in $\mathsf{D}^{\rm cs}_\epsilon(x)$ so that $\int_{W^{\rm sc}(x)} \psi^{\rm cs} = 1$. We can choose such a function so that it moreover satisfies $\mathcal{C}^1(\psi^{\rm s}) \ll \epsilon^{-N_6}$ where N_6 and the implied constant depend on α.

Define Ψ on $\mathsf{FB}_r(x)$ by

$$\Psi(y) = \lambda_{y^{\rm u}}\psi^{\rm cs}\big(p^{\rm cs}_{y^{\rm u}}(y)\big) \cdot \psi^{\rm u}(y^{\rm u}) \tag{26}$$

where $\lambda^{-1}_{y^{\rm u}} = \int_{W^{\rm cs}(y^{\rm u})} \psi^{\rm cs}\big(p^{\rm cs}_{y^{\rm u}}(w)\big)\,{\rm d}\mu^{\rm cs}_{y^{\rm u}}(w)$. Extend Ψ to a smooth function on $\mathcal{Q}_1(\alpha)$ by defining $\Psi(y) = 0$ for all $y \notin \mathsf{FB}_r(x)$; note that $\mu(\Psi) = \mu^{\rm u}_x(\psi^{\rm u})$, see the computation in (30).

We need the following lemma.

Lemma. *There exists κ_6 depending only on α so that*

$$\left|\int_{W^{\rm u}(x)} \phi(a_t y)\psi^{\rm u}(y)\,{\rm d}\mu^{\rm u}_x(y) - \int_{\mathcal{Q}_1(\alpha)} \phi(a_t z)\Psi(z)\,{\rm d}\mu(z)\right| \ll \mathcal{C}^1(\phi)\mathcal{C}^1(\Psi)\epsilon^{\kappa_6} \tag{27}$$

where the implied constant depends only on α.

Let us assume the lemma and finish the proof of the proposition. Optimizing the choice of ϵ to be of size $e^{-\kappa t}$ for some small $0 < \kappa < 1$, the proposition follows from (27) and Theorem 3.1 applied with $\Psi_1 = \phi$ and $\Psi_2 = \Psi$ — recall again that $\mu(\Psi) = \mu^{\rm u}_x(\psi^{\rm u})$. □

Proof of the Lemma. Since Ψ is supported in $\mathsf{FB}_r(x)$, we need to estimate

$$\int_{\mathsf{B}^{\rm u}_r(x)} \phi(a_t y)\psi^{\rm u}(y)\,{\rm d}\mu^{\rm u}_x(y) - \int_{\mathsf{FB}_r(x)} \phi(a_t z)\Psi(z)\,{\rm d}\mu(z). \tag{28}$$

Let $z \in \mathsf{FB}_r(x)$. Recall that $\Phi(z) = \Phi(z^{\rm u}) + w$ where $w \in \mathbb{R}\mathbf{v}(z^{\rm u}) + E^{\rm s}(z^{\rm u})$, indeed $z \in W^{\rm cs}(z^{\rm u})$. In view of (22) we have

$$\|(a_t)_* w\|_{{\rm AGY},a_t x} \leq \|w\|_{{\rm AGY},x}.$$

Thus using the definition of $\mathcal{C}^1(\phi)$, we have

$$|\phi(a_t z) - \phi(a_t z^{\rm u})| \ll \epsilon^{\kappa_6}\mathcal{C}^1(\phi)$$

where κ_6 and the implied constant depend only on α.

In consequence, we may replace $\phi(a_t z)$ by $\phi(a_t z^{\rm u})$ in (28), and use the bound $\|\cdot\|_\infty \le \mathcal{C}^1(\cdot)$, to conclude the following

$$\left|\int_{\mathsf{B}^{\rm u}_r(x)} \phi(a_t y)\psi^{\rm u}(y)\,\mathrm{d}\mu^{\rm u}_x(y) - \int_{\mathsf{FB}_r(x)} \phi(a_t z)\Psi(z)\,\mathrm{d}\mu(z)\right| \ll \mathcal{C}^1(\phi)\mathcal{C}^1(\Psi)\epsilon^{\kappa_6} \quad + \left|\int_{\mathsf{B}^{\rm u}_r(x)} \phi(a_t y)\psi^{\rm u}(y)\,\mathrm{d}\mu^{\rm u}_x(y) - \int_{z\in\mathsf{FB}_r(x)} \phi(a_t z^{\rm u})\Psi(z)\,\mathrm{d}\mu(z)\right| \tag{29}$$

where the implied constant depends on α.

Recall the definition of Ψ from (26), in particular recall the normalizing factor $\lambda_{y^{\rm u}}$. This and the product structure of μ yield the following

$$\begin{aligned}\int_{z\in\mathsf{FB}_r(x)} \phi(a_t z^{\rm u})\Psi(z)\,\mathrm{d}\mu(z) &= \int_{z\in\mathsf{FB}_r(x)} \lambda_{z^{\rm u}}\phi(a_t z^{\rm u})\psi^{\rm cs}\big(p^{\rm cs}_{z^{\rm u}}(z)\big)\psi^{\rm u}(z^{\rm u})\,\mathrm{d}\mu(z)\\ &= \int_{\mathsf{B}^{\rm u}_r(x)} \phi(a_t z^{\rm u})\psi^{\rm u}(z^{\rm u}) \int_{W^{\rm cs}(z^{\rm u})} \lambda_{z^{\rm u}}\psi^{\rm cs}\big(p^{\rm cs}_{z^{\rm u}}(w)\big)\,\mathrm{d}\mu^{\rm cs}_{z^{\rm u}}(w)\,\mathrm{d}\mu^{\rm u}_x(z^{\rm u})\\ &= \int_{\mathsf{B}^{\rm u}_r(x)} \phi(a_t y)\psi^{\rm u}(y)\,\mathrm{d}\mu^{\rm u}_x(y). \qquad (30)\end{aligned}$$

We now combine the estimates in (29) and (30), and get the following.

$$\left|\int_{W^{\rm u}(x)} \phi(a_t y)\psi^{\rm u}(y)\,\mathrm{d}\mu^{\rm u}_x(y) - \int_{\mathcal{Q}_1(\alpha)} \phi(a_t z)\Psi(z)\,\mathrm{d}\mu(z)\right| \ll \mathcal{C}^1(\phi)\mathcal{C}^1(\Psi)\epsilon^{\kappa_6}$$

where the implied constant is absolute. □

Remark 3.3. It is worth mentioning that Proposition 3.2 and its proof hold for any affine invariant manifold, $(\mathcal{M},\mu)$. In the sequel, however, we will only need this result for $\mathcal{Q}_1(\alpha)$; and even more specifically, in our application to counting problems, we will need this result for the principal stratum $\mathcal{Q}_1(1,\ldots,1)$. The main result in [AGY] was generalized to $\mathcal{Q}_1(\alpha)$ in [AR].

Corollary 3.4. *There exist κ_7, κ_8, and N_7 so that the following holds. Let $x,z\in\mathcal{Q}_1(\alpha)$ and suppose $0<r,r'\le 0.01\min\{r(x),r(z)\}$. Let $\mathsf{B}\subset\mathsf{B}_{r'}(z)$ be so that $1_{\mathsf{B}}\in\mathcal{S}(z,r')$ and let $\psi^{\rm u}\in C^\infty_c(\mathsf{B}^{\rm u}_r(x))$. Then for every $\epsilon<r'/L$, see Lemma 2.11, we have*

$$\left|\frac{1}{\mu(\mathsf{B})}\int_{W^{\rm u}(x)} 1_{\mathsf{B}}(a_t y)\psi^{\rm u}(y)\,\mathrm{d}\mu^{\rm u}_x(y) - \int\psi^{\rm u}\,\mathrm{d}\mu^{\rm u}_x\right| \le \epsilon^{-N_7}\mathcal{C}^1(\psi^{\rm u})e^{-\kappa_7 t} + \mathcal{C}^1(\psi^{\rm u})\epsilon^{\kappa_8}.$$

Proof. This follows from Proposition 3.2 by approximating 1_{B} with $\varphi_{\pm,\epsilon}$ and using properties $(\mathcal{S}\text{-}1)$—$(\mathcal{S}\text{-}3)$. □

4. A counting function

Let $x, z \in \mathcal{Q}_1(\alpha)$. Let ψ^{u} be a function which is supported and defined everywhere in $\mathsf{B}^{\mathrm{u}}_{0.1r(x)}(x) = \mathsf{B}_{0.1r(x)}(x) \cap W^{\mathrm{u}}(x)$, and let ϕ^{cs} be a function which is supported and defined everywhere in $\mathsf{B}^{\mathrm{cs}}_{0.1r(z)}(z) = \mathsf{B}_{0.1r(z)}(z) \cap W^{\mathrm{cs}}(z)$. For all $t > 0$, define

$$\mathcal{N}_{\mathrm{nc}}(t, \psi^{\mathrm{u}}, \phi^{\mathrm{cs}}) := \sum \psi^{\mathrm{u}}(y)\phi^{\mathrm{cs}}(a_t y) \tag{31}$$

where the sum is taken over all $y \in \mathsf{B}^{\mathrm{u}}_{0.1r(x)}(x)$ so that $a_t y \in \mathsf{B}^{\mathrm{cs}}_{0.1r(z)}(z)$ — note that the sum is indeed over all $y \in \mathrm{supp}(\psi^{\mathrm{u}})$ so that $a_t y \in \mathrm{supp}(\phi^{\mathrm{cs}})$.

Alternatively, the sum is taken over connected components of $a_t\mathrm{supp}(\psi^{\mathrm{u}})\cap\mathrm{supp}(\phi^{\mathrm{cs}})$ (indeed the subscript nc stands for the number of connected components); this point will be made more explicit later in this section, see e.g. Lemma 4.2 below and recall that W^{u} and W^{cs} are complementary foliations.

The function $\mathcal{N}_{\mathrm{nc}}$ may be thought of as a bisector counting function where one studies the asymptotic behavior of the number of translates of a piece of W^{u} by Mod_g which intersect a cone in the Teichmüller space.

The following proposition is the main result of this section and provides an asymptotic behavior for $\mathcal{N}_{\mathrm{nc}}$. This proposition plays a prime role in the proof of Theorem 1.2 in §7.

Proposition 4.1. *There exist κ_9 and N_8 with the following property. Let $x, z \in \mathcal{Q}_1(\alpha)$, and let $t \geq N_8 \max\{\log u(x), \log u(z)\}$. Let $\psi^{\mathrm{u}} \in C_c^\infty(\mathsf{B}^{\mathrm{u}}_{0.1r(x)}(x))$ with $0 \leq \psi^{\mathrm{u}} \leq 1$, and let $\phi^{\mathrm{cs}} \in C_c^\infty(\mathsf{B}^{\mathrm{cs}}_{0.1r(z)}(z))$. Then*

$$|\mathcal{N}_{\mathrm{nc}}\big(t, \psi^{\mathrm{u}}, \phi^{\mathrm{cs}}\big) - e^{ht}\mu^{\mathrm{u}}_x(\psi^{\mathrm{u}})\mu^{\mathrm{cs}}_z(\phi^{\mathrm{cs}})| \leq \mathcal{C}^1(\psi^{\mathrm{u}})\mathcal{C}^1(\phi^{\mathrm{cs}})e^{(h-\kappa_9)t}$$

where $h = \frac{1}{2}(\dim_{\mathbb{R}} \mathcal{Q}(\alpha) - 2)$.

The proof of this proposition is based on Lemma 4.5 which in turn relies on Proposition 3.2. In particular, the main term is given by Proposition 3.2. However, we need to control the contribution of two types of exceptional points as we now describe.

Similar to Lemma 2.8, given a compact subset $K \supset K_\alpha$, define

$$\mathsf{H}^{\mathrm{u}}_t(x, K) := \big\{y \in \mathsf{B}^{\mathrm{u}}_r(x) : |\{\tau \in [0, t] : a_\tau y \in K\}| \geq t/2\big\}. \tag{32}$$

The first (and more difficult to control) type of exceptional points are $y \in \mathsf{B}^{\mathrm{u}}_r(x)$ so that $a_t y \in \mathsf{B}_{r'}(z)$, however, $y \notin \mathsf{H}^{\mathrm{u}}_t(x, K)$. The contribution coming from these points is controlled using [EMR, Thm. 1.7], see Theorem 4.4 below.

We also need to control the contribution of points $y \in \mathsf{B}^{\mathrm{u}}_r(x)$ which are exponentially close to the boundary of $\mathsf{B}^{\mathrm{u}}_r(x)$. This set has a controlled geometry, and we use a covering argument and Proposition 3.2 to control this contribution. The argument here is standard and will be presented after we establish an essential estimate in (42).

Let us begin with some preliminary statements which are essentially consequences of the fact that $\tilde{W}^{\mathrm{u}}$ and $\tilde{W}^{\mathrm{cs}}$ are complimentary foliations in the spaces marked surfaces $\mathcal{Q}^1\mathcal{T}(\alpha)$.

Recall that for any $\tilde{x} \in \mathcal{Q}^1\mathcal{T}(\alpha)$, $\mathsf{B}^\bullet_r(\tilde{x})$ denotes a ball in $\tilde{W}^\bullet(\tilde{x})$ for $\bullet = \mathrm{u}, \mathrm{s}, \mathrm{cs}, \mathrm{cu}$.

Lemma 4.2. *Let $\tilde{x}, \tilde{x}' \in \mathcal{Q}^1\mathcal{T}(\alpha)$ and let $0 < r \leq 1/2$. Assume there are $\tilde{y}_1, \tilde{y}_2 \in \tilde{W}^{\mathrm{u}}(\tilde{x})$ and some $t \in \mathbb{R}$ so that $a_t\tilde{y}_1$ and $a_t\tilde{y}_2$ belong to $\mathsf{B}^{\mathrm{cs}}_r(\tilde{x}')$. Then $\tilde{y}_1 = \tilde{y}_2$.*

Proof. We present the argument when $\varsigma = -1$, the other case is similar. By the assumption, we have $a_t\tilde{y}_i \in \tilde{W}^{\mathrm{cs}}(\tilde{x}')$ which implies that

$$\tilde{y}_i \in \tilde{W}^{\mathrm{cs}}(\tilde{x}') \text{ for } i = 1, 2.$$

Recall now that $\tilde{y}_1, \tilde{y}_2 \in \tilde{W}^{\mathrm{u}}(\tilde{x})$, hence, by (20) the corresponding abelian differentials at $\tilde{y}_1$ and $\tilde{y}_2$ differ from each other by some $c \in H^1_{\mathrm{odd}}(\hat{M}, \hat{\Sigma}, \mathbb{R})$. However, since $\tilde{y}_1, \tilde{y}_2 \in \tilde{W}^{\mathrm{cs}}(\tilde{x}')$, they differ from each other by some $c \in H^1_{\mathrm{odd}}(\hat{M}, \hat{\Sigma}, \mathbf{i}\mathbb{R}) \oplus \mathbb{R}\mathbf{v}(\tilde{x}')$. Therefore, $\tilde{y}_1 = \tilde{y}_2$. □

Corollary 4.3. *Let $\mathbf{g}_1, \mathbf{g}_2 \in \mathrm{Mod}_g$ be so that $\mathbf{g}_1 \cdot \tilde{W}^{\mathrm{u}}(\tilde{y}) = \tilde{W}^{\mathrm{u}}(\tilde{x}) = \mathbf{g}_2 \cdot \tilde{W}^{\mathrm{u}}(\tilde{y})$. Let $\tilde{x}_1$ and $\tilde{x}_2$ in $\tilde{W}^{\mathrm{u}}(\tilde{x})$. Assume for some $r, b > 0$ that*

$$\mathsf{B}^{\mathrm{cs}}_r(\tilde{x}') \cap \mathbf{g}_i \cdot a_t\mathsf{B}^{\mathrm{u}}_b(\tilde{x}_i) \neq \emptyset \text{ for } i = 1, 2 \text{ and some } t \in \mathbb{R}.$$

Then $\mathsf{B}^{\mathrm{cs}}_r(\tilde{x}') \cap \mathbf{g}_1 \cdot a_t\mathsf{B}^{\mathrm{u}}_b(\tilde{x}_1) = \mathsf{B}^{\mathrm{cs}}_r(\tilde{x}') \cap \mathbf{g}_2 \cdot a_t\mathsf{B}^{\mathrm{u}}_b(\tilde{x}_2)$. In particular, we have

$$\mathbf{g}_1 \cdot \mathsf{B}^{\mathrm{u}}_b(\tilde{x}_1) \cap \mathbf{g}_2 \cdot \mathsf{B}^{\mathrm{u}}_b(\tilde{x}_2) \neq \emptyset.$$

Proof. Let $\tilde{y}_i \in \mathsf{B}^{\mathrm{cs}}_r(\tilde{x}') \cap \mathbf{g}_i \cdot a_t\mathsf{B}_b(\tilde{x}_i)$ for $i = 1, 2$. Then $\tilde{y}_1, \tilde{y}_2 \in \mathsf{B}^{\mathrm{cs}}_r(\tilde{x}') \cap a_t\tilde{W}^{\mathrm{u}}(\tilde{x})$. Hence, by Lemma 4.2 we have $\tilde{y}_1 = \tilde{y}_2$ which implies the claim. □

As was discussed above, there are two types of exceptional points. The first type will be controlled using the following theorem.

Theorem 4.4 (Cf. [EMR], Thm. 1.7)**.** *There exist N_9 and a compact subset $\bar{K}_\alpha \supset K_\alpha$ so that*

$$\#\{y \in \mathsf{B}^{\mathrm{u}}_{0.1r(x)}(x) - \mathsf{H}^{\mathrm{u}}_t(x, \bar{K}_\alpha) : a_t y \in \mathsf{B}^{\mathrm{cs}}_{0.1r(z)}(z)\} \ll u(x)^{N_9} u(z)^{N_9} e^{(h-0.5)t}$$

where the implied constant is absolute.

Proof. Let us write $r = 0.1r(x)$ and $r' = 0.1r(z)$. For a compact subset $K \supset K_\alpha$, put

$$\mathsf{E}_t(x, K) := \{y \in \mathsf{B}^{\mathrm{u}}_{2r}(x) - \mathsf{H}^{\mathrm{u}}_t(x, K) : a_t y \in \mathsf{B}^{\mathrm{cs}}_{r'}(z)\}.$$

In $\mathcal{Q}^1\mathcal{T}(\alpha)$ fix lifts $\mathsf{B}^{\mathrm{u}}_{2r}(\tilde{x})$ and $\mathsf{B}_{r'}(\tilde{z})$ for the sets $\mathsf{B}^{\mathrm{u}}_{2r}(x)$ and $\mathsf{B}^{\mathrm{cs}}_{r'}(z)$, respectively. For every element $y \in \mathsf{B}^{\mathrm{u}}_{2r}(x)$ we fix a lift $\tilde{y} \in \mathsf{B}^{\mathrm{u}}_{2r}(\tilde{x})$. Then for every $y \in \mathsf{E}_t(x, K)$ there exists some $\mathbf{g}_y \in \mathrm{Mod}_g$ and some $\tilde{z}_y \in \mathsf{B}^{\mathrm{cs}}_{r'}(\tilde{z})$ so that $a_t\tilde{y} = \mathbf{g}_y\tilde{z}_y$.

Recall from Lemma 2.6 that the diameter of $\mathsf{B}_{r(q)}(\tilde{q})$ in the Teichmüller metric is at most 1 for all $\tilde{q}$. Hence, for every $y \in \mathsf{E}_t(x, K)$ we have

(1) $\tilde{y}$ is within Teichmüller distance 1 from $\tilde{x}$ and $a_t\tilde{y} = \mathbf{g}_y\tilde{z}_y$ is within Teichmüller distance 1 of $\mathbf{g}_y\tilde{z}$, and
(2) $|\{\tau \in [0, t] : \pi(a_\tau\tilde{y}) \in K)\}| < t/2$.

It is shown in [EMR, Thm. 1.7], see also [EMir], that there exists some K_0 so that if $K \supset K_0$, then the number of $\{\mathbf{g}\tilde{z}\}$ for which such a $\tilde{y}$ exists is

$$\ll u(x)^\star u(z)^\star e^{(h-0.5)t}$$

where the implied constant is absolute — indeed apply with $\delta = 0.1$ and $\theta = 0.9$ and observe that the function G in [EMR, Thm. 1.7] is dominated by our function u here.

We now claim that there exists some C which depends on α and K so that the following holds: the map $y \mapsto \mathbf{g}_y\tilde{z}$ from $\mathsf{E}_t(x, K)$ to $\{\mathbf{g}\tilde{z} : \mathbf{g} \in \mathrm{Mod}_g\}$ is at most C-to-one.

First note that the above discussion together with the claim implies that

$$\#\mathsf{E}_t(x, K) \ll_C u(x)^\star u(z)^\star e^{(h-0.5)t}, \tag{33}$$

as we wanted to show.

To see the claim, let $y_1, y_2 \in \mathsf{E}_t(x, K)$. Then there exists $\mathbf{g}_1, \mathbf{g}_2 \in \mathrm{Mod}_g$ so that

$$\mathbf{g}_i \cdot a_t\tilde{y}_i \in \mathsf{B}^{\mathrm{cs}}_{r'}(\tilde{z}).$$

Therefore, by Corollary 4.3, applied with $\tilde{x}_i = \tilde{x}$ and $b = 2r$, we have

- either $\mathbf{g}_1 \cdot W^{\mathrm{u}}(\tilde{x}) \neq \mathbf{g}_2 \cdot W^{\mathrm{u}}(\tilde{x})$ which in particular implies that $\mathbf{g}_1 \neq \mathbf{g}_2$,
- or $\mathbf{g}_1 \cdot \mathsf{B}^{\mathrm{u}}_{2r}(\tilde{x}) \cap \mathbf{g}_2 \cdot \mathsf{B}^{\mathrm{u}}_{2r}(\tilde{x}) \neq \emptyset$ which implies $\mathbf{g}_1^{-1}\mathbf{g}_2$ belongs to a fixed finite subset of Mod_g.

The claim thus follows and the proof is complete. □

The following lemma will play a crucial role in the proof of Proposition 4.1.

Lemma 4.5. *There exists κ_{10} and N_{10} with the following property. Let $x, z \in \mathcal{Q}_1(\alpha)$, and let $t \geq N_{10}\max\{\log u(x), \log u(z)\}$. Let*

- *$\psi^{\mathrm{u}} \in C^\infty_c(\mathsf{B}^{\mathrm{u}}_{0.1r(x)}(x))$ with $0 \leq \psi^{\mathrm{u}} \leq 1$, and*
- *$\phi^{\mathrm{u}} \in C^\infty_c(\mathsf{B}^{\mathrm{u}}_{0.1r(z)}(z))$ and $\phi^{\mathrm{cs}} \in C^\infty_c(\mathsf{B}^{\mathrm{cs}}_{0.1r(z)}(z))$.*

Put $\phi(y) := \phi^{\mathrm{cs}}(p^{\mathrm{cs}}_{y^{\mathrm{u}}}(y))\phi^{\mathrm{u}}(y^{\mathrm{u}})$, see §3. Define

$$\mathcal{N}'_{\mathrm{nc}}(t, \psi^{\mathrm{u}}, \phi) := \sum \psi^{\mathrm{u}}(y)\mu^{\mathrm{u}}_{a_t y}(\phi) \tag{34}$$

where the sum is taken over all $y \in \mathsf{B}^{\mathrm{u}}_r(x)$ so that $a_t y \in \mathsf{B}^{\mathrm{cs}}_{r'}(z)$. Then

$$|\mathcal{N}'_{\mathrm{nc}}(t, \psi^{\mathrm{u}}, \phi) - e^{ht}\mu^{\mathrm{u}}_x(\psi^{\mathrm{u}})\mu(\phi)| \leq \mathcal{C}^1(\psi^{\mathrm{u}})\mathcal{C}^1(\phi)e^{(h-\kappa_{10})t}$$

where $h = \frac{1}{2}(\dim_{\mathbb{R}} \mathcal{Q}(\alpha) - 2)$.

Proof. We will compute

$$\int_{W^{\mathrm{u}}(x)} \phi(a_t y)\psi^{\mathrm{u}}(y)\,\mathrm{d}\mu^{\mathrm{u}}_x(y)$$

in terms of $\mathcal{N}'_{\mathrm{nc}}$. The claim will then follow from Proposition 3.2.

Let us write $r = 0.1r(x)$ and $r' = 0.1r(z)$. First note that

$$r' \ll \mathrm{diam}\big(W^{\mathrm{u}}(z') \cap \mathsf{B}_{r'}(z)\big) \ll r' \tag{35}$$

where the diameter, diam, is measured with respect to $\|\ \|_{z',\mathrm{AGY}}$ for all $z' \in \mathsf{B}_{r'}(z)$, see [AG, Prop. 5.3].

Let $\bar{K}_\alpha$ be given by Theorem 4.4 and put $\mathsf{H}^{\mathrm{u}}_t(x) := \mathsf{H}^{\mathrm{u}}_t(x, \bar{K}_\alpha)$, see (32) for the notation. Since $K_\alpha \subset \bar{K}_\alpha$, it follows from Lemma 2.8 that

$$\mu^{\mathrm{u}}_x\left(\mathsf{B}^{\mathrm{u}}_r(x) - \mathsf{H}^{\mathrm{u}}_t(x)\right) \le e^{-\kappa_4 t}\mu^{\mathrm{u}}_x(\mathsf{B}^{\mathrm{u}}_r(x)) \tag{36}$$

for every $t \ge t_0$ where t_0 depends only on K_α.

It is more convenient for the proof to treat points in $\mathsf{H}^{\mathrm{u}}_t(x)$ which are *too* close to the boundary of $\mathsf{B}^{\mathrm{u}}_r(x)$ separately. Define

$$\mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}} := \{y \in \mathsf{H}^{\mathrm{u}}_t(x) : \mathsf{B}^{\mathrm{u}}_{10e^{-\kappa_{11}t}}(y) \subset \mathsf{B}^{\mathrm{u}}_r(x)\}$$

where $\kappa_{11} := \kappa_2(\bar{K}_\alpha)/2$, see Proposition 2.7 for the definition of κ_2. The precise radius which is used in the definition of $\mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}$ is motivated by estimates for uniform hyperbolicity of the Teichmüller geodesic flow, see Claim 4.6 below.

Using (36) and the definition of $\mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}$ we have

$$\mu^{\mathrm{u}}_x\left(\mathsf{B}^{\mathrm{u}}_r(x) - \mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}\right) \le e^{-\kappa_{12} t}\mu^{\mathrm{u}}_x(\mathsf{B}^{\mathrm{u}}_r(x)) \tag{37}$$

for some κ_{12} depending on $\bar{K}_\alpha$. The estimate in (37) implies the following:

$$\int_{W^{\mathrm{u}}(x)} \phi(a_t y)\psi^{\mathrm{u}}(y)\,\mathrm{d}\mu^{\mathrm{u}}_x(y) = O(e^{-\kappa_{12}t})\mu^{\mathrm{u}}_x(\mathsf{B}^{\mathrm{u}}_r(x))\mathcal{C}^1(\psi^{\mathrm{u}})\mathcal{C}^1(\phi) + \int_{\mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}} \phi(a_t y)\psi^{\mathrm{u}}(y)\,\mathrm{d}\mu^{\mathrm{u}}_x(y). \tag{38}$$

We now compute the term $\int_{\mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}} \phi(a_t y)\psi^{\mathrm{u}}(y)\,\mathrm{d}\mu^{\mathrm{u}}_x(y)$ appearing in (38).

For every $y \in \mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}$ so that $a_t y \in \mathsf{B}_r(z)$, there is an open neighborhood C_y of y such that $a_t\mathsf{C}_y$ is a connected component of $a_t\mathsf{B}^{\mathrm{u}}_r(x) \cap \mathsf{B}_{r'}(z)$ containing $a_t y$. We note that $\mathcal{C} = \{\mathsf{C}_y\}$ is a disjoint collection of open subsets in $\mathsf{B}^{\mathrm{u}}_r(x)$. Further, in view of (21) we have

$$\mu^{\mathrm{u}}_{a_t y}(\phi) = e^{ht}\mu^{\mathrm{u}}_y\big(a_{-t}\phi\big) = e^{ht}\mu^{\mathrm{u}}_x\big(a_{-t}\phi\big); \tag{39}$$

recall that $a_{-t}\phi(y') = \phi(a_t y')$.

Claim 4.6. *Let $y \in \mathsf{H}^{\mathrm{u}}_t(x)$, then $\mathsf{C}_y \subset \mathsf{B}^{\mathrm{u}}_{10e^{-\kappa_{11}t}}(y)$. If we further assume that $y \in \mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}$, then $\mathsf{C}_y \subset \mathsf{B}^{\mathrm{u}}_{10e^{-\kappa_{11}t}}(y) \subset \mathsf{B}^{\mathrm{u}}_r(x)$.*

Proof of the claim. Let $y' \in \mathsf{C}_y$. It follows from the definition of C_y that $a_t y' \in W^{\mathrm{u}}(a_t y) \cap \mathsf{B}_{r'}(z)$. Let us write $a_t y' = \Phi^{-1}(\Phi(a_t y) + w)$. Then, by (35) we have

$$\|w\|_{\mathrm{AGY},a_t y} \ll r'.$$

This, in view of Corollary 2.9, implies that

$$\|w\|_{\mathrm{AGY},y} \le e^{-\kappa_2 t}\|w\|_{\mathrm{AGY},a_t y} \ll e^{-\kappa_2 t} r'$$

where the implied constant depends only on α. The claim follows from this estimate if we assume t is large enough so that the above estimate implies

$$\|w\|_{\mathrm{AGY},y} < e^{-\kappa_{11} t};$$

recall that $\kappa_{11} = \kappa_2/2$. The final claim follows from the definition of $\mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}$. □

Claim 4.6 in particular implies that

$$|\psi^{\mathrm{u}}(y)-\psi^{\mathrm{u}}(y')| \ll e^{-\kappa_{11}t}\mathcal{C}^1(\psi^{\mathrm{u}}) \quad \text{for all } y'\in \mathsf{C}_y \tag{40}$$

where the implied constant depends only on α.

Returning to (38), we get from (39) and (40) that

$$\int_{\mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}} \phi(a_ty)\psi^{\mathrm{u}}(y)\,\mathrm{d}\mu^{\mathrm{u}}_x(y) = O(e^{-\kappa_{11}t})\mathcal{C}^1(\psi^{\mathrm{u}})\mathcal{C}^1(\phi) + e^{-ht}\sum_{\mathsf{C}_y\in\mathcal{C}}\psi^{\mathrm{u}}(y)\mu^{\mathrm{u}}_{a_ty}(\phi). \tag{41}$$

Combining (38), (41), and Proposition 3.2 we get the following

$$\Big|\sum_{\mathcal{C}}\psi^{\mathrm{u}}(y)\mu^{\mathrm{u}}_{a_ty}(\phi) - e^{ht}\mu^{\mathrm{u}}_x(\psi^{\mathrm{u}})\mu(\phi)\Big| \le \mathcal{C}^1(\psi^{\mathrm{u}})\mathcal{C}^1(\phi)e^{(h-\kappa_{13})t} \tag{42}$$

for some κ_{13} depending on α. Thus, in order to get the conclusion, we need to control the difference between $\mathcal{N}'_{\mathrm{nc}}(t,\psi^{\mathrm{u}},\phi)$ and the summation appearing on the left side of (42). That is: the contribution of points $y\notin \mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}$.

Contribution from points in $\mathsf{H}^{\mathrm{u}}_t(x)$ which are not in $\mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}$. Let $y\in\mathsf{H}^{\mathrm{u}}_t(x)-\mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}$ be so that $a_ty\in\mathsf{B}_r(z)$. We note that C_y is not necessarily contained in $\mathsf{B}^{\mathrm{u}}_r(x)$; however, in view Claim 4.6, we have C_y is contained in $\mathsf{B}_{10e^{-\kappa_{11}t}}(y)$.

The following is a consequence of the definition.

$$\bigcup_{y\in\mathsf{H}^{\mathrm{u}}_t(x)-\mathsf{H}^{\mathrm{u}}_{t,\mathrm{int}}}\mathsf{B}^{\mathrm{u}}_{10e^{-\kappa_{11}t}}(y)\subset \mathsf{B}^{\mathrm{u}}_{r+O(e^{-\kappa_{11}t})}(x)-\mathsf{B}^{\mathrm{u}}_{r-O(e^{-\kappa_{11}t})}(x) =: \mathsf{G}(x)$$

where the implicit multiplicative constant depends only on α.

Let $0<\hat\kappa<\kappa_{11}$ be a small constant which will be optimized later, and let $t\ge\frac{2N_3\log u(x)}{\hat\kappa}$. We can cover $\mathsf{G}(x)$ with period balls $\{\mathsf{B}(y_i):1\le i\le I\}$ centered at y_i and of radius $e^{-\hat\kappa t}$ with multiplicity bounded by $\ll e^{N_6\hat\kappa t}$, see [Hör, Lemma 1.4.9] and also §2.11. We have

$$I\ll e^{N\hat\kappa t} \tag{43}$$

for some N depending only on α.

For every i, let $\hat{\mathsf{B}}(y_i)$ denote the the period ball with the same center y_i and with radius $0.04e^{-\hat\kappa t}$. Note that since $\hat\kappa<\kappa_{11}=\kappa_2/2$ we have

$$2e^{-\hat\kappa t}>e^{-\hat\kappa t}+10e^{-\kappa_2 t}.$$

Therefore, $\cup_i\hat{\mathsf{B}}(y_i)$ covers a set $\mathsf{G}'(x)\supset\mathsf{G}(x)$ so that $\mu^{\mathrm{u}}_x(\mathsf{G}'(x))\ll e^{-\hat\kappa t}$.

Let $0\le\hat\psi^{\mathrm{u}}{}_i\le 1$ be a smooth function which is supported in $\hat{\mathsf{B}}^{\mathrm{u}}(y_i)$ which equals 1 on $\mathsf{B}^{\mathrm{u}}_{2e^{-\hat\kappa t}}(y_i)$ so that

$$\mathcal{C}^1(\hat\psi^{\mathrm{u}}_i)\le e^{N_6\hat\kappa t} \qquad\text{and}\qquad \sum\hat\psi^{\mathrm{u}}_i\le \mathbb{1}_{\mathsf{G}'(x)}, \tag{44}$$

where $N_6\ge N_5$ is chosen to account for the multiplicative constant in Lemma 2.10.

Let $\mathcal{I}_i$ be the contribution coming from $\mathsf{B}(y_i)$ to $\mathcal{N}_{\rm nc}(t,\psi^{\rm u},\phi)$. Then arguing as above and using Proposition 3.2, the choice of $\hat\psi^{\rm u}$ implies that

$$\mathcal{I}_i \le e^{ht}\int_{W^{\rm u}(x)}\phi(a_ty)\hat\psi_i^{\rm u}(y)\,{\rm d}\mu_x^{\rm u}(y) \le e^{ht}\mu(\phi)\int\hat\psi_i^{\rm u}\,{\rm d}\mu_x^{\rm u} + \mathcal{C}^1(\hat\psi_i^{\rm u})\mathcal{C}^1(\phi)e^{(h-\kappa_5)t} \tag{45}$$

Summing (45) over all $1\le i\le I$ and using (44), (43), and $\int\hat\psi^{\rm u}\,{\rm d}\mu_x^{\rm u}\ll e^{-h\hat\kappa t}$ we get

$$\begin{aligned}\sum_i\mathcal{I}_i &\ll e^{ht}\mu_x^{\rm u}(\mathsf{G}'(x)) + e^{N\hat\kappa t}\mathcal{C}^1(\phi)e^{(h-\kappa_5+N_6\hat\kappa)t}\\ &= e^{(h-\hat\kappa)t}+\mathcal{C}^1(\phi)e^{(h-\kappa_5+(N+N_6)\hat\kappa)t}.\end{aligned}$$

Therefore, we can choose $\hat\kappa$ so that the above upper bound yields

$$\sum_i\mathcal{I}_i \ll \mathcal{C}^1(\phi)\mathcal{C}^1(\psi^{\rm u})e^{(h-\kappa_{14})t} \tag{46}$$

for some κ_{14} depending only on α.

Contribution from points in $\mathsf{B}_r^{\rm u}(x)-\mathsf{H}_t^{\rm u}(x)$. Let $\mathcal{J}$ denote the contribution to $\mathcal{N}'_{\rm nc}(t,\psi^{\rm u},\phi)$ coming from points $y\in\mathsf{B}_r^{\rm u}(x)-\mathsf{H}_t^{\rm u}(x)$. Then there is a unique $z_y\in\mathsf{B}^{\rm u}_{r+r'}(x)-\mathsf{H}_t^{\rm u}(x,\bar K_\alpha)$ such that $a_tz_y\in\mathsf{B}^{\rm cs}_{r'}(z)$. In consequence, by Theorem 4.4, we have

$$\mathcal{J}\ll u(x)^{N_9}u(z)^{N_9}\|\phi\|_\infty\|\psi^{\rm u}\|_\infty e^{(h-0.5)t}\ll u(x)^{N_9}u(z)^{N_9}\mathcal{C}^1(\phi)\mathcal{C}^1(\psi^{\rm u})e^{(h-0.5)t}.$$

Assuming $t\gg\max\{\log u(x),\log u(z)\}$, the above implies

$$\mathcal{J}\le\mathcal{C}^1(\phi)\mathcal{C}^1(\psi^{\rm u})e^{(h-0.6)t}. \tag{47}$$

The proposition now follows from (42) in view of (46) and (47). □

Proof of Proposition 4.1. Let $\varrho=e^{-\kappa t}$ and let $\epsilon=\varrho^N$, for two constants $\kappa,N>0$ which will be optimized later. Put $\phi=1_{\mathsf{B}^{\rm u}_\varrho(z)}\phi^{\rm cs}$. Then

$$\mu(\phi)=\varrho^h\mu_z^{\rm cs}(\phi^{\rm cs}) \tag{48}$$

In view of Lemma 2.11, properties ($\mathcal{S}$-1), ($\mathcal{S}$-2), and ($\mathcal{S}$-2) hold with ϵ and $f=1_{\mathsf{B}^{\rm u}_{\varrho-2\epsilon}(z)}$. Let $\phi_1^{\rm u}=\varphi_{+,\epsilon}$ for these choices. Put $\phi_1=\phi_1^{\rm u}\phi^{\rm cs}$; there exists some κ_{15} so that

$$\mu(\phi_1)-\mu(\phi)\le\epsilon^{\kappa_{15}}\mu_z^{\rm cs}(\phi^{\rm cs}). \tag{49}$$

By Lemma 4.5, we have

$$\begin{aligned}\mathcal{N}'_{\rm nc}(t,\psi^{\rm u},\phi_1) &= e^{ht}\mu_x^{\rm u}(\psi^{\rm u})\mu(\phi_1)+O(\mathcal{C}^1(\psi^{\rm u})\mathcal{C}^1(\phi_1)e^{(h-\kappa_{10})t})\\ &\overset{(49)\rightsquigarrow}{=} e^{ht}\mu_x^{\rm u}(\psi^{\rm u})\mu(\phi)+O(\epsilon^{\kappa_{15}}e^{ht}\mu_x^{\rm u}(\psi^{\rm u})+\mathcal{C}^1(\psi^{\rm u})\mathcal{C}^1(\phi^{\rm cs})\epsilon^{-\star}e^{(h-\kappa_{10})t})\\ &\overset{(48)\rightsquigarrow}{=} e^{ht}\varrho^h\mu_x^{\rm u}(\psi^{\rm u})\mu_z^{\rm cs}(\phi^{\rm cs})+O(\epsilon^{\kappa_{15}}e^{ht}\mu_x^{\rm u}(\psi^{\rm u})+\mathcal{C}^1(\psi^{\rm u})\mathcal{C}^1(\phi^{\rm cs})\epsilon^{-\star}e^{(h-\kappa_{10})t}).\end{aligned} \tag{50}$$

Let now $\phi_2^{\rm u}=\varphi_{+,\epsilon}$ for ϵ and $f=1_{\mathsf{B}^{\rm u}_\varrho(z)}$. Put $\phi_2=\phi_2^{\rm u}\phi^{\rm cs}$. Then similar to the above estimate, using Lemma 4.5, we get that

$$\mathcal{N}'_{\rm nc}(t,\psi^{\rm u},\phi_2)=e^{ht}\varrho^h\mu_x^{\rm u}(\psi^{\rm u})\mu_z^{\rm cs}(\phi^{\rm cs})+O(\epsilon^{\kappa_{15}}e^{ht}\mu_x^{\rm u}(\psi^{\rm u})+\mathcal{C}^1(\psi^{\rm u})\mathcal{C}^1(\phi^{\rm cs})\epsilon^{-\star}e^{(h-\kappa_{10})t}). \tag{51}$$

Since $\phi_1 \le \phi \le \phi_2$, we have

$$\mathcal{N}_{\rm nc}(t,\psi^{\rm u},\phi_1) \le \mathcal{N}'_{\rm nc}(t,\psi^{\rm u},\phi) \le \mathcal{N}_{\rm nc}(t,\psi^{\rm u},\phi_2). \tag{52}$$

Moreover, using the definitions of $\mathcal{N}_{\rm nc}$ and $\mathcal{N}'_{\rm nc}$ we have

$$\begin{aligned}\mathcal{N}'_{\rm nc}(t,\psi^{\rm u},\phi) &= \sum \psi^{\rm u}(y)\mu^{\rm u}_{a_ty}(\phi)\\ &= \sum \psi^{\rm u}(y)\phi^{\rm cs}(a_ty)\mu^{\rm u}_z(B^{\rm u}_\varrho(z)) = \varrho^h \sum \psi^{\rm u}(y)\phi^{\rm cs}(a_ty)\\ &= \varrho^h\mathcal{N}_{\rm nc}(t,\psi^{\rm u},\phi^{\rm cs}).\end{aligned}$$

This and (52) imply that

$$\varrho^{-h}\mathcal{N}_{\rm nc}(t,\psi^{\rm u},\phi_1) \le \mathcal{N}_{\rm nc}(t,\psi^{\rm u},\phi^{\rm cs}) \le \varrho^{-h}\mathcal{N}_{\rm nc}(t,\psi^{\rm u},\phi_1).$$

Hence, using (50) and (51), we get that

$$\mathcal{N}_{\rm nc}(t,\psi^{\rm u},\phi^{\rm cs}) = e^{ht}\mu^{\rm u}_x(\psi^{\rm u})\mu^{\rm cs}_z(\phi^{\rm cs}) + O(\varrho^{-h}\epsilon^{\kappa_{15}}e^{ht}\mu^{\rm u}_x(\psi^{\rm u}) + \mathcal{C}^1(\psi^{\rm u})\mathcal{C}^1(\phi^{\rm cs})\epsilon^{-\star}e^{(h-\kappa_{10})t}).$$

We choose N large enough so that $\kappa_{15}N - h > \kappa_{15}N/2$ then choose κ small enough so that $\epsilon^{-\star}e^{(h-\kappa_{10})t} = e^{(h-\kappa_{10}/2)t}$. The proof is complete. □

We end this section with the following corollary.

Corollary 4.7. *There exist κ_{16}, κ_{17}, and N_{11} with the following property. Let $x,z \in \mathcal{Q}_1(\alpha)$. Let $\psi^{\rm u} \in C^\infty_c(\mathsf{B}^{\rm u}_{0.1r(x)}(x))$ with $0 \le \psi^{\rm u} \le 1$ and let $\phi^{\rm cs} \in \mathcal{S}_{W^{\rm cs}(z)}(z, 0.1r(z))$. Then for all $\delta < r(z)/10L$ and all $t \ge N_8 \max\{\log u(x), \log u(z)\}$ we have*

$$|\mathcal{N}_{\rm nc}\big(t,\psi^{\rm u},\phi^{\rm cs}\big) - e^{ht}\mu^{\rm u}_x(\psi^{\rm u})\mu^{\rm cs}_z(\phi^{\rm cs})| \ll \mathcal{C}^1(\psi^{\rm u})\delta^{-N_{11}}e^{(h-\kappa_{16})t} + \delta^{\kappa_{17}}\mathcal{C}^1(\psi^{\rm u})e^{ht}$$

where $h = \frac{1}{2}(\dim_{\mathbb{R}} \mathcal{Q}(\alpha) - 2)$.

In particular, there exists κ_{18}, depending only on α, so that the following holds. Assume further that $t \ge 2|\log r(z)| = 2N_3 \log u(z)$, see (19), then

$$|\mathcal{N}_{\rm nc}\big(t,\psi^{\rm u},\phi^{\rm cs}\big) - e^{ht}\mu^{\rm u}_x(\psi^{\rm u})\mu^{\rm cs}_z(\phi^{\rm cs})| \ll \mathcal{C}^1(\psi^{\rm u})e^{(h-\kappa_{18})t}. \tag{53}$$

Proof. The corollary follows from Proposition 4.1 by approximating $\phi^{\rm cs}$ with smooth functions. Let $\delta < 0.1r(z)/L$ and let $\phi^{\rm cs}_{\pm,\delta}$ be smooth functions satisfying ($\mathcal{S}$-1), ($\mathcal{S}$-2), and ($\mathcal{S}$-2) with δ and $\phi^{\rm cs}$. Hence, we have

$$\phi^{\rm cs}_{-,\delta} \le \phi^{\rm cs} \le \phi^{\rm cs}_{+,\delta} \quad\text{and}\quad \mathcal{C}^1(\phi_{\pm,\delta}) \ll \delta^{-\star}; \tag{54}$$

furthermore, property ($\mathcal{S}$-3) implies that

$$|\mu^{\rm cs}_z(\phi^{\rm cs}_{+,\delta}) - \mu^{\rm cs}_z(\phi^{\rm cs}_{-,\delta})| \ll \delta^\star. \tag{55}$$

With this notation and in view of the first estimate in (54), we have

$$\mathcal{N}_{\rm nc}\big(t,\psi^{\rm u},\phi^{\rm cs}_{-,\delta}\big) \le \mathcal{N}_{\rm nc}\big(t,\psi^{\rm u},\phi^{\rm cs}\big) \le \mathcal{N}_{\rm nc}\big(t,\psi^{\rm u},\phi^{\rm cs}_{+,\delta}\big). \tag{56}$$

In addition we may apply Proposition 4.1 with $\psi^{\rm u}$ and $\phi^+_{\pm\delta}$ and get that

$$\mathcal{N}_{\rm nc}\big(t,\psi^{\rm u},\phi^{\rm cs}_{\pm,\delta}\big) = e^{ht}\mu^{\rm u}_x(\psi^{\rm u})\mu^{\rm cs}_z(\phi^{\rm cs}_{\pm}) + O(\mathcal{C}^1(\psi^{\rm u})\mathcal{C}^1(\phi^{\rm cs}_{\pm,\delta})e^{(h-\kappa_9)t}.$$

This together with (56), (55), and the second estimate in (54) implies the first claim.

The second claim follows from the first claim by optimizing the choice $\delta = e^{-\star t}$. $\square$

5. The space of measured laminations

In this section we recall some basic facts about the space of geodesic measured laminations and train track charts. The basic references for these results are [Th1] and [HP].

The space of geodesic measured laminations on S is denoted by $\mathcal{ML}(S)$; it is a piecewise linear manifold homeomorphic to $\mathbb{R}^{6g-6}$, but it does not have a natural differentiable structure [Th1]. Train tracks were introduced by Thurston as a powerful technical device for understanding measured laminations. Roughly speaking train tracks are induced by squeezing almost parallel strands of a very long simple closed geodesic to simple arcs on a hyperbolic surface. A train track τ on a surface S is a finite closed 1 complex $\tau \subset S$ with vertices (switches) which is

- embedded on S,
- away from its switches, it is C^1,
- it has tangent vectors at every point, and
- for each component R of $S - \tau$, the double of R along the interiors of the edges of $\partial(R)$ has negative Euler characteristic.

The vertices (or switches), V, of a train track are the points where 3 or more smooth arcs come together. Each edge of τ is a smooth path with a well defined tangent vector. That is: all edges at a given vertex are tangent. The inward pointing tangent of an edge divides the branches that are incident to a vertex into incoming and outgoing branches.

A train track τ is called maximal (or generic) if at each vertex there are two incoming edges and one outgoing edge.

5.1. Train track charts.

A lamination λ on S is carried by a train track τ if there is a differentiable map $f : S \to S$ so that

- f is homotopic to the identity,
- the restriction of $\mathrm{d}f$ to a tangent line of λ is nonsingular, and
- f maps λ onto τ.

Every geodesic lamination is carried by some train track. Let λ be a measured lamination with invariant measure μ. If λ is carried by the train track τ, then the carrying map defines a counting measure $\mu(b)$ to each branch line b: $\mu(b)$ is just the transverse measure of the leaves of λ collapsed to a point on b. At a switch, the sum of the entering numbers equals the sum of the exiting numbers.

The piecewise linear integral structure on $\mathcal{ML}(S)$ is induced by train tracks as follows. Let $\mathcal{V}(\tau)$ be the set of measures on a train track τ; more precisely, $u \in \mathcal{V}(\tau)$ is an assignment of positive real numbers to the edges of the train track satisfying the switch condition:

$$\sum_{\text{incoming } e_i} u(e_i) = \sum_{\text{outgoing } e_j} u(e_j).$$

Also, let $\mathcal{W}(\tau)$ be the vector space of all real weight systems on edges of τ satisfying the switch condition, i.e., $u(e_i)$ need not be positive for $u \in \mathcal{W}(\tau)$. Then $\mathcal{V}(\tau)$ is a cone on a finite-sided polyhedron where the faces are of the form $\mathcal{V}(\sigma) \subset \mathcal{V}(\tau)$ where σ is a sub train track of τ.

If τ is *bi-recurrent*, then the natural map $\iota_\tau : \mathcal{V}(\tau) \to \mathcal{ML}(S)$ is continuous and injective, see [HP, §1.7]. Let

$$U(\tau) = \iota_\tau(\mathcal{V}(\tau)) \subset \mathcal{ML}(S). \tag{57}$$

Moreover, we have the following.

Lemma 5.1. *Let $\mathcal{U}_1 \subset \mathcal{V}(\tau_1)$ and $\mathcal{U}_2 \subset \mathcal{V}(\tau_2)$ be such that $\iota_{\tau_1}(\mathcal{U}_1) = \iota_{\tau_2}(\mathcal{U}_2)$. Then the map $\iota_{\tau_2}^{-1} \circ \iota_{\tau_1} : \mathcal{U}_1 \to \mathcal{U}_2$ is a piecewise linear map and hence it is bilipschitz.*

For the proof see [HP, §2 and Thm. 3.1.4].

5.2. **Thurston symplectic form on $\mathcal{ML}(S)$.** We can identify $\mathcal{W}(\tau)$ with the tangent space of $\mathcal{ML}(S)$ at a point $u \in \mathcal{V}(\tau)$, see [HP].

For any train track τ, the integral points in $\mathcal{V}(\tau)$ are in one to one correspondence with the set of integral multicurves in $U(\tau) \subset \mathcal{ML}(S)$. The natural volume form on $\mathcal{V}(\tau)$ defines a mapping class group invariant volume form μ_{Th} in the Lebesgue measure class on $\mathcal{ML}(S)$.

In fact, the volume form on $\mathcal{ML}(S)$ is induced by a mapping class group invariant 2-form ω as follows. Suppose τ is maximal, for $u_1, u_2 \in \mathcal{W}(\tau)$ the symplectic pairing is defined as follows.

$$\omega(u_1, u_2) = \frac{1}{2}\Big(\sum u_1(e_1)\, u_2(e_2) - \, u_1(e_2)\, u_2(e_1)\Big), \tag{58}$$

the sum is over all vertices v of the train track where e_1 and e_2 are the two incoming branches at v such that e_1 is on the right side of the common tangent vector.

This form defines an antisymmetric bilinear form on $\mathcal{W}(\tau)$.

Lemma 5.2. *Let τ be maximal. The Thurston form ω, defined in* (58)*, is non-degenerate. Therefore it gives rise to a symplectic form on the piecewise linear manifold $\mathcal{ML}(S)$.*

See [HP, §3] for a proof and also the relationship between the intersection pairing of $H^1(S, \mathbb{R})$ and Thurston intersection pairing.

5.3. **Combinatorial type of measured laminations and train tracks.** Each component of $S - \lambda$ is a region bounded by closed geodesics and infinite geodesics; further, any such region can be doubled along its boundary to give a complete hyperbolic surface which has finite area.

We say a filling measured lamination λ is of type $\mathbf{a} = (a_1, \ldots a_k)$ if and only if $S - \lambda$ consists of ideal polygons with $a_1, \ldots, a_k$ sides. By extending the measured lamination λ to a foliation with isolated singularities on the complement, we see that $\sum_{i=1}^k a_i = 4g - 4 + 2k$, see [Th1] and [Le].

Similarly, each component of the complement of a filling train track τ is a non-punctured or once-punctured cusped polygon of negative Euler index. We say a train track τ is of type $\mathbf{a} = (a_1, \dots, a_k)$, if and only if $S - \tau$ consists of k polygons with $a_1, \dots, a_k$ sides. Every measured lamination of type $\mathbf{a} = (a_1, \dots, a_k)$ can be carried by a train track of type $\mathbf{a}$.

Lemma 5.3. *For any filling train track τ of type $\mathbf{a} = (a_1, \dots, a_k)$ we have*

$$\dim(V(\tau)) = 2g + k - 1 \quad \text{if } \tau \text{ is orientable;}$$

$$\dim(V(\tau)) = 2g + k - 2 \quad \text{if } \tau \text{ is not orientable.}$$

More generally, a measured lamination λ is said to be of type $\mathbf{a}$ if there exists a quadratic differential $q \in \mathcal{Q}(a_1 - 2, \dots, a_k - 2)$ such that $\lambda = \Re(q)$. It is easy to check that if λ is filling, the above can happen only if $S - \lambda$ consists of ideal polygons with $a_1, \dots, a_k$ sides.

In general, see [HP, §3], we have:

Proposition 5.4. *Given a measured lamination λ of type $\mathbf{a}$, there exists a birecurrent train track of type $\mathbf{a}$ such that λ is an interior point of $U(\tau)$.*

For every $\mathbf{a} = (a_1, \dots, a_k)$ so that $\sum_{i=1}^k a_i = 4g-4+2k$, we can fix a collection $\tau_{\mathbf{a},1}, \dots, \tau_{\mathbf{a},\mathbf{c_a}}$ of train tracks with the following property. Every λ which can be carried by a train track of type $\mathbf{a}$ can be carried by at least one $\tau_{\mathbf{a},i}$ for some i.

5.4. The Hubbard-Masur map.

Let $\mathcal{MF}(S)$ denote the space of measured foliations on S. Define

$$\tilde{\mathcal{P}} : \mathcal{QT}(S) \to \mathcal{MF}(S) \times \mathcal{MF}(S) - \Delta$$

by $\tilde{\mathcal{P}}(q) = (\Re(q^{1/2}), \Im(q^{1/2}))$ where

$$\Delta = \{(\eta, \lambda) : \text{ there exists } \sigma \text{ so that } i(\sigma, \lambda) + i(\sigma, \eta) = 0\}.$$

Theorem 5.5 (Hubbard-Masur, Gardiner)**.** *The map $\tilde{\mathcal{P}}$ is a Mod_g equivariant homeomorphism.*

This gives rise to an equivariant homeomorphism from $\mathcal{QT}(S)$ onto $\mathcal{ML}(S) \times \mathcal{ML}(S) - \Delta$ which we continue to denote by $\tilde{\mathcal{P}}$, see [Th1] and [Le].

Recall that $\mathrm{P}\mathcal{ML}(S)$ denotes the space of *projective* measured lamination. The map $\tilde{\mathcal{P}}$ also gives rise to an equivariant homeomorphism

$$\tilde{\mathcal{P}}_1 : \mathcal{Q}_1\mathcal{T}(S) \to \mathrm{P}\mathcal{ML}(S) \times \mathcal{ML}(S) - \Delta$$

where $\tilde{\mathcal{P}}_1(q) = ([\Re(q^{1/2})], \Im(q^{1/2}))$ and $\Delta = \{([\eta], \lambda) : \exists\, \sigma \text{ so that } i(\sigma, \eta) + i(\sigma, \lambda) = 0\}$.

Recall that π is the natural projection from $\mathcal{Q}_1\mathcal{T}(S)$ to $\mathcal{Q}_1(S)$, then we have the map

$$\pi \circ \tilde{\mathcal{P}}_1^{-1} : \mathrm{P}\mathcal{ML}(S) \times \mathcal{ML}(S) - \Delta \to \mathcal{Q}_1(S). \tag{59}$$

5.5. **Convexity of the hyperbolic length function.** Let $\lambda_1, \lambda_2 \in U(\tau) = \iota_\tau(\mathcal{V}(\tau))$, see §5.1 for the definition of ι_τ. The sum

$$\lambda_1 \oplus_\tau \lambda_2 = \iota_\tau(\iota_\tau^{-1}(\lambda_1) + \iota_\tau^{-1}(\lambda_2))$$

could depend on τ. However, it is proved in [Mir1, App. A] that given a closed curve γ, $i(\gamma, .) : U(\tau) \to \mathbb{R}_+$ defines a convex function from which convexity of the hyperbolic length function is drawn in [Mir1, Thm. A.1]. The following is an extension of [Mir1, Thm. A.1] to the case of variable negative curvature. We are grateful to K. Rafi for providing the proof of this theorem.

Theorem 5.1. *Let X be a compact surface equipped with a Riemannian metric of negative curvature, and let τ be a train track. Let $\ell_X : U(\tau) \to \mathbb{R}^+$ denote the length function. For every pair of measured laminations $\lambda_1, \lambda_2 \in \mathcal{ML}(S)$ carried by τ if $\mu = \lambda_1 \oplus_\tau \lambda_2$, then*

$$\ell_X(\mu) \leq \ell_X(\lambda_1) + \ell_X(\lambda_2).$$

In particular, ℓ_X is convex.

The following lemma is well known

Lemma 5.6. *Let τ a train-track, and let λ_1 and λ_2 be multi-curves carried by τ. Then, there exists a multi-curve μ carried by τ such that $\mu = \lambda_1 + \lambda_2$ in coordinates given by τ. Furthermore, μ can be obtained from λ_1 and λ_2 by a sequence of surgeries.*

We now turn to the proof of the Theorem 5.1.

Proof of Theorem 5.1. Let $\mathcal{C}$ be the space of geodesic currents on X, that is the space of $\pi_1(X)$-invariant Radon measures on the space of geodesics in X. Recall that the space of measured laminations can be topologically embedded into the space of geodesic currents, therefore, we can think of any $\lambda \in \mathcal{ML}(S)$ as a geodesic current, namely, an element of $\mathcal{C}$. Also recall from [Bon1] that there is a continuous intersection pairing

$$i : \mathcal{C} \times \mathcal{C} \to \mathbb{R}.$$

Furthermore, there is a geodesic current $L_X \in \mathcal{C}$ such that for every

$$i(L_X, \lambda) = \ell_X(\lambda), \quad \text{for all } \lambda \in \mathcal{ML}(S),$$

see [Ot]. The set of simple closed curves with rational weights are dense in $\mathcal{ML}(S)$. Therefore, in view of the continuity of intersection pairing i, it is sufficient to check the statement of the theorem for rationally weighted simple closed curves only. Since, length is homogeneous, we can in fact assume the weights are integers or λ_1, λ_2, and μ are multi-curves with the possibility of some curve appearing more than once.

Claim 5.7. *Assume λ_1 and λ_2 are two simple closed curves with $i(\lambda_1, \lambda_2) > 0$. Let β be a curve obtained from λ_1 and λ_2 by a surgery at an intersection point. Then, $\ell_X(\beta) \leq \ell_X(\lambda_1) + \ell_X(\lambda_2)$.*

Proof of the claim. Note that λ_1 and λ_2 have unique geodesic representatives in M. Let p be an intersection point of λ_1 and λ_2 where the surgery takes place. Then the free homotopy class of β can be represented by a traversing λ_1 first (starting from p) then λ_2. Which means β has a representative whose length is $\ell_X(\lambda_1)+\ell_X(\lambda_2)$. This proves the claim. □

Further, we note that, $\mu=\lambda_1\oplus_\tau\lambda_2$ can be obtained from λ_1 and λ_2 by a sequence of surgery maps, see Lemma 5.6. This proves the theorem. □

Let $C\subset\mathbb{R}^n$ be a cone and $f:C\to\mathbb{R}$ be a convex function. Let K be a closed and bounded set contained in the relative interior of the domain of f. Then f is Lipschitz continuous on K. That is: there exists a constant $L=L(K)$ such that for all $x,y\in K$ we have

$$|f(x)-f(y)|\leq L|x-y|.$$

Therefore, we have the following.

Corollary 5.2. *Let X be a compact surface equipped with a Riemannian metric of negative curvature. Then*

$$\ell_X:\mathcal{ML}(S)\to\mathbb{R}^+$$

is locally Lipschitz. In other words, and in view of the fact that $\ell_X(t\,\cdot)=t\ell_X(\cdot)$ for all $t>0$, we can cover $\mathcal{ML}(S)$ with finitely many cones such that ℓ_X is Lipschitz in each cone.

The Lipschitz constant *depends* on X. See also [LS].

6. Linear structure of $\mathcal{ML}(S)$ and $\mathcal{QT}(S)$

Our arguments are based on relating the counting problems in $\mathcal{ML}(S)$ to dynamical results in $\mathcal{Q}_1(1,\ldots,1)$. To that end, we need to compare the linear structure on $\mathcal{Q}_1(1,\ldots,1)$, arising from period coordinates, with the piecewise linear structure on $\mathcal{ML}(S)$, which arises from train track charts. This section establishes required results in this direction.

From this point to the end of the paper, we will be concerned with the principal stratum, i.e., $\mathcal{Q}_1(1,\ldots,1)$. Also $\mathbf{a}=(3,\ldots,3)$ for the rest of the discussion.

Fix once and for all a collection $\tau_1,\ldots,\tau_{\mathsf{c}}$ of train tracks so that every λ can be carried by at least one τ_i for some i, see §5.3.

Given a point $x=(M,q)\in\mathcal{Q}_1(1,\ldots,1)$ we sometimes use q to denote x. We fix a fundamental domain for $\mathcal{Q}_1(1,\ldots,1)$, and unless explicitly stated otherwise, by a lift $\tilde{q}$ of $q\in\mathcal{Q}_1(1,\ldots,1)$ we mean a representative in this fundamental domain.

Let $x=(M,q)\in\mathcal{Q}_1(1,\ldots,1)$. We denote by $\Re(q^{1/2})$ (resp. $\Im(q^{1/2})$) the real (resp. imaginary) foliation induced by q; abusing the notation we will often simply denote these foliations by $\Re(q)$ and $\Im(q)$. Note that $W^{\mathrm{u,s}}(x)$, which we sometimes also denote by $W^{\mathrm{u,s}}(q)$, may alternatively be defined as follows.

$$W^{\mathrm{u}}(q):=\{q'\in\mathcal{Q}_1(1,\ldots,1):\Im(q')=\Im(q)\},$$

and $W^{\mathrm{s}}(q):=\{q'\in\mathcal{Q}_1(1,\ldots,1):\Re(q')=\Re(q)\}$.

Similarly, we will write $\mathsf{B}_r(q)$ and $\mathsf{B}_r^\bullet(q)$ for $\mathsf{B}_r(x)$ and $\mathsf{B}_r^\bullet(x)$, respectively.

Let τ be a maximal train track, i.e., a train track of type $(3,\ldots,3)$, and let $U(\tau)$ be a train track chart, i.e., the set of weights on τ satisfying the switch conditions. Recall from §5.1 that $U(\tau)$ has a linear structure, indeed $U(\tau)$ is a cone on a finite-sided polyhedron. We use the L^1-norm on $\mathcal{W}(\tau)$ to define a norm on $U(\tau)$. That is: for every measured lamination $\lambda\in U(\tau)$, we define $\|\lambda\|_\tau$ to be the sum of the weights of λ. Let us define

$$P(\tau) := \{\lambda \in U(\tau) : \|\lambda\|_\tau = 1\}. \tag{60}$$

For every $\lambda\in U(\tau)$, define

$$\bar\lambda^\tau := \tfrac{1}{\|\lambda\|_\tau}\lambda \in P(\tau);$$

if τ is fixed and clear from the context, we sometimes drop the subscript and the superscript τ and simply write $\|\lambda\|$ and $\bar\lambda$ for $\|\lambda\|_\tau$ and $\bar\lambda^\tau$, respectively.

By a *polyhedron* $\mathcal{U}\subset U(\tau)$, we mean a polyhedron of dimension $\dim U(\tau)-1$ where the angles are bounded below and the number of facets are bounded, both by constants depending only on the genus. We will mainly be concerned with $\dim U(\tau)-1$ dimensional *cubes* in the sequel.

Lemma 6.1 (Cf. [LMir], Thm. 6.4)**.** *Let $\eta\in\mathcal{ML}(S)$ be maximal. There is a compact subset $K\subset\mathcal{Q}_1(1,\ldots,1)$, depending on τ and η, so that $\pi\circ\tilde{\mathcal{P}}_1^{-1}([\eta],P(\tau))\subset K$, see* (59) *for the definition of $\pi\circ\tilde{\mathcal{P}}_1^{-1}$.*

Proof. Recall that we fixed a collection $\tau_1,\ldots,\tau_{\mathsf{c}}$ of train tracks so that every lamination λ is carried by some τ_i. In view of Lemma 5.1, there exists some $L=L(\tau)$ so that

$$P(\tau)\subset\bigcup_{i=1}^{\mathsf{c}}\{\lambda\in U(\tau_i): 1/L\le\|\lambda\|_i\le L\};$$

where $\|\ \|_i=\|\ \|_{\tau_i}$.

For every $1\le i\le\mathsf{c}$, put $U_i:=\{\lambda\in U(\tau_i): 1/L\le\|\lambda\|_i\le L\}$. Since η is a maximal measured lamination, for any $\lambda\in U(\tau_i)$ we have $\pi\circ\tilde{\mathcal{P}}_1^{-1}([\eta],\lambda)\in\mathcal{Q}_1(1,\ldots,1)$. Define

$$K:=\bigcup_i \pi\circ\tilde{\mathcal{P}}_1^{-1}(\{[\eta]\}\times U_i). \tag{61}$$

Then $K\subset\mathcal{Q}_1(1,\ldots,1)$ is a compact subset with the desired property. □

Lemma 6.2. *There is some $N_{12}\ge N_3$ so that the following holds, see* (19) *for the definition of N_3. Let $q\in\mathcal{Q}_1(1,\ldots,1)$. There exists a 1-complex $T\subset S$ with the following properties.*

1. *Every edge of T is a saddle connection of q.*
2. *$|\Im(e)|\ge 0.1\ell_q(e)$ for any $e\in T$.*
3. *$S-T$ is a union of triangles.*
4. *For every edge $e\in T$, we have $u(q)^{-N_{12}}\le\ell_q(e)\le u(q)^{-N_{12}}$.*
5. *We have $u(q)^{-N_{12}}\le r(q)$, moreover, the parallel translate of T to $q'\in\mathsf{B}_{u(q)^{-N_{12}}}(q)$ satisfies* (1), (2), and (3) *above,*

Similar statement holds if we replace $\Im(e)$ in (3) above by $\Re(e)$.

Proof. We will find such a T with $|\Im(e)| \geq 0.1\ell_q(e)$, the proof of the fact that such a T exists with $|\Re(e)| > 0.1\ell_q(e)$ is similar, by replacing $a_t u_s$ with $a_{-t}\bar{u}_s$ in the following argument.

Let K be the compact set given by Theorem 2.5; let $r_0 = \inf\{r(x) : x \in K\}$, see (19). For every $q' \in K$, there is a graph $T' \subset S$ of saddle connections of q' so that

- the q' length of each of these saddle connections is bounded by $L_0 = L_0(K)$, and
- $S - T'$ is a union of triangles.

We will always assume that $L_0 > 2$. Increasing L_0, if necessary, we will also assume that L_0 bounds the lengths of saddle connections obtained by parallel transporting T' to $q'' \in \mathsf{B}_{r_0}(q')$ for all $q' \in K$.

Set $R_q := \{$saddle connections γ of q with $|\Im(\gamma)| < 0.1\ell_q(\gamma)\}$. Note that for all $\gamma \in R_q$ and all $0 \leq s \leq 1$, we have $|\Re(u_s\gamma)| \geq \ell_q(\gamma)/2$. Define the function

$$f(q) := \max\{1, \max\{1/\ell_q(\gamma) : \gamma \in R_q\}\}.$$

Apply Theorem 2.5 with $t_0 = L_0 \log f(q)$. There exists some

$$t_0 < t \leq \max\{2t_0, N_2 \log u(q)\} \tag{62}$$

and some $s \in [0, 1]$ so that $q' = a_t u_s q \in K$.

Let now T' be a graph of saddle connections for q' defined as above. We claim that for every $e \in T'$, we have $e \notin a_t u_s R_q$. To see the claim, first note that for every $\gamma \in R_q$ we have

$$\begin{aligned}
\ell_{q'}(a_t u_s \gamma) &\geq e^t \Re(u_s\gamma) \\
&\geq e^t \ell_q(\gamma)/2 && |\Re(u_s\gamma)| \geq \ell_q(\gamma)/2 \\
&\geq e^{L_0} f(q)\ell_q(\gamma)/2 > L_0 && t > L_0 \log f(q) \;\;\&\;\; f(q)\ell_q(\gamma) \geq 1.
\end{aligned}$$

Hence $a_t u_s \gamma$ is not contained in T'. In consequence, $T = u_{-s}a_{-t}T'$ satisfies (1), (2), (3), and (4). Note that for every $e \in T$, we have $u(q)^{-\star} \ll \ell_q(e) \ll u(q)^{\star}$ where the implied constants depend only on the genus.

We now turn to the proof of part (5). First note that there is N' so that $u(q)^{N'} \geq f(q)^{2L_0}$; put $N := \max\{2N', 2N_2, N_3\}$. Let $N_{12} > N$ be so that

$$e^2 \cdot 2^{N - N_{12}} \leq r_0/2. \tag{63}$$

Let us write $0 < r = u(q)^{-N_{12}}$, then $0 < r \leq r(q)$, recall that $N_{12} \geq N_3$. For every $z \in \mathsf{B}_r(q)$, we have $z = \Phi^{-1}(\Phi(q') + v)$ where $\|v\|_{\mathrm{AGY},q} \leq u(q)^{-N_{12}}$.

Let $t \leq \max\{2L_0 f(q), N_2 \log u(q)\}$ and $0 \leq s \leq 1$ be so that $q' = a_t u_s q \in K$; see the preceding discussion. Note that in view the choice of t and N, we have

$$e^{2t} \leq u(q)^N. \tag{64}$$

Now for all v so that $\Phi^{-1}(\Phi(q') + v) \in \mathsf{B}_r(q)$ we have

$$
\begin{aligned}
\|v\|_{\mathrm{AGY},a_t u_s q} &\le e^{2+2t}\|v\|_{\mathrm{AGY},q} && \text{by (15)}\\
&\le e^2 \cdot u(q)^N \|v\|_{\mathrm{AGY},q} && \text{by (64)}\\
&\le e^2 \cdot u(q)^{N-N_{12}} && \|v\|_{\mathrm{AGY},q} \le u(q)^{-N_{12}} \text{ by the choice of } r\\
&\le e^2 \cdot 2^{N-N_{12}} \le r_0/2 && \text{since } u(q) \ge 2 \text{ and using (63).}
\end{aligned}
$$

Hence $a_t u_s \mathsf{B}_r(q) \subset \mathsf{B}_{r_0}(q')$ which gives the claim in view of the definitions of T and T'.

Increasing N_{12} if necessary part (4) also holds for this exponent. □

Lemma 6.3 (Cf. [Mir3], Lemma 4.3)**.** *Let $q \in \mathcal{Q}_1(1,\dots,1)$, and let $\tilde{q}$ be a lift of q in our fixed fundamental domain. Let $r = 0.01u(q)^{-2N_{12}}$, there is a maximal train track σ with the following properties*

1. *$\mathsf{B}_r(\tilde{q})$ projects homeomorphically onto $\mathsf{B}_r(q) \subset \mathcal{Q}_1(1,\dots,1)$.*
2. *The restriction of $\tilde{\mathcal{P}}_1$ to $\mathsf{B}_r(\tilde{q})$ is a homeomorphism.*
3. *$\{\Im(\tilde{p}) : \tilde{p} \in \mathsf{B}_r(\tilde{q})\}$ is contained in one train track chart $U(\sigma)$.*
4. *The linear structure on $U_{\Im}(\tilde{q}) := \{\Im(\tilde{p}) : \tilde{p} \in \mathsf{B}_r(\tilde{q}), \Re(\tilde{p}) = \Re(\tilde{q})\}$ as a subset of $U(\sigma)$ agrees with the linear structure on $U_{\Im}(\tilde{q})$ which is induced by the restriction of $\tilde{\mathcal{P}}_1$ to $\{\tilde{p} \in \mathsf{B}_r(\tilde{q}) : \Re(\tilde{p}) = \Re(\tilde{q})\} \subset W^{\mathrm{s}}(\tilde{q})$.*

Moreover, the radius r of $\mathsf{B}_r(\tilde{q})$ can be taken to be uniform on compact subsets of $\mathcal{Q}_1(1,\dots,1)$.

Proof. Let T be a triangulation of q given by Lemma 6.2. In particular,

(i) every edge of T is a saddle connection,
(ii) $|\Im(e)| \ge 0.1\ell_q(e)$ for any $e \in T$,
(iii) $S - T$ is a union of triangles, and
(iv) $A_q \le \ell_q(e) \le A_q^{-1}$ for every edge $e \in T$ where $A_q = u(q)^{-N_{12}}$

Our construction of the train track σ will depend on T.

Recall that $r = 0.01A_q^2$. Then the balls $\mathsf{B}_r(\tilde{q})$ and $\mathsf{B}_r(q)$ satisfy (1) and (2) in the lemma by Lemma 6.2(5).

Let σ' be the null-gon dual graph to T, in particular, there is one triangle of σ' in each component of $S - T$. Let σ be the train track obtained from σ' as follows. If Δ is a triangle in T with edges $e_1^\Delta, e_2^\Delta, e_3^\Delta$, then there is a permutation $\{i_1, i_2, i_3\}$ of $\{1,2,3\}$ so that

$$|\Im(e_{i_1}^\Delta)| = |\Im(e_{i_2}^\Delta)| + |\Im(e_{i_3}^\Delta)|; \tag{65}$$

put $\sigma := \sigma' - \bigcup\{\text{the edge corresponding to } e_{i_1}^\Delta \text{ in } \sigma'\}$.

We claim the lemma holds with σ. To see the claim, first note that σ is a maximal train track. Assign the weight $|\Im(e_b)|$ to each branch $b \in \sigma$ where $e_b \in T$ is the edge which intersects b. In view of (65) and the fact that $|\Im(\gamma)| = i(\gamma, \Re(\tilde{q}))$ for every saddle connection γ, we get that $\lambda = \Im(\tilde{q})$ is carried by σ.

By Lemma 6.2, for any $\tilde{p} \in \mathsf{B}_r(\tilde{q})$ we identify T with its image (under parallel transport) on $\tilde{p}$. Let $\tilde{p} \in \mathsf{B}(\tilde{q})$ and write $\tilde{p} = \tilde{q} + w$ for some w with $\|w\|_{\mathrm{AGY},q} \le 0.01A_q^2$. Then

$$|\Im(\mathrm{hol}_{\tilde{p}}(e_b))| = |\Im(\mathrm{hol}_{\tilde{q}}(e_b)) + \Im(w(e_b))|.$$

Further, we have $|w(e_b)| \le 0.01A_q^2\ell_q(e_b) < 0.01A_q \le 0.1|\Im(\mathrm{hol}_{\tilde{q}}(e_b))|$; we used (ii) and (iv) in the last inequality. Hence, $|\Im(\mathrm{hol}_{\tilde{p}}(e_b))| > 0$ and $\Im(\tilde{p})$ is carried by the train track σ.

Taking $w \in \mathbf{i}H_1(M, \Sigma, \mathbb{R})$, the above discussion also implies that σ satisfies (3) and (4). □

7. Counting integral points in $\mathcal{ML}(S)$

Let the notation be as in §6. In particular, τ is a maximal train track. Also recall that $P(\tau)$ denotes the finite-sided polyhedron in $U(\tau)$ corresponding to laminations with $\|\lambda\|_\tau = 1$.

The smallest t so that a lamination $\lambda \in U(\tau)$ lies in

$$[0, e^t]P(\tau) = \{\lambda' \in U(\tau) : \|\lambda'\|_\tau \le e^t\}$$

can be thought of as a measure of complexity (or length) for the lamination λ. In this section we obtain an effective counting result with respect to this complexity. In §8 we will use the convexity of the hyperbolic length function in $U(\tau)$ to relate the counting problem in Theorem 1.1 to this counting problem.

Let $\mathcal{U} \subset P(\tau)$ be a cube. For every $t \ge 0$, define

$$\mathcal{O}_\tau(\gamma_0, e^t, \mathcal{U}) := \big\{\gamma \in \mathrm{Mod}_g . \gamma_0 \cap [0, e^t]\mathcal{U}\big\}. \tag{66}$$

The following strengthening of Theorem 1.2 is the main result of this section.

Theorem 7.1. *There exist κ_{19} and κ_{20} so that the following holds. Let $t \ge 1$, and let $\mathcal{U} \subset P(\tau)$ be a cube of size $\ge e^{-\kappa_{19}t}$. Then*

$$\#\mathcal{O}_\tau(\gamma_0, e^t, \mathcal{U}) = \mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}([0,1]\mathcal{U})e^{ht} + O_{\tau,\gamma_0}(e^{(h-\kappa_{20})t})$$

where $\mathrm{v}(\gamma_0)$ is defined as in (69) and $h = 6g - 6$.

The basic tool in the proof of Theorem 7.1 is Proposition 4.1. We relate the counting problem in Theorem 7.1 to a counting problem for translations of $W^{\mathrm{u}}(q_0)$ in Lemma 7.2. Proposition 4.1 studies a more local version of this latter counting problem. That is: one works with translations of a small region in $W^{\mathrm{u}}(q_0)$. Using Corollary 4.3, we will reduce to this local analysis. The main step in the proof of Theorem 7.1 is Lemma 7.6 below.

Let us begin with some preparation. Recall that $\mathcal{ML}(S)$ does not have a natural differentiable structure, in particular, $\tilde{\mathcal{P}}_1$ is only a homeomorphism. The situation however drastically improves so long as we restrict to one train track chart and fix a transversal lamination. Therefore, we fix a maximal lamination η which is carried by τ for the rest of the discussion.

Let $\delta > 0$, and let $\mathcal{U} \subset P(\tau)$ be a cube of size $\ge \delta$ centered at λ. Let $\epsilon \le \delta$. We always assume $\tilde{\mathcal{P}}_1^{-1}$ is a homeomorphism on $\{[\eta]\} \times \{e^r\mathcal{U} : |r| \le \delta\}$. Put $\tilde{W}^{\mathrm{cs}}_{\mathcal{U}} = \tilde{\mathcal{P}}_1^{-1}(\{[\eta]\} \times \mathcal{U})$ and

$$\tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon} = \tilde{\mathcal{P}}_1^{-1}\big(\{[\eta]\} \times \{e^r\mathcal{U} : -\epsilon \le r \le 0\}\big). \tag{67}$$

Let $\gamma_0 \in U(\tau)$ be a rational multicurve. For all $t \geq 0$ and $0 < \epsilon < 1$, define

$$\mathcal{O}_\tau(\gamma_0, t, \mathcal{U}, \epsilon) := \left\{\gamma \in U(\tau) \cap \mathrm{Mod}_g .\gamma_0 : e^{t-\epsilon} \leq \|\gamma\|_\tau \leq e^t \text{ and } \overline{\gamma}^\tau \in \mathcal{U}\right\}. \tag{68}$$

Put $\tilde{q}_0 := \tilde{\mathcal{P}}_1^{-1}([\eta], \overline{\gamma}_0^\tau)$. Without loss of generality we assume γ_0 and η are so that $\tilde{q}_0$ belongs to our fixed fundamental domain.

Lemma 7.2. *Let $\delta > 0$, and let $\mathcal{U} \subset P(\tau)$ be a cube of size $\geq \delta$. Let λ denote the center of $\mathcal{U}$. For all $\epsilon \leq \delta$ and all large enough $t \geq 0$ we have:*

$$\mathbf{g}\gamma_0 \in \mathcal{O}_\tau(\gamma_0, t, \mathcal{U}, \epsilon) \ \textit{ if and only if } \ \tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon} \cap \mathbf{g} \cdot a_t \tilde{W}^{\mathrm{u}}(\tilde{q}_0) \neq \emptyset.$$

Proof. Since τ is fixed throughout, we drop it from the subscript and superscript for the norm and the normalization.

Suppose $\gamma = \mathbf{g}\gamma_0 \in \mathcal{O}_\tau(\gamma_0, t, \mathcal{U}, \epsilon)$ for some $\mathbf{g} \in \mathrm{Mod}_g$; such $\mathbf{g}$ is not unique, however, for any other $\mathbf{g}' \in \mathrm{Mod}_g$ with $\mathbf{g}\gamma_0 = \mathbf{g}'\gamma_0$ we have $\mathbf{g} \cdot \tilde{W}^{\mathrm{u}}(\tilde{q}_0) = \mathbf{g}' \cdot \tilde{W}^{\mathrm{u}}(\tilde{q}_0)$. Put $\tilde{q} = \mathbf{g} \cdot \tilde{q}_0$. Then $\mathbf{g}\gamma = \mathfrak{I}(\tilde{q})$, moreover,

$$\mathbf{g} \cdot a_t \tilde{W}^{\mathrm{u}}(\tilde{q}_0) = a_t \tilde{W}^{\mathrm{u}}(\tilde{q}).$$

Recall that $\overline{\gamma} \in \mathcal{U}$ and put $\tilde{p}' := \tilde{\mathcal{P}}_1^{-1}([\eta], \overline{\gamma})$. Then, $\tilde{p}' \in \tilde{W}^{\mathrm{cs}}_{\mathcal{U}}$; moreover, it follows from the definition that $\mathfrak{I}(\tilde{p}') = \overline{\gamma}$. Hence, $\tilde{p}' \in a_{t_1} \tilde{W}^{\mathrm{u}}(\tilde{q})$ where $t_1 = \log \|\gamma\|$.

Put $s = t_1 - t$; since $\gamma \in \mathcal{O}_\tau(\gamma_0, t, \mathcal{U}, \epsilon)$ we have $-\epsilon \leq s \leq 0$. We get from the above and the definition of $\tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon}$ that $a_s \tilde{p}' \in a_t \tilde{W}^{\mathrm{u}}(\tilde{q}) \cap \tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon}$. In particular,

$$\tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon} \cap a_t \tilde{W}^{\mathrm{u}}(\tilde{q}) = \tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon} \cap \mathbf{g} \cdot a_t \tilde{W}^{\mathrm{u}}(\tilde{q}_0) \neq \emptyset.$$

Conversely, suppose that for some $\mathbf{g} \in \mathrm{Mod}_g$ we have $\tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon} \cap \mathbf{g} \cdot a_t \tilde{W}^{\mathrm{u}}(\tilde{q}_0) \neq \emptyset$. Put $\gamma = \mathbf{g}\gamma_0$; we claim that $\gamma \in \mathcal{O}_\tau(\gamma_0, t, \mathcal{U}, \epsilon)$.

Set $\tilde{q} = \mathbf{g} \cdot \tilde{q}_0$. Then $\mathfrak{I}(\tilde{q}) = \gamma$, and as above we have $\mathbf{g} \cdot a_t \tilde{W}^{\mathrm{u}}(\tilde{q}_0) = a_t \tilde{W}^{\mathrm{u}}(\tilde{q})$. Let now $\lambda \in \mathcal{U}$ and $-\epsilon \leq s \leq 0$ be so that

$$\tilde{\mathcal{P}}_1^{-1}([\eta], e^s \lambda) \in \tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon} \cap a_t \tilde{W}^{\mathrm{u}}(\tilde{q}).$$

Let us write $\tilde{\mathcal{P}}_1^{-1}([\eta], e^s \lambda) = a_t \tilde{q}'$ where $\tilde{q}' \in \tilde{W}^{\mathrm{u}}(\tilde{q})$. Then, we have

$$e^{-t}\gamma = \mathfrak{I}(a_t \tilde{q}') = e^s \lambda \in e^s \mathcal{U}.$$

This gives $\bar{\gamma} = \lambda$, hence, $\bar{\gamma} \in \mathcal{U}$ and $\|\gamma\| = e^{t+s}$; we get $\gamma \in \mathcal{O}_\tau(\gamma_0, t, \mathcal{U}, \epsilon)$ as we claimed. $\square$

7.1. **Strebel differentials.** Problems related to the existence and uniqueness of Jenkins-Strebel differentials have been extensively studied.

Theorem 7.3 (Cf. [Str], Thm. 20.3)**.** *Let $\gamma = \sqcup_{i=1}^d \gamma_i$ be a rational multi-geodesic on M, and let $r_1, \ldots, r_d$ be positive real numbers. Then there exists a unique holomorphic quadratic differential q on M (Jenkins-Strebel differential) with the following properties.*

(1) *If* Γ *is the critical graph*[1] *of* q*, then* $M - \Gamma = \cup_{i=1}^d \Omega_i$*, where* Ω_i *is either empty or a cylinder whose core curve is* γ_i*.*
(2) *If* Ω_i *is not empty, it is swept out by trajectories whose* q *length is* r_i*.*

The following lemma will be used in the sequel.

Lemma 7.4. *Let* $\gamma \in U(\tau)$ *be rational, and let* $\tilde{q} = \tilde{\mathcal{P}}_1^{-1}([\eta], \gamma) \in \mathcal{Q}^1\mathcal{T}(\alpha)$ *be a quadratic differential so that* $\Im(\tilde{q}) = \gamma$*; put* $q := \pi(\tilde{q})$*. Then*

(1) $W^{\mathrm{u}}(q) \subset \mathcal{Q}_1(1, \dots, 1)$ *is a properly immersed, affine submanifold which carries a natural finite Borel measure* ν*.*
(2) *There exists some* $\epsilon_0 = \epsilon_0(\tau, \eta, \|\gamma\|_\tau) > 0$ *so that the following holds. Let* $0 < \hat{\epsilon} < \epsilon_0$ *and let*
$$K(\hat{\epsilon}) = \{q : \textit{all saddle connections on } q \textit{ are} \geq \hat{\epsilon}\}.$$
Put $\mathsf{D}(\hat{\epsilon}) = \mathsf{D}_{\mathrm{cusp}}(\hat{\epsilon}) := W^{\mathrm{u}}(q) \cap K(\hat{\epsilon})^{\complement}$*. There are constants* κ_{21} *and* N_{13}*, and a smooth function* $0 \leq \psi^{\mathrm{u}}_{\hat{\epsilon}} \leq 1$ *supported on* $W^{\mathrm{u}}(q)$ *so that*
(a) $\mathcal{C}^1(\psi^{\mathrm{u}}_{\hat{\epsilon}}) \ll \hat{\epsilon}^{-N_{13}}$*,*
(b) $\|\psi^{\mathrm{u}}_{\hat{\epsilon}}\|_{2,\nu} \ll \hat{\epsilon}^{\kappa_{21}}$*,*
(c) $\psi^{\mathrm{u}}_{\hat{\epsilon}}|_{\mathsf{D}(\hat{\epsilon})} = 1$*, and* $\|1_{\mathsf{D}(\hat{\epsilon})} - \psi^{\mathrm{u}}_{\hat{\epsilon}}\|_{2,\nu} \ll \hat{\epsilon}^{\kappa_{21}}$*.*
In particular, we have $\nu(\mathsf{D}(\hat{\epsilon})) \leq \hat{\epsilon}^{\kappa_{21}}$ *for all small enough* $\hat{\epsilon}$*.*

Proof. We first show that $W^{\mathrm{u}}(q)$ is a properly immersed submanifold of $\mathcal{Q}_1(1, \dots, 1)$. This is equivalent to showing the following two statements.

(i) $\mathbf{g}_1 \cdot \tilde{W}^{\mathrm{u}}(\tilde{q}) \cap \mathbf{g}_2 \cdot \tilde{W}^{\mathrm{u}}(\tilde{q}) \neq \emptyset$ if and only if $\mathbf{g}_1 \cdot \tilde{W}^{\mathrm{u}}(\tilde{q}) = \mathbf{g}_2 \cdot \tilde{W}^{\mathrm{u}}(\tilde{q})$.
(ii) $\bigcup_{\mathbf{g} \in \mathrm{Mod}_g} \mathbf{g} \cdot \tilde{W}^{\mathrm{u}}(\tilde{q}) \subset \mathcal{Q}^1\mathcal{T}(\alpha)$ is closed.

Recall that $\tilde{W}^{\mathrm{u}}(\tilde{p}) = \{\tilde{p}' : \Im(\tilde{p}') = \Im(\tilde{p})\}$ and that $\mathbf{g} \cdot \tilde{W}^{\mathrm{u}}(\tilde{p}) = \tilde{W}^{\mathrm{u}}(\mathbf{g} \cdot \tilde{p})$ for all $\tilde{p} \in \mathcal{Q}^1\mathcal{T}(\alpha)$. These imply (i). To see (ii), note further that the set
$$\bigcup_{\mathbf{g} \in \mathrm{Mod}_g} \mathbf{g} \cdot \tilde{W}^{\mathrm{u}}(\tilde{q})$$
is the set of quadratic differentials $\tilde{p} \in \mathcal{Q}^1\mathcal{T}(1, \dots, 1)$ so that $\Im(\tilde{p}) \in \mathrm{Mod}_g . \gamma$. Since γ is rational, $\mathrm{Mod}_g . \gamma$ is a discrete Mod_g-invariant set; (ii) follows.

Let γ be as in the statement. Write $\gamma = \sum_i a_i \gamma_i$ where each γ_i is a simple closed curve and $a_i \in \mathbb{Q}$. By Theorem 7.3 we have: the locus $W^{\mathrm{u}}(q) \cap \mathcal{Q}_1(1, \dots, 1)$ is identified with a linear subspace $\mathcal{W} = \{(x_{ij}) : \sum_j x_{ij} = r_i, x_{ij} > 0\}$ in the period coordinates, where $r_1, \dots, r_d$ are positive real numbers. Moreover, the measure ν is the pull back of the Lebesgue measure from $\mathcal{W}$ to $W^{\mathrm{u}}(q)$. This finishes the proof of (1).

To see part (2), let ϵ_0 be so that $\pi \circ \tilde{\mathcal{P}}_1^{-1}([\eta], \gamma) \in K(\epsilon_0)$, recall from Lemma 6.1 that ϵ_0 depends only on τ, η, and $\|\gamma\|_\tau$. For any $0 < \hat{\epsilon} < \epsilon_0$ put
$$\mathcal{W}(\hat{\epsilon}) = \{(x_{ij}) \in \mathcal{W} : 0 < x_{ij} < \hat{\epsilon} \text{ for some } (i, j)\}.$$

[1]Recall that the critical graph of a quadratic differential is the union of the compact leaves of the measured foliation induced by q which contain a singularity of q.

Using Theorem 7.3, we have $W^{\mathrm{u}}(q) \cap K(\hat\epsilon)^{\complement} \subset \Phi^{-1}(\mathcal{W}(\hat\epsilon))$. The claims in part (2) now follow from Lemma 2.10. Indeed apply Lemma 2.10 with $D = \mathsf{D}(2\hat\epsilon) - \mathsf{D}(\hat\epsilon/2)$, and let $\{\varphi_i\}$ be the collection of functions obtained by that lemma. Define

$$\psi^{\mathrm{u}}_{\hat\epsilon}(p) = \begin{cases} \sum \varphi_i(p) & \text{if } p \in W^{\mathrm{u}}(q) - \mathsf{D}(\hat\epsilon/2) \\ 1 & \text{if } p \in \mathsf{D}(\hat\epsilon/2) \end{cases}.$$

This function satisfies the claims. □

Let γ_0 and $\tilde q_0 \in \mathcal{Q}^1\mathcal{T}(1,\ldots,1)$ be as in Lemma 7.2 and put $q_0 := \pi(\tilde q_0)$. Then by Lemma 7.4 we have $W^{\mathrm{u}}(q_0)$ is an affine submanifold of $\mathcal{Q}_1(1,\ldots,1)$. We will put

$$\mathrm{v}(\gamma_0) = \nu(W^{\mathrm{u}}(q_0)) \tag{69}$$

where ν is the finite measure in Lemma 7.4.

Let $b > 0$; this choice will be optimized later. Apply Lemma 7.4(2) with $\hat\epsilon = 10b$ and let $\mathsf{D}_{\mathrm{cusp}}(10b)$ be as in that lemma. Put $\mathsf{D}_b := W^{\mathrm{u}}(q) - \mathsf{D}_{\mathrm{cusp}}(10b)$.

Lemma 7.5. *For every b there exists some $N(b) \ll b^{-N_{14}}$ so that the following holds. There exists a collection of functions $\{\psi^{\mathrm{u}}_i : 0 \le i \le N(b)\}$ with the following properties:*

1. *$\psi^{\mathrm{u}}_0 = \psi^{\mathrm{u}}_{10b}$ where ψ^{u}_{10b} is given by Lemma 7.4(2).*
2. *$0 \le \psi^{\mathrm{u}}_i \le 1$ for all $i \ge 0$.*
3. *For all $i \ge 1$, ψ^{u}_i is supported in $\mathsf{B}^{\mathrm{u}}_b(y_i)$ where $y_i \in \mathsf{D}_b$; furthermore, the multiplicity of $\{\mathsf{B}^{\mathrm{u}}_b(y_i)\}$ is at most N_5.*
4. *$\sum_{i=1}^{N(b)} \psi^{\mathrm{u}}_i \le 1$ and $\sum_{i=1}^{N(b)} \psi^{\mathrm{u}}_i = 1$ on $\cup_{i=1}^{N(b)} \mathsf{B}_b(y_i)$.*

Moreover, we have

$$\mathcal{C}^1(\psi^{\mathrm{u}}_i) \le N_{16} b^{-N_{15}} \text{ for all } 0 \le i \le N(b) \tag{70}$$

where N_{15} is an absolute constant and N_{16} is allowed to depend on q_0.

Proof. This follows from Lemma 2.10 applied with $D = \mathsf{D}_b$ and Lemma 7.4. □

Let us also fix a fundamental domain $\tilde{\mathsf{D}} \subset \tilde W^{\mathrm{u}}(\tilde q_0)$ which projects to $W^{\mathrm{u}}(q_0)$. For each $i \ge 1$, we let $\tilde y_i \in \tilde{\mathsf{D}}$ be a lift of y_i, see Lemma 7.5. Let $N'(b)$ be so that

$$\mathsf{B}^{\mathrm{u}}_b(\tilde y_i) \subset \tilde{\mathsf{D}} \text{ for all } N'(b) < i \le N(b). \tag{71}$$

For simplicity in notation, let $\mathsf{B}^{\mathrm{u}}_b(\tilde y_0) \subset \tilde{\mathsf{D}}$ denote the lift of $\mathsf{D}_{\mathrm{cusp}}(10b)$. Increasing $N'(b)$, if necessary, we assume that $\mathsf{B}^{\mathrm{u}}_b(\tilde y_i) \cap \mathsf{B}^{\mathrm{u}}_b(\tilde y_0) = \emptyset$ for all $i \ge N'(b)$.

7.2. **Counting in linear sectors in $\mathcal{ML}(S)$.** Recall from the beginning of this section that $\mathcal{U} \subset P(\tau)$ is a box of size $\ge \delta$. Let λ be the center of $\mathcal{U}$, and let $\epsilon \le \delta$. Let $\eta \in \mathcal{ML}(S)$ be fixed as in the beginning of this section. We always assume $0 < \delta < 1/2$ and η are so that $\tilde{\mathcal{P}}_1^{-1}$ is a homeomorphism on $\{[\eta]\} \times \{e^r\mathcal{U} : |r| < \delta\}$. Recall also our notation $\tilde W^{\mathrm{cs}}_{\mathcal{U}} = \tilde{\mathcal{P}}_1^{-1}(\{[\eta]\} \times \mathcal{U})$ and

$$\tilde W^{\mathrm{cs}}_{\mathcal{U},\epsilon} = \tilde{\mathcal{P}}_1^{-1}\big(\{[\eta]\} \times \{e^r\mathcal{U} : -\epsilon < r \le 0\}\big).$$

Abusing the notation, we denote by $\mu_{\mathrm{Th}}(\mathcal{U})$ the measure induced from μ_{Th} on $P(\tau)$. The following lemma is a crucial step in the proof of Theorem 7.1.

Lemma 7.6. *There exist κ_{22} and κ_{23} so that the following holds. Let $t \geq 0$ and in the above notation, define*

$$\mathcal{N}(\tilde{q}_0, t, \mathcal{U}, \epsilon) := \big\{\mathbf{g} \cdot \tilde{W}^{\mathrm{u}}(\tilde{q}_0) : \mathbf{g} \in \mathrm{Mod}_g \text{ and } \tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon} \cap \mathbf{g} \cdot a_t W^{\mathrm{u}}(\tilde{q}_0) \neq \emptyset\big\}.$$

Suppose $\epsilon \geq e^{-\kappa_{22} t}$, then

$$\#\mathcal{N}(\tilde{q}_0, t, \mathcal{U}, \epsilon) = \mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht} + O_{\tau,\gamma_0}((1-e^{-h\epsilon})e^{(h-\kappa_{23})t}).$$

We will prove Lemma 7.6 using Proposition 4.1, more precisely Corollary 4.7. In order to use those results we need to control the *geometry* of $\tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon}$.

Lemma 7.7. *The characteristic function of*

$$\tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon} = \tilde{\mathcal{P}}_1^{-1}(\{[\eta]\} \times \{e^s\mathcal{U} : |s| \leq \epsilon\})$$

belongs to $\mathcal{S}_{\tilde{W}^{\mathrm{cs}}(\tilde{q}_j)}(\tilde{p}, \epsilon)$ where $\tilde{p} = \tilde{\mathcal{P}}_1^{-1}([\eta], \lambda)$.

Proof. Apply Lemma 6.1 with τ and let $K = K(\tau)$ be defined as in (61). Then

$$\pi \circ \tilde{\mathcal{P}}_1^{-1}([\eta], P(\tau)) \subset K.$$

Let $\{\mathsf{B}_{r_p}(p) : p \in K\}$ be the covering of K by period boxes given by Lemma 6.3. Let $\mathsf{B}_{\cdot}(q_1), \ldots, \mathsf{B}_{\cdot}(q_{\mathsf{b}'})$ be a finite subcover of this covering. Consider all lifts of $\mathsf{B}(q_j)$ to period boxes based at lifts $\tilde{q}_j$ of q_j in our fixed (weak) fundamental domain. Denote these lifts by $\mathsf{B}_{r_1}(\tilde{q}_1), \ldots, \mathsf{B}_{r_{\mathsf{b}}}(\tilde{q}_{\mathsf{b}})$ — note that we only fixed a weak fundamental domain, hence there might be more than one lift, however, there is a universal bound on the number of lifts.

For every $1 \leq j \leq \mathsf{b}$, let σ_j be a train track obtained by applying Lemma 6.3 to $\mathsf{B}_{r_j}(\tilde{q}_j)$. Assume ϵ is smaller than the radius of $\mathsf{B}_{r_j}(\tilde{q}_j)$ for all j. Write $\mathcal{U} = \cup \hat{\mathcal{U}}_i$ where

$$\hat{\mathcal{U}}_i = \mathcal{U} \cap U(\sigma_j).$$

By Lemma 5.1 each $\hat{\mathcal{U}}_i$ is a piecewise linear subset of $\mathcal{U}_i$. The claim now follows from Lemma 6.3(4) if we ignore those $\hat{\mathcal{U}}_i$'s which have size less than ϵ^N for some $N > 1$ depending only on the dimension. □

Proof of Lemma 7.6. Recall that λ is the center of $\mathcal{U}$; put $\tilde{p} = \tilde{\mathcal{P}}_1^{-1}([\eta], \lambda)$ and $p = \pi(\tilde{p})$. Let $\tilde{\phi}^{\mathrm{cs}}$ be the characteristic function of $\tilde{W}^{\mathrm{cs}}_{\mathcal{U},\epsilon} \subset \tilde{W}^{\mathrm{cs}}(\tilde{p})$. Define

$$\phi^{\mathrm{cs}} := \tilde{\phi}^{\mathrm{cs}} \circ \left(\pi^{-1}|_{\pi(\mathrm{supp}(\tilde{\phi}^{\mathrm{cs}}))}\right)$$

— the push-forward of $\tilde{\phi}^{\mathrm{cs}}$ to $W^{\mathrm{cs}}(p)$. Recall from Lemma 7.7 that $\phi^{\mathrm{cs}} \in \mathcal{S}_{W^{\mathrm{cs}}(p)}(p, \epsilon)$.

Recall from §2 that μ denotes the $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure on $\mathcal{Q}_1(1, \ldots, 1)$ which is in the Lebesgue measure class. The measures μ^{u}_x and μ^{s}_x are the conditional measures of μ along $W^{\mathrm{u}}(x)$ and $W^{\mathrm{s}}(x)$; μ^{cs}_x and μ^{cu}_x are defined accordingly.

Recall also that $\mu_{\rm Th}(\{e^s\mathcal{U} : -\epsilon < s \le 0\}) = \frac{1-e^{-h\epsilon}}{h}\mu_{\rm Th}(\mathcal{U})$. Therefore, we have

$$\mu_p^{\rm cs}(\phi^{\rm cs}) = \tfrac{1-e^{-h\epsilon}}{h}\mu_{\rm Th}(\mathcal{U}). \tag{72}$$

For simplicity in notation, let us write $\tilde W^{\rm cs} = \tilde W^{\rm cs}_{\mathcal{U},\epsilon}$ and put

$$\mathcal{N} = \mathcal{N}(\tilde q_0, t, \mathcal{U}, \epsilon).$$

Let $\mathbf{g} \in {\rm Mod}_g$ be so that $\tilde W^{\rm cs} \cap \mathbf{g}\cdot a_t \tilde W^{\rm u}(\tilde q_0) \neq \emptyset$. Recall that $\{\mathsf{B}^{\rm u}_i(\tilde y_i) : 0 \le i \le N(b)\}$ cover $\tilde{\mathsf{D}} \subset \tilde W^{\rm u}(\tilde q_0)$, see Lemma 7.5 and the paragraph following that lemma; there exists some $\mathbf{g'} \in {\rm Mod}_g$ so that $\mathbf{g'}\cdot \tilde W^{\rm u}(\tilde q_0) = \tilde W^{\rm u}(\tilde q_0)$ and some $0 \le i \le N(b)$ so that

$$\tilde W^{\rm cs} \cap \mathbf{g}\mathbf{g'}\cdot a_t \mathsf{B}^{\rm u}_b(\tilde y_i) \neq \emptyset. \tag{73}$$

Let $N'(b)$ be defined in (71). We claim that the following holds:

$$\#\{\mathbf{g}\cdot \tilde W^{\rm u}(\tilde q_0) :\ (73) \text{ holds for some } 0 \le i \le N'(b)\ \} \ll \epsilon^{-\star} b^{-\star} {\rm v}(\gamma_0) e^{(h-\kappa_5)t} + b^\star {\rm v}(\gamma_0) e^{ht} \tag{74}$$

where the implied constants depend on the genus.

Let us assume (74) and finish the proof. Let

$$\mathcal{N}' := \{\mathbf{g}\cdot \tilde W^{\rm u}(\tilde q_0) \in \mathcal{N} :\ (73) \text{ does } \textit{not} \text{ hold for any } 0 \le i \le N'(b)\}$$

i.e, the contribution to $\mathcal{N}$ coming from $N'(b) < i \le N(b)$. We claim that

$$\Big|\#\mathcal{N}' - \sum_i \sum_y \psi^{\rm u}_i(y)\Big| \ll \epsilon^{-\star} b^{-\star} {\rm v}(\gamma_0) e^{(h-\kappa_5)t} + b^\star {\rm v}(\gamma_0) e^{ht} \tag{75}$$

where the outer summation is over all $N'(b) < i \le N(b)$ and the inner summation is over all $y \in \mathsf{B}^{\rm u}_b(y_i)$ so that $a_t y \in \pi(\tilde W^{\rm cs})$.

To see the claim, first note that by the definition of $\mathcal{N}'$, if $\mathbf{g}\cdot \tilde W^{\rm u}(\tilde q_0) \in \mathcal{N}'$, then (73) holds with some $N'(b) < i \le N(b)$. Let now $\mathbf{g}_1, \mathbf{g}_2 \in {\rm Mod}_g$ and $N'(b) \le i_1, i_2 \le N(b)$ be so that

$$\tilde W^{\rm cs} \cap \mathbf{g}\mathbf{g}_j \cdot a_t \mathsf{B}^{\rm u}_b(\tilde y_{i_j}) \neq \emptyset.$$

Then $\mathbf{g}_j \tilde W^{\rm u}(q_0) = \tilde W^{\rm u}(q_0)$ for $j = 1, 2$, see the discussion preceding (73); hence by Corollary 4.3 we have

$$\tilde W^{\rm cs} \cap \mathbf{g}\mathbf{g}_1 \cdot a_t \mathsf{B}^{\rm u}_b(\tilde y_{i_1}) = \tilde W^{\rm cs} \cap \mathbf{g}\mathbf{g}_2 \cdot a_t \mathsf{B}^{\rm u}_b(\tilde y_{i_2}).$$

In particular, $\mathbf{g}_1 \mathsf{B}^{\rm u}_b(\tilde y_{i_1}) \cap \mathbf{g}_2 \mathsf{B}^{\rm u}_b(\tilde y_{i_2}) \neq \emptyset$. Since $\mathsf{B}^{\rm u}_b(\tilde y_{i_j}) \subset \tilde{\mathsf{D}}$ for $j = 1, 2$ — recall that $N'(b) \le i_1, i_2 \le N(b)$ — we get that $\mathbf{g}_1 = \mathbf{g}_2$. Therefore,

$$\tilde W^{\rm cs} \cap \mathbf{g}\mathbf{g}_1 \cdot a_t \mathsf{B}^{\rm u}_b(\tilde y_{i_1})$$

corresponds to points lying in the intersection $\mathsf{B}^{\rm u}_b(\tilde y_{i_1}) \cap \mathsf{B}^{\rm u}_b(\tilde y_{i_2})$ but not in $\cup_{i=0}^{N'(b)} \mathsf{B}^{\rm u}_b(\tilde y_i)$. Recall from Lemma 7.5 that $\sum_i \psi^{\rm u} = 1$ on $\cup_{i=1}^{N(b)} \mathsf{B}_b(y_i)$, hence $\sum_{N'(b)<i\le N(b)} \psi^{\rm u}_i = 1$ on $\mathsf{D}_b - \cup_{i=1}^{N'(b)} \mathsf{B}^{\rm u}_b(y_i)$. In particular, since $\psi^{\rm u}_i \ge 0$, we get that

$$\#\mathcal{N}' \le \sum_i \sum_y \psi^{\rm u}_i(y)$$

where the outer summation is over all $N'(b) < i \le N(b)$ and the inner summation is over all $y \in \mathsf{B}^{\rm u}_b(y_i)$ so that $a_t y \in \pi(\tilde W^{\rm cs})$. Moreover, in view of the fact that $\mathsf{B}^{\rm u}_b(\tilde y_i) \cap \mathsf{B}^{\rm u}_b(\tilde y_0) = \emptyset$ for all $i \ge N'(b)$ and using Lemma 7.5(2) and (4), we have

$$\sum_i \sum_y \psi_i^{\rm u}(y) - \#\mathcal{N}' \ll \#\{\mathbf{g}\cdot \tilde W^{\rm u}(\tilde q_0) :\ (73) \text{ holds for some } 1 \le i \le N'(b)\ \}$$

where the implied constant depends on α. The claim in (75) thus follows in view of the estimate in (74).

Let us now investigate $\sum_i \sum_y \psi_i^{\rm u}(y)$. Using the definition of $\mathcal{N}_{\rm nc}$ in (34), we have

$$\begin{aligned}\mathcal{N}_{\rm nc}\big(t, \psi_i^{\rm u}, \phi^{\rm cs}\big) &= \sum \psi_i^{\rm u}(y)\phi^{\rm cs}(a_t y)\\ &= \sum \psi_i^{\rm u}(y) \qquad\qquad \phi^{\rm cs}(a_t y) = 0,1\end{aligned}$$

where the summations are over all $y \in \mathsf{B}^{\rm u}_b(y_i)$ so that $a_t y \in \pi(\tilde W^{\rm cs}) = \operatorname{supp}(\phi^{\rm cs})$. Now apply Corollary 4.7, see in particular (53), with $\psi_i^{\rm u}$ and $\phi^{\rm cs}$, and get that

$$\Big|\sum \psi_i^{\rm u}(y) - \mu^{\rm u}_{q_0}(\psi_i^{\rm u})\mu^{\rm cs}_p(\phi^{\rm cs})e^{ht}\Big| \le \mathcal{C}^1(\psi_i^{\rm u})e^{(h-\kappa_{18})t}. \tag{76}$$

In view of (72) and the estimate $\mathcal{C}^1(\psi_i^{\rm u}) \le N_{16} b^{-N_{15}}$, see (70), we get the following from (76).

$$\Big|\sum_y \psi_i^{\rm u}(y) - \mu^{\rm u}_{q_0}(\psi_i^{\rm u})\mu_{\rm Th}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht}\Big| \ll N_{16}\epsilon^{-\star} b^{-\star} e^{(h-\kappa_{18})t}. \tag{77}$$

Summing up (77) over all $N'(b) \le i \le N(b)$ and using the fact that $N(b) \ll b^{-\star}$, we get that

$$|\textstyle\sum_i \sum_y \psi_i^{\rm u}(y) - \sum_i \mu^{\rm u}_{q_0}(\psi_i^{\rm u})\mu_{\rm Th}(\mathcal{U})(\frac{1-e^{-h\epsilon}}{h})e^{ht}| \ll N_{16}\epsilon^{-\star} b^{-\star} e^{(h-\kappa_{18})t}. \tag{78}$$

We now compare $\sum_i \mu^{\rm u}_{q_0}(\psi_i^{\rm u})$ and $\mathrm{v}(\gamma_0)$. Indeed, using Lemma 7.4, see also (69), and the relationship between ν and $\mu^{\rm u}_{q_0}$ we get the following:

$$(1 - b^{\kappa_{21}})\mathrm{v}(\gamma_0) \le \mathrm{v}(\gamma_0) - \nu(\mathsf{D}'_b) \le \sum_{i=N'(b)}^{N(b)} \mu^{\rm u}_{q_0}(\psi_i^{\rm u}) \le \mathrm{v}(\gamma_0) \tag{79}$$

where $\mathsf{D}'_b = \mathsf{D}_{\rm cusp}(10b) \cup (\cup_{i=1}^{N'(b)} \mathsf{B}_b(y_i))$. The estimate in (79) implies that

$$\begin{aligned}\big|\textstyle\sum_i \sum_y \psi_i^{\rm u}(y) - \mathrm{v}(\gamma_0)\mu_{\rm Th}(\mathcal{U})(\frac{1-e^{-h\epsilon}}{h})e^{ht}\big| &\le b^{\kappa_{21}}\mathrm{v}(\gamma_0)\mu_{\rm Th}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht}\ +\\ &\big|\textstyle\sum_i \sum_y \psi_i^{\rm u}(y) - \sum_i \mu^{\rm u}_{q_0}(\psi_i^{\rm u})\mu_{\rm Th}(\mathcal{U})(\frac{1-e^{-h\epsilon}}{h})e^{ht}\big|.\end{aligned} \tag{80}$$

We now use these estimates to get an estimate for $\#\mathcal{N}'$. First note that

$$\begin{aligned}\big|\#\mathcal{N}' - \mathrm{v}(\gamma_0)\mu_{\rm Th}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht}\big| &\le \big|\#\mathcal{N}' - \textstyle\sum_i \sum_y \psi_i^{\rm u}(y)\big| \quad +\\ &\big|\textstyle\sum_i \sum_y \psi_i^{\rm u}(y) - \mathrm{v}(\gamma_0)\mu_{\rm Th}(\mathcal{U})(\frac{1-e^{-h\epsilon}}{h})e^{ht}\big|\\ &\overset{(75)\rightsquigarrow}{\ll} \epsilon^{-\star} b^{-\star}\mathrm{v}(\gamma_0)e^{(h-\kappa_5)t} + b^{\star}\mathrm{v}(\gamma_0)e^{ht} +\\ &\big|\textstyle\sum_i \sum_y \psi_i^{\rm u}(y) - \mathrm{v}(\gamma_0)\mu_{\rm Th}(\mathcal{U})(\frac{1-e^{-h\epsilon}}{h})e^{ht}\big|\end{aligned}$$

where the implied constant depends only on the genus. This estimate and (80) imply that

$$\begin{aligned}\big|\#\mathcal{N}' - \mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht}\big| \ll \epsilon^{-\star}b^{-\star}\mathrm{v}(\gamma_0)e^{(h-\kappa_5)t} \quad + \quad b^{\star}\mathrm{v}(\gamma_0)e^{ht} \qquad + \\ b^{\kappa_{21}}\mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht} + \big|\textstyle\sum_i\sum_y \psi_i^{\mathrm{u}}(y) - \sum_i \mu_{q_0}^{\mathrm{u}}(\psi_i^{\mathrm{u}})\mu_{\mathrm{Th}}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht}\big|.\end{aligned}$$

Putting this estimate and (78) together we get that

$$\begin{aligned}\big|\#\mathcal{N}' - \mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht}\big| \ll \epsilon^{-\star}b^{-\star}\mathrm{v}(\gamma_0)e^{(h-\kappa_5)t} + b^{\star}\mathrm{v}(\gamma_0)e^{ht} \\ b^{\star}\mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht} + N_{16}\epsilon^{-\star}b^{-\star}e^{(h-\kappa_{18})t}.\end{aligned} \tag{81}$$

We now choose ϵ and b of size $e^{-\star t}$ so that $\epsilon^{-\star}b^{-\star}e^{(h-\kappa_5)t}$ in (74) is $< e^{(h-\star)t}$ and so that $N_{16}\epsilon^{-\star}b^{-\star}e^{-\kappa_{18}t}$ on the right side of (81) is $< (1-e^{-h\epsilon})e^{-\star t}$. The lemma follows from this in view of (74).

Let us now turn to the proof of (74). The argument is similar to the one that was used in the proof of (45). For $1 \le i \le N'(b)$, let $\hat\psi_i^{\mathrm{u}}$ be so that $\mathrm{supp}(\hat\psi_i^{\mathrm{u}}) \subset \mathsf{B}_{2b}(y_i)$, $\hat\psi_i^{\mathrm{u}}|_{\mathsf{B}_b(y_i)} = 1$, and $\mathcal{C}^1(\hat\psi_i^{\mathrm{u}}) \ll b^{-\star}$, see Lemma 2.10. Let $\hat\psi_0^{\mathrm{u}} = \psi_0^{\mathrm{u}}$.

Let $\varrho > 0$ be small enough so that 10ϱ-neighborhood of $\mathrm{supp}(\phi^{\mathrm{cs}})$ embeds in $\mathcal{Q}(1,\ldots,1)$, and let $\kappa > 0$ be a constant which will be chosen later. In view of Lemma 2.11, we have

$$1_{\mathsf{B}_\varrho^{\mathrm{u}}(p)} \in \mathcal{S}(\mathsf{B}_\varrho^{\mathrm{u}}(p), \varrho/10).$$

Therefore, properties ($\mathcal{S}$-1), ($\mathcal{S}$-2), and ($\mathcal{S}$-2) hold with $\epsilon = 0.1\varrho e^{-\kappa t}$ and $f = 1_{\mathsf{B}_\varrho^{\mathrm{u}}(p)}$. Let $\phi_1^{\mathrm{u}} = \varphi_{+,0.1\varrho e^{-\kappa t}}$ for these choices.

Similarly, using Lemma 2.11 (this time, it is applied to the function ϕ^{cs} with $\epsilon = 0.1\varrho e^{-\kappa t}$) we let $\phi_1^{\mathrm{cs}} = \varphi_{+,0.1\varrho e^{-\kappa t}}$.

Put $\phi_1 := \phi_1^{\mathrm{u}}\phi_1^{\mathrm{cs}}$. Note that $1_{\mathsf{B}_\varrho^{\mathrm{u}}(p)}\phi^{\mathrm{cs}} \le \phi_1 \le 1_{\mathsf{B}_{2\varrho}^{\mathrm{u}}(p)}\phi^{\mathrm{cs}}$. Therefore,

$$\mu_p^{\mathrm{u}}\big(\mathsf{B}_\varrho^{\mathrm{u}}(p)\big)\mu_p^{\mathrm{cs}}(\phi^{\mathrm{cs}}) \le \mu(\phi_1) \le \mu_p^{\mathrm{u}}\big(\mathsf{B}_{2\varrho}^{\mathrm{u}}(p)\big)\mu_p^{\mathrm{cs}}(\phi^{\mathrm{cs}}). \tag{82}$$

Moreover, $\mu_p^{\mathrm{u}}(\phi_1) \ge \mu_p^{\mathrm{u}}(\mathsf{B}_\varrho^{\mathrm{u}}(p))$.

Since $\hat\psi_i^{\mathrm{u}}|_{\mathsf{B}_b(y_i)} = 1$ and $\mu_p^{\mathrm{u}}(\phi_1) \ge \mu_p^{\mathrm{u}}(\mathsf{B}_\varrho^{\mathrm{u}}(p))$, we have

$$\begin{aligned}\#\{\mathbf{g}\cdot W^{\mathrm{u}}(\tilde q_0) :\ (73) \text{ holds with } 0 \le i \le N'(b)\} \ll \\ \tfrac{e^{ht}}{\mu_p^{\mathrm{u}}(\mathsf{B}_\varrho^{\mathrm{u}}(p))}\sum_i \int_{W^{\mathrm{u}}(q_0)} \phi_1(a_t y)\hat\psi_i^{\mathrm{u}}(y)\,\mathrm{d}\mu_{q_0}^{\mathrm{u}}(y).\end{aligned}$$

Moreover, by Proposition 3.2 we have

$$\int_{W^{\mathrm{u}}(q_0)} \phi_1(a_t y)\hat\psi_i^{\mathrm{u}}(y)\,\mathrm{d}\mu_{q_0}^{\mathrm{u}}(y) = \mu(\phi_1)\mu_{q_0}^{\mathrm{u}}(\psi_i^{\mathrm{u}}) + O(\mathcal{C}^1(\psi_i^{\mathrm{u}})\mathcal{C}^1(\phi_1)e^{(h-\kappa_5)t})$$

for all $0 \le i \le N'(b)$.

Combining these two estimates and using the fact that in view of the estimates in (82) we have $\mu(\phi_1)/\mu^{\rm u}_p(\mathsf{B}_\varrho(p)) \ll 1$ we conclude that

$$\#\{\mathbf{g}\cdot W^{\rm u}(\tilde q_0):\ (73)\text{ holds with } 0\le i\le N'(b)\} \ll e^{ht}\sum \mu^{\rm u}_{q_0}(\psi^{\rm u}_i)+ O(\mathcal{C}^1(\psi^{\rm u}_i)\mathcal{C}^1(\phi_1)e^{(h-\kappa_5)t})N'(b). \tag{83}$$

In view of (70) we have $\mathcal{C}^1(\psi^{\rm u}_i)\ll b^{-\star}{\rm v}(\gamma_0)$; moreover, $\mathcal{C}^1(\phi)\ll\epsilon^{-\star}$ and $N'(b)\ll N(b)\ll b^{-\star}$. Recall also from (79) that $\sum_{i=0}^{N'(b)}\mu^{\rm u}_{q_0}(\psi^{\rm u}_i)\ll b^{\kappa_{21}}{\rm v}(\gamma_0)$.

If we now choose κ small enough, (74) follows from (83) and the proof of complete. □

Corollary 7.8. *There exist some κ_{24} and κ_{25} so that the following holds. Let $t\ge 0$ and let $\epsilon\ge e^{-\kappa_{24}t}$. Then*

$$\#\mathcal{O}_\tau(\gamma_0,t,\mathcal{U},\epsilon)={\rm v}(\gamma_0)\mu_{\rm Th}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht}+O_{\gamma_0}((1-e^{-h\epsilon})e^{(h-\kappa_{25})t}) \tag{84}$$

where as in (68) *we have*

$$\mathcal{O}_\tau(\gamma_0,t,\mathcal{U},\epsilon)=\{\gamma\in{\rm Mod}_g.\gamma_0\cap\big([0,e^t]\mathcal{U}-[0,e^{t-\epsilon}]\mathcal{U}\big)\}.$$

Proof. We will show this holds with $\kappa_{24}=\kappa_{22}/2$. By Lemma 7.2 we have $\gamma\in\mathcal{O}_\tau(\gamma_0,e^t,\mathcal{U},\epsilon)$ if and only if

$$\mathbf{g}\cdot a_t\tilde W^{\rm u}(\tilde q_0)\cap\tilde W^{\rm cs}_{\mathcal{U},\epsilon}\ne\emptyset.$$

Therefore, it suffices to show that

$$\#\mathcal{N}(\tilde q_0,t,\mathcal{U},\epsilon)={\rm v}(\gamma_0)\mu_{\rm Th}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht}+O_{\gamma_0}((1-e^{-h\epsilon})e^{(h-\star)t}).$$

This last statement is proved in Lemma 7.6. □

Proof of Theorem 7.1. Let $\epsilon\ge e^{-\kappa_{24}t}$, and for every $n\ge0$ define $t_n:=t-n\epsilon$. Then (84) applied with $t=t_n$ implies that

$$\begin{aligned}\#\mathcal{O}_\tau(\gamma_0,t_n,\mathcal{U},\epsilon)&={\rm v}(\gamma_0)\mu_{\rm Th}(\mathcal{U})(\tfrac{1-e^{-h\epsilon}}{h})e^{ht_n}+O_{\gamma_0}((1-e^{-h\epsilon})e^{(h-\kappa_{25})t_n})\\&={\rm v}(\gamma_0)\mu_{\rm Th}(\mathcal{U})(\tfrac{e^{-nh\epsilon}-e^{(-n-1)h\epsilon}}{h})e^{ht}+O_{\gamma_0}((1-e^{-h\epsilon})(e^{(h-\kappa_{25})t-(h-\kappa_{25}n\epsilon}).\end{aligned}$$

Summing these up over all $n\ge0$ so that $t_n\ge\frac{h-1}{h}t$ we get that

$$\#\{\gamma\in{\rm Mod}_g.\gamma_0\cap([0,e^t]\mathcal{U}-[0,e^{\frac{h-1}{h}t}]\mathcal{U})\}={\rm v}(\gamma_0)\mu_{\rm Th}(\mathcal{U})(\tfrac{1-e^{\frac{h-1}{h}t}}{h})e^{ht}+O_{\gamma_0}(e^{(h-\star)t}).$$

This implies the proposition — note that by basic lattice point count in Euclidean spaces[2], we have the number of integral points $\gamma\in U(\tau)$ so that $\|\gamma\|\le e^{\frac{h-1}{h}t}$ is $\ll e^{(h-1)t}$. □

[2]As we remarked in the introduction, the point here is that we are counting the number of point in one ${\rm Mod}_g$-orbit.

8. Proof of Theorem 1.1

We are now in the position to prove Theorem 1.1. The proof relies on Theorem 7.1. We cover $\mathcal{ML}(S)$ with finitely many train track charts $U(\tau_1), \ldots, U(\tau_{\mathsf{c}})$. Using the convexity of the hyperbolic length function, we can reduce the counting problem in Theorem 1.1 to an orbital counting in sectors on $U(\tau_i)$, with respect to linear structure, where the length function ℓ_X is well approximated by the $\| \|_{\tau_i}$. Theorem 7.1 is then brought to bear in the study of the latter counting problem.

Let X be a compact surface equipped with a Riemannian metric of negative curvature. Recall that $\ell_X : \mathcal{ML}(S) \to \mathcal{ML}(S)$ denotes the length function. It satisfies $\ell_X(t\lambda) = t\ell_X(\lambda)$ for any $t > 0$.

Let τ be a maximal train track. By Corollary 5.2, ℓ_X is Lipschitz in $U(\tau)$. Let L_τ be the Lipschitz constant, hence

$$|\ell_X(\lambda) - \ell_X(\lambda')| \le L_\tau \|\lambda - \lambda'\|_\tau. \tag{85}$$

Recall that $U(\tau)$ is a cone on the polyhedron $P(\tau)$.

Lemma 8.1. *There exists a constant $\hat{L}_\tau$, depending on L_τ, with the following property. For every $\lambda, \lambda' \in P(\tau)$ we have $|\frac{1}{\ell_X(\lambda)} - \frac{1}{\ell_X(\lambda')}| \le \hat{L}_\tau \delta$.*

Proof. First note that there exists some $\ell_{X,\tau} > 1$ so that $1/\ell_{X,\tau} \le \ell_X(\lambda) \le \ell_{X,\tau}$ for all $\lambda \in P(\tau)$. The claim thus follows from (85). □

For any $T > 0$, let $C_X(\tau, T) = \{\lambda \in U(\tau) : \ell_X(\lambda) \le T\}$. To simplify the notation, we will write $C_X(\tau)$ for $C_X(\tau, 1)$. Let $S_X(\tau) = \{\lambda \in U(\tau) : \ell_X(\lambda) = 1\}$. Then

$$C_X(\tau, T) = TC_X(\tau) = [0, T]S_X(\tau).$$

Proof of Theorem 1.1. Let X be as above. Let $\tau_1, \ldots, \tau_{\mathsf{c}}$ be finitely many maximal train tracks with the following properties.

- $\mathcal{ML}(S) = \cup_{i=1}^{\mathsf{c}} U(\tau_i)$, and
- $\ell_X : U(\tau_i) \to \mathbb{R}$ is L_i-Lipschitz for all $1 \le i \le \mathsf{c}$.

Let $L = \max L_i$; increasing L if necessary we will also assume that the conclusion of Lemma 8.1 holds with L.

Let us fix some $1 \le i \le \mathsf{c}$ and write $\tau = \tau_i$; when there is no confusion we drop τ from the notation for the norm and normalization in $U(\tau)$. We will first consider the contribution coming from $U(\tau)$ and then will combine contributions of different τ_i for $1 \le i \le \mathsf{c}$.

In the following we will use the following upper bound estimate for the number of integral point in a Euclidean region: the number of lattice points in a Euclidean region is $\ll$ the volume of the 1-neighborhood of the region.

Let γ_0 be a rational (multi) geodesic. For every $T > 0$ define

$$\mathcal{N}_\tau(\gamma_0, T) = \#\{\mathbf{g}\gamma_0 \in U(\tau) : \ell_X(\mathbf{g}\gamma_0) \le T\}. \tag{86}$$

Fix some $\delta > 0$; this will be optimized later and will be chosen to be of size $T^{-\star}$. Define

$$P_{\geq\delta}(\tau) := \{(b_i) \in P(\tau) : b_i \geq 2\delta \text{ for all } i\}. \tag{87}$$

Cover $P(\tau)$ with cubes of size δ with disjoint interior. Let $\{U_j : j \in J_\delta\}$ be the subcollection of these cubes so that $U_j \cap P_{\geq\delta}(\tau) \neq \emptyset$

For every j, let $\lambda_j \in U_j$ be the center of U_j. The number of U_j's required to cover $P(\tau)$ is $\ll \delta^{-N_{17}}$ for some N_{17} depending on τ.

There is some κ_{26}, depending only on the dimension, with the following property. If $\delta \geq T^{-\kappa_{26}}$, then the number of integral points $\gamma \in U(\tau)$ with $\|\gamma\| \leq \ell_{X,\tau} T$ and

$$\bar\gamma = \gamma/\|\gamma\| \in P(\tau) - P_{\geq\delta}(\tau) \tag{88}$$

is $\ll \delta T^h$.

For each j, let $U_{j,-}$ denote the cube which has the same center λ_j as U_j, but has size $\delta - \delta^{N_{18}}$ where $N_{18} = N_{17} + 1$.

Then, if $\delta^{N_{18}} \geq T^{-\kappa_{26}}$, the number of integral points $\gamma \in U(\tau)$ with $\|\gamma\| \leq \ell_{X,\tau} T$ and

$$\bar\gamma \in \bigcup_j U_j - U_{j,-} \tag{89}$$

is $\ll \delta^{-N_{17}}\delta^{N_{18}} T^h \ll \delta T^h$.

Altogether, we have: if $\delta^{N_{18}} \geq T^{-\kappa_{26}}$, then

$$\#\{\gamma \in \mathrm{Mod}_g .\gamma_0 \cap U(\tau) : \ell_X(\gamma) \leq T, \bar\gamma \text{ satisfies (88) or (89)}\} \ll \delta T^h. \tag{90}$$

We now find an estimate for

$$\#\{\gamma \in \mathrm{Mod}_g .\gamma_0 \cap C_X(\tau, T) : \bar\gamma \in \cup U_{j,-}\}.$$

Put $U_{j,-,+} = \{\frac{\lambda}{\ell_X(\lambda_j) - L\delta} : \lambda \in U_{j,-}\}$ and $U_{j,-,-} = \{\frac{\lambda}{\ell_X(\lambda_j) + L\delta} : \lambda \in U_{j,-}\}$. Then it follows from (85) that

$$[0,1]U_{j,-,-} \subset \{\lambda \in C_X(1,\tau) : \bar\lambda \in U_{j,-}\} \subset [0,1]U_{j,-,+}$$

Therefore, applying Theorem 7.1, with $U = U_{j,-,\pm}$, we get that

$$\frac{\mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}(U_{j,-})}{h(\ell_X(\lambda_j)+L\delta)^h} T^h + O_{\tau,\gamma_0}(T^{h-\kappa_{20}}) \leq$$
$$\#\{\gamma \in \mathrm{Mod}_g .\gamma_0 : \gamma \in C_X(\tau, T), \bar\gamma \in U_{j,-}\} \leq \frac{\mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}(U_{j,-})}{h(\ell_X(\lambda_j)-L\delta)^h} T^h + O_{\tau,\gamma_0}(T^{h-\kappa_{20}});$$

this estimate implies that

$$\#\{\gamma \in \mathrm{Mod}_g .\gamma_0 : \gamma \in C_X(\tau, T), \bar\gamma \in U_{j,-}\} = \frac{\mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}(U_{j,-})}{h(\ell_X(\lambda_j))^h} T^h + O_{\tau,\gamma_0}(\delta\mu_{\mathrm{Th}}(U_{j,-})T^h + T^{h-\kappa_{20}}). \tag{91}$$

Let us put $S_X(\tau, j) = \{\lambda \in S_X(\tau) : \bar\lambda \in U_{j,-}\}$. Then by Lemma 8.1 we have

$$\mu_{\mathrm{Th}}([0,1]S_X(\tau, j)) = \int_{U_{j,-}} \frac{1}{h\ell_X(\lambda)^h}\, \mathrm{d}\mu_{\mathrm{Th}} = \frac{\mu_{\mathrm{Th}}(U_{j,-})}{h(\ell_X(\lambda_j))^h} + O(\delta)\mu_{\mathrm{Th}}(U_{j,-}).$$

This observation together with (91) gives that

$$\#\{\gamma \in \mathrm{Mod}_g .\gamma_0 : \gamma \in C_X(\tau,T), \bar\gamma \in U_{j,-}\} = \mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}([0,1]S_X(\tau,j))T^h + O_{\tau,\gamma_0}(\delta\mu_{\mathrm{Th}}(U_{j,-})T^h + T^{h-\kappa_{20}}). \tag{92}$$

Recall also that $\ell_X^{\pm 1}$ is bounded on $P(\tau)$; we have $\sum \mu_{\mathrm{Th}}([0,1]S_X(\tau,j)) = \mu_{\mathrm{Th}}([0,1]S_X(\tau))+ O(\delta^\star)$. Hence, summing (92) over all j's we get

$$\#\{\gamma \in \mathrm{Mod}_g .\gamma_0 : \gamma \in C_X(\tau,T), \bar\gamma \in \cup U_{j,-}\} = \mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}([0,1]S_X(\tau))T^h + O_{\tau,\gamma_0}(\delta^\star T^h + \delta^{-N_{17}}T^{h-\kappa_{20}}). \tag{93}$$

Now choose $\delta = T^\star$ so that $\delta^\star T^h + \delta^{-N_{17}}T^{h-\kappa_{20}} = T^{h-\kappa_{27}}$. Then we get from (93) and (90) that

$$\#\{\gamma \in \mathrm{Mod}_g .\gamma_0 : \gamma \in C_X(\tau,T)\} = \mathrm{v}(\gamma_0)\mu_{\mathrm{Th}}([0,1]S_X(\tau))T^h + O(T^{h-\kappa_{27}}). \tag{94}$$

This conclude the contribution arising from a single train track chart $U(\tau)$.

Recall now that the regions in $U(\tau_i)$ which are carried by other $U(\tau_{i'})$ are finite sided polyhedra, see Lemma 5.1. We may thus find disjoint finite sided polyhedra $\mathcal{U}_i \subset P(\tau_i)$ to the $\cup\mathbb{R}^+.\mathcal{U}_i = \mathcal{ML}(S)$. Repeating the above argument for each $\mathcal{U}_i$, the theorem follows from the estimate in (94). □

We conclude with the following which are of independent interest. Let $\Gamma \subset \mathrm{Mod}_g$ be a finite index subgroup and let τ be a maximal train track. Define

$$\mathcal{N}_{\Gamma,\tau}(\gamma_0, T) := \{\gamma \in \Gamma.\gamma_0 \cap U(\tau) : \|\gamma\|_\tau \le T\}.$$

Theorem 8.2. *There exists some $\kappa_{28} = \kappa_{28}(\Gamma)$ so that the following holds. For every rational multi curve $\gamma_0 \in U(\tau)$, there exists some constant $c_{\Gamma,\tau}(\gamma_0)$ so that*

$$\#\mathcal{N}_{\Gamma,\tau}(\gamma_0,T) = c_{\Gamma,\tau}(\gamma_0)T^{6g-6} + O_{\gamma_0,\tau,\Gamma}(T^{6g-6-\kappa_{28}})$$

Proof. The argument is similar to our argument in the proof of Theorem 1.2. Recall that we normalized the Masur-Veech measure to be a probability measure on $\mathcal{Q}_1(1,\ldots,1)$. Let μ_Γ denote the lift of the Masur-Veech measure to $\mathcal{Q}^1\mathcal{T}(1,\ldots,1)/\Gamma$, then $\mu_\Gamma(\mathcal{Q}^1\mathcal{T}(1,\ldots,1)/\Gamma) = [\mathrm{Mod}_g : \Gamma]$.

Similar to (69), define $\mathrm{v}_\Gamma(\gamma_0)$ to be the measure of the lift of $W^{\mathrm{u}}(q_0)$ to $\mathcal{Q}^1\mathcal{T}(1,\ldots,1)/\Gamma$ where $\Im(q_0) = \gamma_0$.

Now, by virtue of Theorem 7.1, we have

$$\#\{\gamma \in \Gamma.\gamma_0 \cap U(\tau) : \|\gamma\|_\tau \le T\} = \mathrm{v}'_\Gamma(\gamma_0)\mu_{\mathrm{Th}}([0,1]U(\tau))T^h + O_{\gamma_0,\tau,\Gamma}(T^{h-\kappa_{28}})$$

where $\mathrm{v}'_\Gamma(\gamma_0) = \mathrm{v}_\Gamma(\gamma_0)/[\mathrm{Mod}_g : \Gamma]$ and $\mathrm{v}_\Gamma(\gamma_0)$ is as above.

The exponent κ_{28} depends on the exponential mixing rate for the Teichmüller geodesic flow on $(\mathcal{Q}^1\mathcal{T}(1,\ldots,1)/\Gamma, \mu_\Gamma)$. □

Let $\Gamma \subset \mathrm{Mod}_g$ be a finite index subgroup. Given a rational multi-geodesics γ_0 on X define

$$s_{X,\Gamma}(\gamma_0,T) := \#\{\gamma \in \Gamma.\gamma_0 : \ell_X(\gamma) \le T\}$$

We also have the following generalization of Theorem 1.1.

Theorem 8.3. *There exists some $\kappa_{29} = \kappa_{29}(\Gamma) > 0$, dependence on Γ is related to the exponential mixing rate for the Teichmüller geodesic flow on $\mathcal{Q}^1\mathcal{T}(1, \dots, 1)/\Gamma$, and some $c = c(\gamma_0, X, \Gamma)$ so that the following holds.*

$$s_{X,\Gamma}(\gamma_0, T) = c\,T^{6g-6} + O_{\gamma_0,X,\Gamma}(T^{6g-6-\kappa_{29}})$$

Proof. Similar to the discussion in the proof of Theorem 8.2, the proof of Theorem 1.1 applies mutatis mutandis to $s_{X,\Gamma}(\gamma_0, T)$. □

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