ISOLATION, EQUIDISTRIBUTION, AND ORBIT CLOSURES FOR THE $SL(2,\mathbb{R})$ ACTION ON MODULI SPACE.

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Abstract. We prove results about orbit closures and equidistribution for the $SL(2,\mathbb{R})$ action on the moduli space of compact Riemann surfaces, which are analogous to the theory of unipotent flows. The proofs of the main theorems rely on the measure classification theorem of [EMi2] and a certain isolation property of closed $SL(2,\mathbb{R})$ invariant manifolds developed in this paper.

1. Introduction

Suppose $g \geq 1$, and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a partition of $2g-2$, and let $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials, i.e. the space of pairs (M, ω) where M is a Riemann surface and ω is a holomorphic 1-form on M whose zeroes have multiplicities $\alpha_1 \dots \alpha_n$. The form ω defines a canonical flat metric on M with conical singularities at the zeros of ω. Thus we refer to points of $\mathcal{H}(\alpha)$ as *flat surfaces* or *translation surfaces*. For an introduction to this subject, see the survey [Zo].

The space $\mathcal{H}(\alpha)$ admits an action of the group $\mathrm{SL}(2,\mathbb{R})$ which generalizes the action of $\mathrm{SL}(2,\mathbb{R})$ on the space $\mathrm{GL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ of flat tori.

Affine measures and manifolds. The area of a translation surface is given by

$$a(M,\omega) = \frac{i}{2}\int_M \omega \wedge \bar{\omega}.$$

A "unit hyperboloid" $\mathcal{H}_1(\alpha)$ is defined as a subset of translation surfaces in $\mathcal{H}(\alpha)$ of area one. For a subset $\mathcal{N}_1 \subset \mathcal{H}_1(\alpha)$ we write

$$\mathbb{R}\mathcal{N}_1 = \{(M, t\omega) \mid (M,\omega) \in \mathcal{N}_1, \quad t \in \mathbb{R}\} \subset \mathcal{H}(\alpha).$$

Definition 1.1. An ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure ν_1 on $\mathcal{H}_1(\alpha)$ is called *affine* if the following hold:

(i) The support $\mathcal{N}_1$ of ν_1 is an suborbitfold of $\mathcal{H}_1(\alpha)$. Locally in period coordinates (see §5 below), $\mathcal{N} = \mathbb{R}\mathcal{N}_1 \subset \mathbb{C}^n$ is defined by complex linear equations with real coefficients.

(ii) Let ν be the measure supported on $\mathcal{N}$ so that $d\nu = d\nu_1 da$. Then ν is an affine linear measure in the period coordinates on $\mathcal{N}$, i.e. it is (up to normalization) the restriction of Lebesgue measure to the subspace $\mathcal{N}$.

Definition 1.2. We say that any suborbitfold $\mathcal{N}_1$ for which there exists a measure ν_1 such that the pair $(\mathcal{N}_1, \nu_1)$ satisfies (i) and (ii) an *affine invariant submanifold.*

Note that in particular, any affine invariant submanifold is a closed subset of $\mathcal{H}_1(\alpha)$ which is invariant under the $\mathrm{SL}(2,\mathbb{R})$ action, and which in period coordinates looks like an affine subspace.

We also consider the entire stratum $\mathcal{H}(\alpha)$ to be an (improper) affine invariant submanifold.

Notational Conventions. In case there is no confusion, we will often drop the subscript 1, and denote an affine manifold by $\mathcal{N}$. Also we will always denote the affine probability measure supported on $\mathcal{N}$ by $\nu_{\mathcal{N}}$.

Let $P \subset SL(2,\mathbb{R})$ denote the subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. In this paper we prove statements about the action of P and $SL(2,\mathbb{R})$ on $\mathcal{H}_1(\alpha)$ which are analogous to the statements proved in the theory of unipotent flows on homogeneous spaces. For some additional results in this direction, see also [CE].

The following theorem is the main result of [EMi2]:

Theorem 1.3. *Let ν be any P-invariant probability measure on $\mathcal{H}_1(\alpha)$. Then ν is $SL(2,\mathbb{R})$-invariant and affine.*

Theorem 1.3 is a partial analogue of Ratner's celebrated measure classification theorem in the theory of unipotent flows, see [Ra6].

2. The Main Theorems

2.1. **Orbit Closures.**

Theorem 2.1. *Suppose $x \in \mathcal{H}_1(\alpha)$. Then, the orbit closure $\overline{Px} = \overline{SL(2,\mathbb{R})x}$ is an affine invariant submanifold of $\mathcal{H}_1(\alpha)$.*

The analogue of Theorem 2.1 in the theory of unipotent flows is due in full generality to M. Ratner [Ra7]. See also the discusion in §2.8 below.

Theorem 2.2. *Any closed P-invariant subset of $\mathcal{H}_1(\alpha)$ is a finite union of affine invariant submanifolds.*

2.2. **The space of ergodic P-invariant measures.**

Theorem 2.3. *Let $\mathcal{N}_n$ be a sequence of affine manifolds, and suppose $\nu_{\mathcal{N}_n} \to \nu$. Then ν is a probability measure. Furthermore, ν is the affine measure $\nu_{\mathcal{N}}$, where $\mathcal{N}$ is the smallest submanifold with the following property: there exists some $n_0 \in \mathbb{N}$ such that $\mathcal{N}_n \subset \mathcal{N}$ for all $n > n_0$.*

In particular, the space of ergodic P-invariant probability measures on $\mathcal{H}_1(\alpha)$ is compact in the weak-star topology.

Remark 2.4. In the setting of unipotent flows, Theorem 2.3 is due to Mozes and Shah [MS].

We state a direct corollary of Theorem 2.3:

Corollary 2.5. *Let $\mathcal{M}$ be an affine invariant submanifold, and let $\mathcal{N}_n$ be a sequence of affine invariant submanifolds of $\mathcal{M}$ such that no infinite subsequence is contained in any proper affine invariant submanifold of $\mathcal{M}$. Then the sequence of affine measures $\nu_{\mathcal{N}_n}$ converges to $\nu_{\mathcal{M}}$.*

2.3. **Equidistribution for sectors.** Let $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Theorem 2.6. *Suppose $x \in \mathcal{H}_1(\alpha)$ and let $\mathcal{M}$ be the affine invariant submanifold of minimum dimension which contains x. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$, and any interval $I \subset [0, 2\pi)$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{|I|}\int_I \varphi(a_t r_\theta x)\, d\theta\, dt = \int_{\mathcal{M}} \varphi\, d\nu_{\mathcal{M}}.$$

We also have the following uniform version: (cf. [DM4, Theorem 3])

Theorem 2.7. *Let $\mathcal{M}$ be an affine invariant submanifold. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and any $\epsilon > 0$ there are affine invariant submanifolds $\mathcal{N}_1, \dots, \mathcal{N}_\ell$ properly contained in $\mathcal{M}$ such that for any compact subset $F \subset \mathcal{M} \setminus (\cup_{j=1}^{\ell} \mathcal{N}_j)$ there exits T_0 so that for all $T > T_0$ and any $x \in F$,*

$$\left| \frac{1}{T}\int_0^T \frac{1}{|I|}\int_I \varphi(a_t r_\theta x)\, d\theta\, dt - \int_{\mathcal{M}} \varphi\, d\nu_{\mathcal{M}} \right| < \epsilon.$$

We remark that the analogue of Theorem 2.7 for unipotent flows, due to Dani and Margulis [DM4] plays a key role in the applications of the theory.

2.4. **Equidistribution for Random Walks.** Let μ be a measure on $SL(2,\mathbb{R})$ which is compactly supported and is absolutely continuous with respect to the Haar measure. Even though it is not necessary, for clarity of presentation, we will also assume that μ is $SO(2)$-bi-invariant. Let $\mu^{(k)}$ denote the k-fold convolution of μ with itself.

We now state "random walk" analogues of Theorem 2.6 and Theorem 2.7.

Theorem 2.8. *Suppose $x \in \mathcal{H}_1(\alpha)$, and let $\mathcal{M}$ be the affine submanifold of minimum dimension which contains x. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \int_{SL(2,\mathbb{R})} \varphi(gx)\, d\mu^{(k)}(g) = \int_{\mathcal{M}} \varphi\, d\nu_{\mathcal{M}}.$$

We also have the following uniform version, similar in spirit to [DM4, Theorem 3]:

Theorem 2.9. *Let $\mathcal{M}$ be an affine invariant submanifold. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and any $\epsilon > 0$ there are affine invariant submanifolds $\mathcal{N}_1, \dots, \mathcal{N}_\ell$ properly contained in $\mathcal{M}$ such that for any compact subset $F \subset \mathcal{M} \setminus (\cup_{j=1}^{\ell} \mathcal{N}_j)$ there exits n_0 so that for all $n > n_0$ and any $x \in F$,*

$$\left| \frac{1}{n}\sum_{k=1}^{n} \int_{SL(2,\mathbb{R})} \varphi(gx)\, d\mu^{(k)}(g) - \int_{\mathcal{M}} \varphi\, d\nu_{\mathcal{M}} \right| < \epsilon.$$

2.5. **Equidistribution for some Fölner sets.** Let $u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

Theorem 2.10. *Suppose $x \in \mathcal{H}_1(\alpha)$ and let $\mathcal{M}$ be the affine invariant submanifold of minimum dimension which contains x. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$, and any $r > 0$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{r}\int_0^r \varphi(a_t u_s x)\, ds\, dt = \int_{\mathcal{M}} \varphi\, d\nu_{\mathcal{M}}.$$

We also have the following uniform version (cf. [DM4, Theorem 3]):

Theorem 2.11. *Let $\mathcal{M}$ be an affine invariant submanifold. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and any $\epsilon > 0$ there are affine invariant submanifolds $\mathcal{N}_1, \dots, \mathcal{N}_\ell$ properly contained in $\mathcal{M}$ such that for any compact subset $F \subset \mathcal{M} \setminus (\cup_{j=1}^{\ell} \mathcal{N}_j)$ there exits T_0 so that for all $T > T_0$ and any $x \in F$,*

$$\left| \frac{1}{T}\int_0^T \frac{1}{r}\int_0^r \varphi(a_t u_s x)\, ds\, dt - \int_{\mathcal{M}} \varphi\, d\nu_{\mathcal{M}} \right| < \epsilon.$$

2.6. **Counting periodic trajectories in rational billiards.** Let Q be a rational polygon, and let $N(Q,T)$ denote the number of cylinders of periodic trajectories of length at most T for the billiard flow on Q. By a theorem of H. Masur [Mas2] [Mas3], there exist c_1 and c_2 depending on Q such that for all $t > 1$,

$$c_1 e^{2t} \le N(Q, e^t) \le c_2 e^{2t}.$$

As a consequence of Theorem 2.7 we get the following "weak asymptotic formula" (cf. [AEZ]):

Theorem 2.12. *For any rational polygon Q, the exists a constant $c = c(Q)$ such that*
$$\lim_{t\to\infty} \frac{1}{t}\int_0^t N(Q, e^s)e^{-2s}\, ds = c.$$

The constant c in Theorem 2.12 is the Siegel-Veech constant (see [Ve], [EMZ]) associated to the affine invariant submanifold $\mathcal{M} = \overline{SL(2,\mathbb{R})S}$ where S is the flat surface obtained by unfolding Q.

It is natural to conjecture that the extra averaging on Theorem 2.12 is not necessary, and one has $\lim_{t\to\infty} N(Q, e^t)e^{-2t} = c$. This can be shown if one obtains a classification of the measures invariant under the subgroup N of $SL(2,\mathbb{R})$. Such a result is in general beyond the reach of the current methods. However it is known in a few very special cases, see [EMS], [EMaMo], [CW] and [Ba].

2.7. **The Main Proposition and Countability.** For a function $f : \mathcal{H}_1(\alpha) \to \mathbb{R}$, let
$$(A_t f)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(a_t r_\theta x).$$
Following the general idea of Margulis, the strategy of the proof is to define a function which will satisfy a certain inequality involving A_t. In fact, the main technical result of this paper is the following:

Proposition 2.13. *Let $\mathcal{M} \subset \mathcal{H}_1(\alpha)$ be an affine submanifold. (In this proposition $\mathcal{M} = \emptyset$ is allowed). Then there exists an $SO(2)$-invariant function $f_{\mathcal{M}} : \mathcal{H}_1(\alpha) \to [1,\infty]$ with the following properties:*

(a) *$f_{\mathcal{M}}(x) = \infty$ if and only if $x \in \mathcal{M}$, and $f_{\mathcal{M}}$ is bounded on compact subsets of $\mathcal{H}_1(\alpha) \setminus \mathcal{M}$. For any $\ell > 0$, the set $\overline{\{x \ :\ f(x) \le \ell\}}$ is a compact subset of $\mathcal{H}_1(\alpha) \setminus \mathcal{M}$.*

(b) *There exists $b > 0$ (depending on $\mathcal{M}$) and for every $0 < c < 1$ there exists $t_0 > 0$ (depending on $\mathcal{M}$ and c) such that for all $x \in \mathcal{H}_1(\alpha)$ and all $t > t_0$,*
$$(A_t f_{\mathcal{M}})(x) \le c f_{\mathcal{M}}(x) + b.$$

(c) *There exists $\sigma > 1$ such that for all $g \in SL(2,\mathbb{R})$ with $\|g\| \le 1$ and all $x \in \mathcal{H}_1(\alpha)$,*
$$\sigma^{-1} f_{\mathcal{M}}(x) \le f_{\mathcal{M}}(gx) \le \sigma f_{\mathcal{M}}(x).$$

The proof of Proposition 2.13 consists of §4-§10. It is based on the recurrence properties of the $SL(2,\mathbb{R})$-action proved by Athreya in [Ath], and also the fundamental result of Forni on the uniform hyperbolicity in compact sets of the Teichmüller geodesic flow [Fo, Corollary 2.1].

Remark 2.14. In the case $\mathcal{M}$ is empty, a function satisfying the conditions of Proposition 2.13 has been constructed in [EMas] and used in [Ath].

Remark 2.15. In fact, we show that the constant b in Proposition 2.13 (b) depends only on the "complexity" of $\mathcal{M}$ (defined in §8). This is used in §11 for the proof of the following:

Proposition 2.16. *There are at most countably many affine manifolds in each stratum.*

Another proof of Proposition 2.16 is given in [Wr], where it is shown that any affine manifold is defined over a number field.

2.8. **Analogy with unipotent flows and historical remarks.** In the context of unipotent flows, i.e. the left-multiplication action of a unipotent subgroup U of a Lie group G on the space G/Γ where Γ is a lattice in G, the analogue of Theorem 2.1 was conjectured by Raghunathan. In the literature the conjecture was first stated in the paper [Dan2] and in a more general form in [Mar2] (when the subgroup U is not necessarily unipotent but generated by unipotent elements). Raghunathan's conjecture was eventually proved in full generality by M. Ratner (see [Ra4], [Ra5], [Ra6] and [Ra7]). Earlier it was known in the following cases: (a) G is reductive and U is horospherical (see [Dan2]); (b) $G = SL(3, \mathbb{R})$ and $U = \{u(t)\}$ is a one-parameter unipotent subgroup of G such that $u(t) - I$ has rank 2 for all $t \neq 0$, where I is the identity matrix (see [DM2]); (c) G is solvable (see [Sta1] and [Sta2]). We remark that the proof given in [Dan2] is restricted to horospherical U and the proof given in [Sta1] and [Sta2] cannot be applied for nonsolvable G.

However the proof in [DM2] together with the methods developed in [Mar3], [Mar4], [Mar5] and [DM1] suggest an approach for proving the Raghunathan conjecture in general by studying the minimal invariant sets, and the limits of orbits of sequences of points tending to a minimal invariant set. This program was being actively pursued at the time Ratner's results were announced (cf. [Sh]).

3. Proofs of the Main Theorems

In this section we derive all the results of §2.1-§2.6 from Theorem 1.3, Proposition 2.13 and Proposition 2.16.

The proofs are much simpler then the proofs of the analogous results in the theory of unipotent flows. This is related to Proposition 2.16. In the setting of unipotent flows there may be continuous families of invariant manifolds (which involve the centralizer and normalizer of the acting group).

3.1. **Random Walks.** Many of the arguments work most naturally in the random walk setting. But first we need to convert Theorem 1.3 and Proposition 2.13 to the random walk setup.

Stationary measures. Recall that μ is a compactly supported measure on $SL(2,\mathbb{R})$ which is $SO(2)$-bi-invariant and is absolutely continuous with respect to Haar measure. A measure ν on $\mathcal{H}_1(\alpha)$ is called *μ-stationary* if $\mu * \nu = \nu$, where

$$\mu * \nu = \int_{SL(2,\mathbb{R})} (g_* \nu)\, d\mu(g).$$

Recall that by a theorem of Furstenberg [F1], [F2], restated as [NZ, Theorem 1.4], μ-stationary measures are in one-to-one correspondence with P-invariant measures. Therefore, Theorem 1.3 can be reformulated as the following:

Theorem 3.1. *Any μ-stationary measure on $\mathcal{H}_1(\alpha)$ is $SL(2,\mathbb{R})$ invariant and affine.*

The operator $\mathbb{A}_\mu$. Let $\mathbb{A}_\mu : C_c(\mathcal{H}_1(\alpha)) \to C_c(\mathcal{H}_1(\alpha))$ denote the linear operator

$$(\mathbb{A}_\mu f)(x) = \int_{SL(2,\mathbb{R})} f(gx)\, d\mu(g)$$

Lemma 3.2. *Let $f_{\mathcal{M}}$ be as in Proposition 2.13. Then there exists $b > 0$ and for any $c > 0$ there exists $n_0 > 0$ such that for $n > n_0$, and any $x \in \mathcal{H}_1(\alpha)$,*

$$(\mathbb{A}_\mu^n f_{\mathcal{M}})(x) \le c f_{\mathcal{M}}(x) + b.$$

Proof. Since μ is assumed to be $SO(2)$-bi-invariant, we have

$$(\mathbb{A}_\mu^n f_{\mathcal{M}})(x) = \int_0^\infty K_n(t)(A_t f_{\mathcal{M}})(x)\, dt, \tag{1}$$

where the compactly supported function $K_n(t)$ satisfies $K_n(t) \ge 0$, $\int_0^\infty K_n(t)\, dt = 1$. Also, for any $t_0 > 0$ and any $\epsilon > 0$ there exists n_0 such that for $n > n_0$,

$$\int_0^{t_0} K_n(t)\, dt < \epsilon. \tag{2}$$

Now let t_0 be as in Proposition 2.13 (b) for $c/2$ instead of c. By Proposition 2.13 (c), there exists $R > 0$ such that

$$f_{\mathcal{M}}(a_t r_\theta x) < R f_{\mathcal{M}}(x) \qquad \text{when } t < t_0. \tag{3}$$

Then let n_0 be such that (2) holds with $\epsilon = c/(2R)$. Then, for $n > n_0$,

$$\begin{aligned}
(\mathbb{A}_\mu^n f_{\mathcal{M}})(x) &= \int_0^{t_0} K_n(t)(A_t f_{\mathcal{M}})(x)\, dt + \int_{t_0}^\infty K_n(t)(A_t f_{\mathcal{M}})(x)\, dt && \text{by (1)} \\
&\le \int_0^{t_0} K_n(t)(R f_{\mathcal{M}}(x))\, dt + \int_{t_0}^\infty ((c/2) f_{\mathcal{M}}(x) + b)\, dt && \text{by (3) and Proposition 2.13 (b)} \\
&\le (c/2R) R f_{\mathcal{M}}(x) + ((c/2) f_{\mathcal{M}}(x) + b) && \text{by (2)} \\
&= c f_{\mathcal{M}}(x) + b.
\end{aligned}$$

$\square$

Notational conventions. Let

$$\bar{\mu}^{(n)} = \frac{1}{n}\sum_{k=1}^{n}\mu^{(k)}.$$

For $x \in \mathcal{H}_1(\alpha)$ let δ_x denote the Dirac measure at x, and let $*$ denote convolution of measures.

We have the following:

Proposition 3.3. *Let $\mathcal{N}$ be a (possibly empty) proper affine invariant submanifold. Then for any $\epsilon > 0$, there exists an open set $\Omega_{\mathcal{N},\epsilon}$ containing $\mathcal{N}$ with $(\Omega_{\mathcal{N},\epsilon})^c$ compact such that for any compact $F \subset \mathcal{H}_1(\alpha) \setminus \mathcal{N}$ there exists $n_0 \in \mathbb{N}$ so that for all $n > n_0$ and all $x \in F$, we have*

$$(\bar{\mu}^{(n)} * \delta_x)(\Omega_{\mathcal{N},\epsilon}) < \epsilon.$$

Proof. Let $f_{\mathcal{N}}$ be the function of Proposition 2.13. Let $b > 0$ be as in Lemma 3.2, and let

$$\Omega_{\mathcal{N},\epsilon} = \{p : f_{\mathcal{N}}(p) > (b+1)/\epsilon\}^0,$$

where E^0 denotes the interior of E.

Suppose F is a compact subset of $\mathcal{H}_1(\alpha) \setminus \mathcal{N}$. Let $m_F = \sup\{f_{\mathcal{N}}(x) : x \in F\}$. Let $n_0 \in \mathbb{N}$ be as in Lemma 3.2 for $c = 1/m_F$. Then, by Lemma 3.2,

$$(\mathbb{A}_\nu^n f_{\mathcal{N}})(x) < \frac{1}{m_F} f_{\mathcal{N}}(x) + b \le 1 + b, \qquad \text{for all } n > n_0 \text{ and all } x \in F.$$

Equivalently, for all $x \in F$ and all $n > n_0$,

$$(\bar{\mu}^{(n)} * \delta_x)(f_{\mathcal{N}}) \le 1 + b.$$

Thus for any $x \in F$ and $L > 0$ we have

$$(\bar{\mu}^{(n)} * \delta_x)(\{p : f_{\mathcal{N}}(p) > L\}) < \frac{b+1}{L}. \tag{4}$$

Then (4) implies that $(\bar{\mu}^{(n)} * \delta_x)(\Omega_{\mathcal{N},\epsilon}) < \epsilon$. Also, Proposition 2.13 (a) implies that $\Omega_{\mathcal{N},\epsilon}$ is a neighborhood of $\mathcal{N}$ and

$$(\Omega_{\mathcal{N},\epsilon})^c = \overline{\{p \; : \; f_{\mathcal{N}}(p) \le (b+1)/\epsilon\}}$$

is compact. $\square$

Proof of Theorem 2.8. Suppose this is not the case. Then there exist a $\varphi \in C_c(\mathcal{H}_1(\alpha))$, $\epsilon > 0$, $x \in \mathcal{M}$ and a sequence $n_k \to \infty$ such that

$$|(\bar{\mu}^{(n_k)} * \delta_x)(\varphi) - \mu_{\mathcal{M}}(\varphi)| \ge \epsilon.$$

Recall that the space of measures on $\mathcal{H}_1(\alpha)$ of total mass at most 1 is compact in the weak star topology. Therefore, after passing to a subsequence if necessary, we may and will assume that $\bar{\mu}^{(n_k)} * \delta_x \to \nu$ where ν is some measure on $\mathcal{H}_1(\alpha)$ (which could a priori be the zero measure). Below, we will show that in fact ν is the probability measure $\nu_{\mathcal{M}}$, which leads to a contradiction.

First note that it follows from the definition that ν is an μ-stationary measure. Therefore, by Theorem 3.1, ν is $SL(2,\mathbb{R})$-invariant. Also since $\mathcal{M}$ is $\mathrm{SL}(2,\mathbb{R})$-invariant we get $\mathrm{supp}(\nu) \subset \mathcal{M}$. The measure ν need not be ergodic, but by Theorem 1.3, all of its ergodic components are affine measures supported on affine invariant submanifolds of $\mathcal{M}$. By Proposition 2.16 there are only countably many affine invariant submanifolds of $\mathcal{M}$. Therefore, we have the ergodic decomposition:

$$\nu = \sum_{\mathcal{N} \subseteq \mathcal{M}} a_{\mathcal{N}} \nu_{\mathcal{N}}, \tag{5}$$

where the sum is over all the affine invariant submanifolds $\mathcal{N} \subset \mathcal{M}$ and $a_{\mathcal{N}} \in [0,1]$. To finish the proof we will show that ν is a probability measure, and that $a_{\mathcal{N}} = 0$ for all $\mathcal{N} \subsetneq \mathcal{M}$.

Suppose $\mathcal{N} \subsetneq \mathcal{M}$ and apply Proposition 3.3 with $\mathcal{N}$ and the compact set $F = \{x\}$. We get for any $\epsilon > 0$, there exists some n_0 so that if $n > n_0$, then $(\bar{\mu}^{(n)} * \delta_x)((\Omega_{\mathcal{N},\epsilon})^c) \geq 1-\epsilon$. Therefore, passing to the limit, we get

$$\nu((\Omega_{\mathcal{N},\epsilon})^c) \geq 1 - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this implies that ν is a probability measure and $\nu(\mathcal{N}) = 0$. Hence $a_{\mathcal{N}} = \nu(\mathcal{N}) = 0$. □

Proof of Theorem 2.3. Since the space of measures of mass at most 1 on $\mathcal{H}_1(\alpha)$ is compact in the weak-star topology, the second statement in Theorem 2.3 follows from the first.

Suppose that $\nu_{\mathcal{N}_n} \to \nu$. We first prove that ν is a probability measure. Let $\Omega_{\emptyset,\epsilon}$ be as in Proposition 3.3 with $\mathcal{M} = \emptyset$. By the random ergodic theorem [Fu, Theorem 3.1], for a.e $x_n \in \mathcal{N}_n$,

$$\lim_{m\to\infty} (\bar{\mu}^{(m)} * \delta_{x_n})((\Omega_{\emptyset,\epsilon})^c) = \nu_{\mathcal{N}_n}((\Omega_{\emptyset,\epsilon})^c). \tag{6}$$

Choose x_n such that (6) holds. By Proposition 3.3, for all m large enough (depending on x_n),

$$(\bar{\mu}^{(m)} * \delta_{x_n})((\Omega_{\emptyset,\epsilon})^c) \geq 1 - \epsilon.$$

Passing to the limit as $n \to \infty$, we get

$$\nu((\Omega_{\emptyset,\epsilon})^c) \geq 1 - \epsilon.$$

Since ϵ is arbitrary, this shows that ν is a probability measure.

In view of the fact that the ν_n are invariant under $\mathrm{SL}(2,\mathbb{R})$, the same is true of ν. As in (5), let

$$\nu = \sum_{\mathcal{N}\subseteq\mathcal{H}_1(\alpha)} a_{\mathcal{N}}\nu_{\mathcal{N}}$$

be the ergodic decomposition of ν, where $a_{\mathcal{N}} \in [0,1]$. By Theorem 1.3, all the measures $\nu_{\mathcal{N}}$ are affine and by Proposition 2.16, the number of terms in the ergodic decomposition is countable.

For any affine invariant submanifold $\mathcal{N}$ let

$$X(\mathcal{N}) = \bigcup \{\mathcal{N}' \subsetneq \mathcal{N} : \mathcal{N}' \text{ is an affine invariant submanifold}\}.$$

Let $\mathcal{N} \subseteq \mathcal{H}_1(\alpha)$ be a submanifold such that $\nu(X(\mathcal{N})) = 0$ and $\nu(\mathcal{N}) > 0$. This implies $a_{\mathcal{N}} = \nu(\mathcal{N})$.

Let K be a large compact set, such that $\nu(K) > (1 - a_{\mathcal{N}}/4)$. Then, $\nu(K \cap \mathcal{N}) > (3/4)a_{\mathcal{N}}$. Let $\epsilon = a_{\mathcal{N}}/4$, and let $\Omega_{\mathcal{N},\epsilon}$ be as in Proposition 3.3. Since $K\cap\mathcal{N}$ and $(\Omega_{\mathcal{N},\epsilon})^c$ are both compact sets, we can choose a continuous compactly supported function φ such that $0 \le \varphi \le 1$, $\varphi = 1$ on $K \cap \mathcal{N}$ and $\varphi = 0$ on $(\Omega_{\mathcal{N},\epsilon})^c$. Then,

$$\nu(\varphi) \ge \nu(K \cap \mathcal{N}) > (3/4)a_{\mathcal{N}}.$$

Since $\nu_{\mathcal{N}_n}(\varphi) \to \nu(\varphi)$, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\nu_{\mathcal{N}_n}(\varphi) > a_{\mathcal{N}}/2.$$

For each n let $x_n \in \mathcal{N}_n$ be a generic point for $\nu_{\mathcal{N}_n}$ for the random ergodic theorem [Fu, Theorem 3.1] i.e.

$$\lim_{m\to\infty} (\bar{\mu}^{(m)} * \delta_{x_n})(\varphi) = \nu_{\mathcal{N}_n}(\varphi) \text{ for all } \varphi \in C_c(\mathcal{H}_1(\alpha)). \tag{7}$$

Suppose $n > n_0$. Then, by (7), we get

$$\text{if } m \text{ is large enough, then } (\bar{\mu}^{(m)} * \delta_{x_n})(\varphi) > a_{\mathcal{N}}/4.$$

Therefore, since $0 \le \varphi \le 1$ and $\varphi = 0$ outside of $\Omega_{\mathcal{N},\epsilon}$, we get

$$\text{if } m \text{ is large enough, then } (\bar{\mu}^{(m)} * \delta_{x_n})(\Omega_{\mathcal{N},\epsilon}) > a_{\mathcal{N}}/4.$$

Proposition 3.3, applied with $\epsilon = a_{\mathcal{N}}/4$ now implies that $x_n \in \mathcal{N}$, which, in view of the genericity of x_n implies that $\mathcal{N}_n \subset \mathcal{N}$ for all $n > n_0$. This implies $\nu(\mathcal{N}) = 1$, and since $\nu(X(\mathcal{N})) = 0$, we get $\nu = \nu_{\mathcal{N}}$. Also, since $\nu(X(\mathcal{N})) = 0$, $\mathcal{N}$ is the minimal affine invariant manifold which eventually contains the $\mathcal{N}_n$. $\square$

Lemma 3.4. *Given any $\varphi \in C_c(\mathcal{H}_1(\alpha))$, any affine invariant submanifold $\mathcal{M}$ and any $\epsilon > 0$, there exists a finite collection $\mathcal{C}$ of proper affine invariant submanifolds of $\mathcal{M}$ with the following property; if $\mathcal{N}'$ is so that $|\nu_{\mathcal{N}'}(\varphi) - \nu_{\mathcal{M}}(\varphi)| > \epsilon$, then there exists some $\mathcal{N} \in \mathcal{C}$ such that $\mathcal{N}' \subset \mathcal{N}$.*

Proof. Let φ and $\epsilon > 0$ be given. We will prove this by inductively choosing $\mathcal{N}_j$'s as follows. Suppose $k > 0$, and put

$$\mathcal{A}_k = \{\mathcal{N} \subseteq \mathcal{M} : \mathcal{N} \text{ has codimension } k \text{ in } \mathcal{M} \text{ and } |\nu_{\mathcal{N}}(\varphi) - \nu_{\mathcal{M}}(\varphi)| \geq \epsilon\}.$$

Let $\mathcal{B}_1 = \mathcal{A}_1$, and define

$$\mathcal{B}_k = \{\mathcal{N} \in \mathcal{A}_k : \text{such that } \mathcal{N} \text{ is not contained in any } \mathcal{N}' \in \mathcal{A}_\ell \text{ with } \ell < k\}.$$

Claim. $\mathcal{B}_k$ is a finite set for each k.

We will show this inductively. Note that by Corollary 2.5 we have $\mathcal{A}_1$, and hence $\mathcal{B}_1$, is a finite set. Suppose we have shown $\{\mathcal{B}_j : 1 \leq j \leq k-1\}$ is a finite set. Let $\{\mathcal{N}_j\}$ be an infinite collection of elements in $\mathcal{B}_k$. By Theorem 2.3 we may pass to subsequence (which we continue to denote by $\mathcal{N}_j$) such that $\nu_{\mathcal{N}_j} \to \nu$. Theorem 2.3 also implies that $\nu = \nu_{\mathcal{N}}$ for some affine invariant submanifold $\mathcal{N}$, and that there exists some j_0 such that $\mathcal{N}_j \subset \mathcal{N}$ for all $j > j_0$. Note that $\mathcal{N}$ has codimension $\ell \leq k - 1$.

Since $\nu_{\mathcal{N}_j} \to \nu_{\mathcal{N}}$, and $\mathcal{N}_j \in \mathcal{B}_k \subset \mathcal{A}_k$, we have $|\nu_{\mathcal{N}}(\varphi) - \nu_{\mathcal{M}}(\varphi)| \geq \epsilon$. Therefore $\mathcal{N} \in \mathcal{A}_\ell$. But this is a contradiction to the definition of $\mathcal{B}_k$ since $\mathcal{N}_j \subset \mathcal{N}$ and $\mathcal{N}_j \in \mathcal{B}_k$. This completes the proof of the claim.

Now let

$$\mathcal{C} = \{\mathcal{N} : \mathcal{N} \in \mathcal{B}_k, \text{ for } 0 < k \leq \dim \mathcal{M}\}.$$

This is a finite set which satisfies the conclusion of the lemma. □

Proof of Theorem 2.9. Let φ and $\epsilon > 0$ be given, and let $\mathcal{C}$ be given by Lemma 3.4. Write $\mathcal{C} = \{\mathcal{N}_1, \ldots, \mathcal{N}_\ell\}$. We will show the theorem holds with this choice of the $\mathcal{N}_j$.

Suppose not, then there exists a compact subset $F \subset K \setminus \bigcup_{j=1}^{\ell} \mathcal{N}_j$ such that for all $m_0 \geq 0$,

$$\{x \in F : |(\bar{\mu}^{(m)} * \delta_x)(\varphi) - \nu_{\mathcal{M}}(\varphi)| > \epsilon, \text{ for some } m > m_0\} \neq \emptyset.$$

Let $m_n \to \infty$ and $\{x_n\} \subset F$ be a sequence such that $|(\bar{\mu}^{(m_n)} * \delta_{x_n})(\varphi) - \nu_{\mathcal{M}}(\varphi)| > \epsilon$.

Since the space of measures on $\mathcal{H}_1(\alpha)$ of total mass at most 1 is compact in the weak star topology, after passing to a subsequence if necessary, we may and will assume that $\bar{\mu}^{(m_n)} * \delta_{x_n} \to \nu$ where ν is some measure on $\mathcal{M}$ (which could a priori be the zero measure). We will also assume that $x_n \to x$ for some $x \in F$.

Note that that ν is $\mathrm{SL}(2,\mathbb{R})$-stationary. Let

$$\nu = \sum_{\mathcal{N} \subseteq \mathcal{M}} a_{\mathcal{N}} \nu_{\mathcal{N}}$$

be the ergodic decomposition of ν, as in (5).

We claim that ν is a probability measure and $\nu(\mathcal{N}) = 0$ for all $\mathcal{N} \in \mathcal{C}$. To see this, suppose $\mathcal{N} \in \mathcal{C}$ and apply Proposition 3.3 with $\mathcal{N}$ and F. We get for any $\epsilon > 0$, there exists some n_0 so that if $n > n_0$, then $(\bar{\mu}^{(n)} * \delta_y)((\Omega_{\mathcal{N},\epsilon})^c) \geq 1 - \epsilon$ for all $y \in F$. Therefore, passing to the limit, we get

$$\nu((\Omega_{\mathcal{N},\epsilon})^c) \geq 1 - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this implies that ν is a probability measure and $\nu(\mathcal{N}) = 0$. The claim now follows since $\mathcal{C}$ is a finite family.

The claim and Lemma 3.4 imply that $|\nu(\varphi) - \nu_{\mathcal{M}}(\varphi)| < \epsilon$. This and the definition of ν imply that $|(\bar{\mu}^{(m_n)} * \delta_{x_n})(\varphi) - \nu_{\mathcal{M}}(\varphi)| < \epsilon$ for all large enough n. This contradicts the choice of x_n and m_n and completes the proof. □

The only properties of the measures $\bar{\mu}^{(n)}$ which were used in this subsection were Proposition 3.3 and the fact that any limit of the measures $\bar{\mu}^{(n)} * \delta_x$ is $SL(2,\mathbb{R})$ invariant. In fact, we proved the following theorem, which we will record for future use:

Theorem 3.5. *Suppose $\{\eta_t \;:\; t \in \mathbb{R}\}$ is a family of probability measures on $SL(2,\mathbb{R})$ with the following properties:*

- (a) *Proposition 3.3 holds for η_t instead of $\bar{\mu}^{(n)}$ (and t instead of n).*
- (b) *Any weak-star limit of measures of the form $\eta_{t_i} * \delta_{x_i}$ as $t_i \to \infty$ is $SL(2,\mathbb{R})$-invariant.*

Then,

- (i) *(cf. Theorem 2.8) Suppose $x \in \mathcal{H}_1(\alpha)$, and let $\mathcal{M}$ be the smallest affine invariant submanifold containing x. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$,*
$$\lim_{t\to\infty} (\eta_t * \delta_x)(\varphi) = \mu_{\mathcal{M}}(\varphi)$$
- (ii) *(cf. Theorem 2.9) Let $\mathcal{M}$ be an affine invariant submanifold. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and any $\epsilon > 0$ there are affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_\ell$ properly contained in $\mathcal{M}$ such that for any compact subset $F \subset \mathcal{M} \setminus (\cup_{j=1}^{\ell} \mathcal{N}_j)$ there exits T_0 so that for all $T > T_0$ and any $x \in F$,*
$$|(\eta_t * \delta_x)(\varphi) - \nu_{\mathcal{M}}(\varphi)| < \epsilon.$$

3.2. **Equidistribution for sectors.** We define a sequence of probability measures ϑ_t on $SL(2,\mathbb{R})$ by

$$\vartheta_t(\varphi) = \frac{1}{t}\int_0^t \frac{1}{2\pi}\int_0^{2\pi} \varphi(a_s r_\theta)\, d\theta\, ds.$$

More generally, if $I \subset [0, 2\pi]$ is an interval, then we define

$$\vartheta_{t,I}(\varphi) = \frac{1}{t}\int_0^t \frac{1}{|I|}\int_I \varphi(a_s r_\theta)\, d\theta\, ds.$$

We have the following:

Proposition 3.6. *Let $\mathcal{N}$ be a (possibly empty) proper affine invariant submanifold. Then for any $\epsilon > 0$, there exists an open set $\Omega_{\mathcal{N},\epsilon}$ containing $\mathcal{N}$ with $(\Omega_{\mathcal{N},\epsilon})^c$ compact such that for any compact $F \subset \mathcal{H}_1(\alpha) \setminus \mathcal{N}$ there exists $t_0 \in \mathbb{R}$ so that for all $t > t_0$ and all $x \in F$, we have*

$$(\vartheta_{t,I} * \delta_x)(\Omega_{\mathcal{N},\epsilon}) < \epsilon.$$

Proof. This proof is virtually identical to the proof of Proposition 3.3. It is enough to prove the statement for the case $I = [0, 2\pi]$. Let $f_{\mathcal{N}}$ be the function of Proposition 2.13. Let $b > 0$ be as in Proposition 2.13 (b), and let

$$\Omega_{\mathcal{N},\epsilon} = \{p : f_{\mathcal{N}}(p) > (b+1)/\epsilon\}^0,$$

where E^0 denotes the interior of E.

Suppose F is a compact subset of $\mathcal{H}_1(\alpha) \setminus \mathcal{N}$. Let $m_F = \sup\{f_{\mathcal{N}}(x) : x \in F\}$. By Proposition 2.13 (b) with $c = \frac{1}{2m_F}$, there exists $t_1 > 0$ such that

$$(A_t f_{\mathcal{N}})(x) < \frac{1}{m_F} f_{\mathcal{N}}(x) + b \leq 1 + b, \qquad \text{for all } t > t_1 \text{ and all } x \in F.$$

By Proposition 2.13 (a) there exists $R > 0$ such that $f_{\mathcal{N}}(a_t x) \leq R f_{\mathcal{N}}(x)$ for $0 \leq t \leq t_1$. Now choose t_0 so that $t_1 R / t_0 < m_F/2$. Then, for $t > t_0$,

$$\begin{aligned}(\vartheta_t * \delta_x)(f_{\mathcal{N}}) = \frac{1}{t}\int_0^t (A_s f_{\mathcal{N}})(x)\,ds &= \frac{1}{t}\int_0^{t_1} (A_s f_{\mathcal{N}})(x)\,ds + \frac{1}{t}\int_{t_1}^t (A_s f_{\mathcal{N}})(x)\,ds \\ &\leq \frac{t_1 R}{t} f_{\mathcal{N}}(x) + (\frac{m_F}{2} f_{\mathcal{N}}(x) + b) \leq m_F f_{\mathcal{N}}(x) + b \leq 1 + b.\end{aligned}$$

Thus for any $x \in F$, $t > t_0$ and $L > 0$ we have

$$(\vartheta_t * \delta_x)(\{p : f_{\mathcal{N}}(p) > L\}) < (b+1)/L. \tag{8}$$

Then (8) implies that $(\vartheta_t * \delta_x)(\Omega_{\mathcal{N},\epsilon}) < \epsilon$. Also, Proposition 2.13 (a) implies that $\Omega_{\mathcal{N},\epsilon}$ is a neighborhood of $\mathcal{N}$ and

$$(\Omega_{\mathcal{N},\epsilon})^c = \overline{\{p \ : \ f_{\mathcal{N}}(p) \leq (b+1)/\epsilon\}}$$

is compact. $\square$

Lemma 3.7. *Suppose $t_i \to \infty$, $x_i \in \mathcal{H}_1(\alpha)$, and $\vartheta_{t_i,I} * \delta_{x_i} \to \nu$. Then ν is invariant under P (and then by Theorem 1.3 also invariant under $SL(2,\mathbb{R})$).*

Proof. Let A denote the diagonal subgroup of $SL(2,\mathbb{R})$, and let $U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. From the definition it is clear that ν is A-invariant. We will show it is also U-invariant; indeed it suffices to show this for $u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ with $0 \leq s \leq 1$.

First note that for any $0 < \theta < \pi/2$ we have

$$r_\theta = g_\theta u_{\tan\theta}, \text{ where } g_\theta = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 1/\cos\theta \end{pmatrix}. \tag{9}$$

Therefore, for all $\tau > 0$ we have $a_\tau g_\theta a_\tau^{-1} = \begin{pmatrix} \cos\theta & 0 \\ e^{-2\tau}\sin\theta & 1/\cos\theta \end{pmatrix}$. We have

$$a_\tau r_\theta = a_\tau g_\theta u_{\tan\theta} = a_\tau g_\theta a_\tau^{-1} u_{e^{2\tau}\tan\theta}\, a_\tau. \tag{10}$$

Fix some $0 < s < 1$, and define s_τ by $e^{2\tau}\tan s_\tau = s$. Then, (10) becomes

$$a_\tau r_{s_\tau} = (a_\tau g_{s_\tau} a_\tau^{-1}) u_s a_\tau. \tag{11}$$

For any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and all x we have

$$\varphi(u_s a_\tau r_\theta x) - \varphi(a_\tau r_\theta x) = (\varphi(u_s a_\tau r_\theta x) - \varphi(a_\tau r_{\theta+s_\tau} x)) + (\varphi(a_\tau r_{\theta+s_\tau} x) - \varphi(a_\tau r_\theta x)). \tag{12}$$

We compute the contribution from the two parentheses separately. Note that terms in the first parenthesis are close to each other thanks to (11) and the definition of s_τ. The contribution from the second is controlled as the integral over I and a "small" translate of I are close to each other.

We carry out the computation here. First note that $s_\tau \to 0$ as $\tau \to \infty$. Furthermore, this and (9) imply that $a_\tau g_{s_\tau} a_\tau^{-1}$ tends to the identity matrix as $\tau \to \infty$. Therefore, given $\epsilon > 0$, thanks to (11) and the uniform continuity of φ we have

$$|\varphi(u_s a_\tau r_\theta x) - \varphi(a_\tau r_{\theta+s_\tau} x)| \le \epsilon$$

for all large enough τ and all $x \in \mathcal{H}_1(\alpha)$. Thus, for large enough n (depending on ϵ and φ), we get

$$\frac{1}{t_n}\int_0^{t_n} \frac{1}{|I|}\int_I |\varphi(u_s a_\tau r_\theta x_n) - \varphi(a_\tau r_{\theta+s_\tau} x_n)|\, d\theta\, d\tau \le 2\epsilon. \tag{13}$$

As for the second parentheses on the right side of (12), we have

$$\begin{aligned}
&\left|\frac{1}{t_n}\int_0^{t_n} \frac{1}{|I|}\int_I (\varphi(a_\tau r_{\theta+s_\tau} x_n) - \varphi(a_\tau r_\theta x_n))\, d\theta\, d\tau\right| \le \\
&\le \frac{1}{t_n}\int_0^{t_n} \left|\frac{1}{|I|}\int_{I+s_\tau} \varphi(a_\tau r_\theta x_n) d\theta - \frac{1}{|I|}\int_I \varphi(a_\tau r_\theta x_n)\, d\theta\right| d\tau \le \frac{C_\varphi}{t_n}\int_0^{t_n} s_\tau\, d\tau \\
&\le \frac{C'_\varphi}{t_n}, \qquad \text{since } s_\tau = O(e^{-2\tau}) \text{ and thus the integral converges.}
\end{aligned}$$

This, together with (13) and (12), implies $|\nu(u_s\varphi) - \nu(\varphi)| \le 2\epsilon$; the lemma follows. $\square$

Now in view of Proposition 3.6 and Lemma 3.7, Theorem 2.6 and Theorem 2.7 hold by Theorem 3.5.

3.3. **Equidistribution for some Fölner sets.** In this subsection, we prove Theorem 2.10 and Theorem 2.11. These theorems can be easily derived from Theorem 2.6 and Theorem 2.7, but we choose to derive them directly from Theorem 3.5.

Fix $r > 0$, and define a family of probability measures $\lambda_{t,r}$ on $SL(2,\mathbb{R})$ by

$$\lambda_{t,r}(\varphi) = \frac{1}{rt}\int_0^t \int_0^r \varphi(a_\tau u_s)\, d\tau ds$$

The supports of the measures $\lambda_{t,r}$ form a Fölner family as $t \to \infty$ (and r is fixed). Thus, any limit measure of the measures $\lambda_{t_i,r} * \delta_{x_i}$ is P-invariant (and thus $SL(2,\mathbb{R})$-invariant by Theorem 1.3). Therefore it remains to prove:

Proposition 3.8. *Let $\mathcal{N}$ be a (possibly empty) proper affine invariant submanifold. Then for any $\epsilon > 0$, there exists an open set $\Omega_{\mathcal{N},\epsilon}$ containing $\mathcal{N}$ with $(\Omega_{\mathcal{N},\epsilon})^c$ compact such that for any compact $F \subset \mathcal{H}_1(\alpha) \setminus \mathcal{N}$ there exists $t_0 \in \mathbb{R}$ so that for all $t > t_0$ and all $x \in F$, we have*

$$(\lambda_{t,r} * \delta_x)(\Omega_{\mathcal{N},\epsilon}) < \epsilon.$$

Proof. It is enough to prove the statements for $r = \tan 0.01$. We may write as in the proof of Lemma 3.7

$$r_\theta = g_\theta u_{\tan\theta}$$

and thus

$$a_t u_{\tan\theta} = a_t g_\theta^{-1} r_\theta = (a_t g_\theta^{-1} a_t^{-1}) a_t r_\theta$$

Let $I = (0, 0.01)$. Note that $a_t g_\theta^{-1} a_t^{-1}$ remains bounded for $\theta \in I$ as $t \to \infty$. Also, the derivative of $\tan\theta$ is bounded between two non-zero constants for $\theta \in I$. Therefore, by Proposition 2.13 (c), for all t and x,

$$(\lambda_{t,r} * \delta_x)(f_{\mathcal{N}}) \le C(\vartheta_{t,I} * \delta_x)(f_{\mathcal{N}}),$$

where C depends only on the constant σ in Proposition 2.13 (c). Therefore, for all t and x,

$$(\lambda_{t,r} * \delta_x)(f_{\mathcal{N}}) \le C'(\vartheta_t * \delta_x)(f_{\mathcal{N}}),$$

where $C' = C/|I|$. Now let

$$\Omega_{\mathcal{N},\epsilon} = \{p : f_{\mathcal{N}}(p) > C(b+1)/\epsilon\}^0.$$

The rest of the proof is exactly as in Proposition 3.6. □

Now Theorem 2.10 and Theorem 2.11 follow from Theorem 3.5.

3.4. **Proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.12.**

Proof of Theorem 2.1. This is an immediate consequence of Theorem 2.10. □

Proof of Theorem 2.2. Suppose $\mathcal{A} \subset \mathcal{H}_1(\alpha)$ is a closed P-invariant subset. Let Y denote the set of affine invariant manifolds contained in $\mathcal{A}$, and let Z consist of the set of maximal elements of Y (i.e. elements of Y which are not properly contained in another element of Y). By Theorem 2.1,

$$\mathcal{A} = \bigcup_{\mathcal{N} \in Y} \mathcal{N} = \bigcup_{\mathcal{N} \in Z} \mathcal{N}.$$

We now claim that Z is finite. Suppose not, then there exists an infinite sequence $\mathcal{N}_n$ of distinct submanifolds in Z. Then by Theorem 2.3 there exists a subsequence $\mathcal{N}_{n_j}$ such that $\nu_{\mathcal{N}_{n_j}} \to \nu_{\mathcal{N}}$ where $\mathcal{N}$ is another affine invariant manifold which contains all but finitely many $\mathcal{N}_{n_j}$. Without loss of generality, we may assume that $\mathcal{N}_{n_j} \subset \mathcal{N}$ for all j.

Since $\nu_{\mathcal{N}_{n_j}} \to \nu_{\mathcal{N}}$, the union of the $\mathcal{N}_{n_j}$ is dense in $\mathcal{N}$. Since $\mathcal{N}_{n_j} \subset \mathcal{A}$ and $\mathcal{A}$ is closed, $\mathcal{N} \subset \mathcal{A}$. Therefore $\mathcal{N} \in Y$. But $\mathcal{N}_{n_j} \subset \mathcal{N}$, therefore $\mathcal{N}_{n_j} \notin Z$. This is a contradiction. □

Proof of Theorem 2.12. This is a consequence of Theorem 2.6; see [EMas, §3-§5] for the details. See also [EMaMo, §8] for an axiomatic formulation and an outline of the argument.

We note that since we do not have a convergence theorem for averages of the form

$$\lim_{t\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \varphi(a_t r_\theta x)\, d\theta$$

and therefore we do not know that e.g. assumption (C) of [EMaMo, Theorem 8.2] is satisfied. But by Theorem 2.6 we do have convergence for the averages

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{1}{2\pi} \int_0^{2\pi} \varphi(a_s r_\theta x)\, d\theta\, ds.$$

Since we also have an extra average on the right-hand side of Theorem 2.12, the proof goes through virtually without modifications. □

4. Recurrence Properties

Recall that for a function $f : \mathcal{H}_1(\alpha) \to \mathbb{R}$,

$$(A_t f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(a_t r_\theta x).$$

Theorem 4.1. *There exists a continuous, proper, $SO(2)$-invariant function $u : \mathcal{H}_1(\alpha) \to [2, \infty)$ such that*

(i) *There exists $m \in \mathbb{R}$ such that for all $x \in \mathcal{H}_1(\alpha)$ and all $t > 0$,*

$$e^{-mt} u(x) \le u(a_t x) \le e^{mt} u(x) \tag{14}$$

(ii) *There exists constants $t_0 > 0$, $\tilde{\eta} > 0$ and $\tilde{b} > 0$ such that for all $t \ge t_0$ and all $x \in \mathcal{H}_1(\alpha)$ we have*

$$A_t u(x) \le \tilde{c} u(x) + \tilde{b}, \qquad \text{with } \tilde{c} = e^{-\tilde{\eta} t} \tag{15}$$

We state some consequences of Theorem 4.1, from [Ath]:

Theorem 4.2. *For any $\rho > 0$, there exists a compact $K_\rho \subset \mathcal{H}_1(\alpha)$ such that for any $SL(2, \mathbb{R})$-invariant probability measure μ,*

$$\mu(K_\rho) > 1 - \rho.$$

Theorem 4.3. *Let K_ρ be as in Theorem 4.2. Then, if $\rho > 0$ is sufficiently small, there exists a constant $m'' > 0$ such that for all $x \in \mathcal{H}_1(\alpha)$ there exists $\theta \in [0, 2\pi]$ and $\tau \le m'' \log u(x)$ such that $x' \equiv a_\tau r_\theta x \in K$.*

Theorem 4.4. *For $x \in \mathcal{H}_1(\alpha)$ and a compact set $K_* \subset \mathcal{H}_1(\alpha)$ define*

$$\mathcal{I}_1(t) = \{\theta \in [0, 2\pi] : |\{\tau \in [0, t] : a_\tau r_\theta x \in K_*\}| > t/2\},$$

and

$$\mathcal{I}_2(t) = [0, 2\pi] \setminus \mathcal{I}_1(t).$$

Then, there exists some $\eta_1 > 0$, a compact subset K_*, and constants $L_0 > 0$ and $\eta_0 > 0$ such that for any $t > 0$,

$$\text{if } \log u(x) < L_0 + \eta_0 t, \qquad \text{then } |\mathcal{I}_2(t)| < e^{-\eta_1 t}. \tag{16}$$

Theorem 4.4 is not formally stated in [Ath], but is a combination of [Ath, Theorem 2.2] and [Ath, Theorem 2.3]. (In the proof of [Ath, Theorem 2.3], one should use [Ath, Theorem 2.2] to control the distribution of τ_0).

5. Period Coordinates and the Kontsevich-Zorich cocycle

Let $\Sigma \subset M$ denote the set of zeroes of ω. Let $\{\gamma_1, \ldots, \gamma_k\}$ denote a symplectic $\mathbb{Z}$-basis for the relative homology group $H_1(M, \Sigma, \mathbb{Z})$. We can define a map $\Phi : \mathcal{H}(\alpha) \to \mathbb{C}^k$ by

$$\Phi(M, \omega) = \left(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_k} w \right)$$

The map Φ (which depends on a choice of the basis $\{\gamma_1, \ldots, \gamma_n\}$) is a local coordinate system on (M, ω). Alternatively, we may think of the cohomology class

$[\omega] \in H^1(M, \Sigma, \mathbb{C})$ as a local coordinate on the stratum $\mathcal{H}(\alpha)$. We will call these coordinates *period coordinates*.

The $\mathrm{SL}(2,\mathbb{R})$-action and the Kontsevich-Zorich cocycle. We write $\Phi(M,\omega)$ as a $2 \times n$ matrix x. The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ in these coordinates is linear. We choose some fundamental domain for the action of the mapping class group, and think of the dynamics on the fundamental domain. Then, the $\mathrm{SL}(2,\mathbb{R})$ action becomes

$$x = \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} \to gx = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} A(g,x),$$

where $A(g,x) \in \mathrm{Sp}(2g,\mathbb{Z}) \ltimes \mathbb{R}^{n-1}$ is the *Kontsevich-Zorich cocycle*. Thus, $A(g,x)$ is change of basis one needs to perform to return the point gx to the fundamental domain. It can be interpreted as the monodromy of the Gauss-Manin connection (restricted to the orbit of $SL(2,\mathbb{R})$).

6. The Hodge norm

Let M be a Riemann surface. By definition, M has a complex structure. Let $\mathcal{H}_M$ denote the set of holomorphic 1-forms on M. One can define *Hodge inner product* on $\mathcal{H}_M$ by

$$\langle \omega, \eta \rangle = \frac{i}{2} \int_M \omega \wedge \bar{\eta}.$$

We have a natural map $r : H^1(M,\mathbb{R}) \to \mathcal{H}_X$ which sends a cohomology class $\lambda \in H^1(M,\mathbb{R})$ to the holomorphic 1-form $r(\lambda) \in \mathcal{H}_X$ such that the real part of $r(\lambda)$ (which is a harmonic 1-form) represents λ. We can thus define the Hodge inner product on $H^1(M,\mathbb{R})$ by $\langle \lambda_1, \lambda_2 \rangle = \langle r(\lambda_1), r(\lambda_2) \rangle$. We have

$$\langle \lambda_1, \lambda_2 \rangle = \int_X \lambda_1 \wedge *\lambda_2,$$

where $*$ denotes the Hodge star operator, and we choose harmonic representatives of λ_1 and $*\lambda_2$ to evaluate the integral. We denote the associated norm by $\| \cdot \|_M$. This is the *Hodge norm*, see [FK].

If $x = (M,\omega) \in \mathcal{H}_1(\alpha)$, we will often write $\| \cdot \|_x$ to denote the Hodge norm $\| \cdot \|_M$ on on $H^1(M,\mathbb{R})$. Since $\| \cdot \|_x$ depends only on M, we have $\|\lambda\|_{kx} = \|\lambda\|_x$ for all $\lambda \in H^1(M,\mathbb{R})$ and all $k \in SO(2)$.

Let $E(x) = \mathrm{span}\{\Re(\omega), \Im(\omega)\}$.

For any $v \in E(x)$ and any point y in the $SL(2,\mathbb{R})$ orbit of x, the Hodge norm $\|v\|_y$ of v at y can be explicitly computed. In fact, the following elementary lemma holds:

Lemma 6.1. *Suppose* $x \in \mathcal{H}_1(\alpha)$*,* $g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL(2,\mathbb{R})$*,*

$$v = v_1 \mathfrak{R}(\omega) + v_2 \mathfrak{I}(\omega) \in E(x).$$

Let

$$\begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \tag{17}$$

Then,

$$\|v\|_{gx} = \|u_1^2 + u_2^2\|^{1/2}. \tag{18}$$

Proof. Let

$$c_1 = a_{11}\mathfrak{R}(\omega) + a_{12}\mathfrak{I}(\omega) \qquad c_2 = a_{21}\mathfrak{R}(\omega) + a_{22}\mathfrak{I}(\omega) \tag{19}$$

By the definition of the $SL(2,\mathbb{R})$ action, $c_1 + ic_2$ is holomorphic on gx. Therefore, by the definition of the Hodge star operator, at gx,

$$*c_1 = c_2, \qquad *c_2 = -c_1.$$

Therefore,

$$\|c_1\|_{gx}^2 = c_1 \wedge *c_1 = c_1 \wedge c_2 = (\det g)\mathfrak{R}(\omega) \wedge \mathfrak{I}(\omega) = 1,$$

where for the last equality we used the fact that $x \in \mathcal{H}_1(\alpha)$. Similarly, we get

$$\|c_1\|_{gx} = 1, \qquad \|c_2\|_{gx} = 1, \qquad \langle c_1, c_2\rangle_{gx} = 0. \tag{20}$$

Write

$$v = v_1\mathfrak{R}(\omega) + v_2\mathfrak{I}(\omega) = u_1 c_1 + u_2 c_2.$$

Then, in view of (19), u_1 and u_2 are given by (17). The equation (18) follows from (20). □

On the complementary subspace to $E(x)$ there is no explicit formula comparable to Lemma 6.1. However, we have the following fundamental result due to Forni [Fo, Corollary 2.1], see also [FoMZ, Corollary 2.1]:

Lemma 6.2. *There exists a continuous function* $\Lambda : \mathcal{H}_1(\alpha) \to (0,1)$ *such that; for any* $c \in H^1(M,\mathbb{R})$ *with* $c \wedge E(x) = 0$*, any* $x \in \mathcal{H}_1(\alpha)$ *and any* $t > 0$ *we have*

$$\|c\|_x e^{-\beta_t(x)} \le \|c\|_{a_t x} \le \|c\|_x e^{\beta_t(x)}$$

where $\beta_t(x) = \int_0^t \Lambda(a_\tau x)\, d\tau$*.*

Let $\mathcal{I}_1(t)$ and $\mathcal{I}_2(t)$ be as in Theorem 4.4. Now compactness of K_* and Lemma 6.2 imply that:

(21) there exists $\eta_2 > 0$ such that if $t > t_0$ and $\theta \in \mathcal{I}_1(t)$, then $\beta_t(r_\theta x) < (1 - \eta_2)t$.

7. Expansion on average of the Hodge norm.

We let $p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R})$ denote the natural projection. Let $\mathcal{M}_1$ be an affine invariant suborbitfold of $\mathcal{H}_1(\alpha)$ and let $\mathcal{M} = \mathbb{R}\mathcal{M}_1$ be as above. Then $\mathcal{M}$ is given by complex linear coordinated in period coordinates and is GL-invariant. We let L denote this subspace in $H^1(M, \Sigma, \mathbb{R})$.

Recall that $H^1(M, \mathbb{R})$ is endowed with a natural symplectic structure given by the wedge product on de Rham cohomology and also the Hodge inner product. It is shown in [AEM] that the wedge product restricted to $p(L)$ is non-degenerate. Therefore, there exists an $SL(2, \mathbb{R})$-invariant complement for $p(L)$ in $H^1(M, \mathbb{R})$ which we denote by $p(L)^\perp$.

We will use the following elementary lemma with $d = 2, 3$:

Lemma 7.1. *Let V be a d-dimensional vector space on which $SL(2, \mathbb{R})$ acts irreducibly, and let $\|\cdot\|$ be any $SO(2)$-invariant norm on V. Then there exists $\delta_0(d) > 0$ (depending on d), such that for any $\delta < \delta_0(d)$ any $t > 0$ and any $v \in V$,*

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\|a_t r_\theta v\|^\delta} \le \frac{e^{-k_d t}}{\|v\|^\delta},$$

where $k_d = k_d(\delta) > 0$.

Proof. This is a special case of [EMM, Lemma 5.1]. □

The space $H'(x)$ and the function ψ_x. For $x = (M, \omega)$, let

$$H'(x) = \{v \in H^1(M, \mathbb{C}) \;\;:\;\; v \wedge \overline{\omega} + \omega \wedge \overline{v} = 0\}.$$

We have, for any $x = (M, \omega)$,

$$H^1(M, \mathbb{C}) = \mathbb{R}\omega \oplus H'(x).$$

For $v \in H^1(M, \mathbb{C})$, let

$$\psi_x(v) = \frac{\|v\|_x}{\|v'\|_x} \quad \text{where } v = \lambda\omega + v',\ \lambda \in \mathbb{R},\ v' \in H'(x).$$

Then $\psi_x(v) \ge 1$, and $\psi_x(v)$ is bounded if v is bounded away from $\mathbb{R}\omega$.

7.1. Absolute Cohomology.

Fix some $\delta \le 0.1 \min(\eta_1, \eta_2, \delta_0(2), \delta_0(3))$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $v \in H^1(M, \mathbb{C})$, we write

$$gv = a\,\mathfrak{R}(v) + b\,\mathfrak{I}(v) + i(c\,\mathfrak{R}(v) + d\,\mathfrak{I}(v)). \tag{22}$$

Lemma 7.2. *There exists $C_0 > 0$ such that for all $x = (M, \omega) \in \mathcal{H}_1(\alpha)$, all $t > 0$ and all $v \in H^1(M, \mathbb{C})$ we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(\|a_t r_\theta v\|_{a_t r_\theta x})^{\delta/2}} \leq \min\left(\frac{C_0}{\|v\|_x^{\delta/2}}, \frac{\psi_x(v)^{\delta/2}\kappa(x,t)}{\|v\|_x^{\delta/2}} \right), \tag{23}$$

where

(a) $\kappa(x,t) \leq C_0$ *for some $C_0 > 1$ and for all x and all t, and*
(b) *There exists $C > 0$, $\eta > 0$, $L_0 > 0$ and $\eta_0 > 0$ such that*

$$\kappa(x,t) \leq Ce^{-\eta t}, \qquad \textit{provided } \log u(x) < L_0 + \eta_0 t$$

Proof. For $x = (M, \omega) \in \mathcal{H}_1(\alpha)$ we have an $\mathrm{SL}(2,\mathbb{R})$-invariant and Hodge-orthogonal decomposition

$$H^1(M,\mathbb{R}) = E(x) \oplus H^1(M,\mathbb{R})^\perp,$$

where $E(x) = \mathrm{span}\{\Re(\omega), \Im(\omega)\}$ and $H^1(M,\mathbb{R})^\perp(x) = \{c \in H^1(M,\mathbb{R}) : c \wedge E(x) = 0\}$. For a subspace $V \subset H^1(M,\mathbb{R})$, let $V_\mathbb{C} \subset H^1(M,\mathbb{C})$ denote its complexification. Then, we have

$$H^1(M,\mathbb{C}) = E_\mathbb{C}(x) + H^1(M,\mathbb{R})^\perp_\mathbb{C}(x). \tag{24}$$

Note that $H^1(M,\mathbb{R})^\perp_\mathbb{C}(x) \subset H'(x)$. We can write

$$v = \lambda\omega + u + w,$$

where $\lambda \in \mathbb{R}$, $u \in E_\mathbb{C}(x) \cap H'(x)$, $w \in H^1(M,\mathbb{R})^\perp_\mathbb{C}(x)$. Since $u \in E_\mathbb{C}(x)$, we may write

$$u = u_{11}\Re(\omega) + u_{12}\Im(\omega) + i(u_{21}\Re(\omega) + u_{22}\Im(\omega)).$$

Since $u \in H'(x)$,

$$u_{11} + u_{22} = 0. \tag{25}$$

Recall that the Hilbert-Schmidt norm $\|\cdot\|_{HS}$ of a matrix is the square root of the sum of the squares of the entries. Then,

$$\begin{aligned}(\|a_t r_\theta(\lambda\omega + u)\|_{a_t r_\theta x})^2 &= \left\| (a_t r_\theta) \begin{pmatrix} \lambda + u_{11} & u_{12} \\ u_{21} & \lambda + u_{22} \end{pmatrix} (a_t r_\theta)^{-1} \right\|_{HS}^2 && \text{by Lemma 6.1 and (22)} \\ &= \lambda^2 + \left\| (a_t r_\theta) \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} (a_t r_\theta)^{-1} \right\|_{HS}^2 && \text{by (25).} \qquad (26)\end{aligned}$$

Since the decomposition (24) is Hodge orthogonal, it follows that for all t and all θ,

$$(\|a_t r_\theta v\|_{a_t r_\theta x})^2 = \lambda^2 + (\|a_t r_\theta u\|_{a_t r_\theta x})^2 + (\|a_t r_\theta w\|_{a_t r_\theta x})^2. \tag{27}$$

By (26), (25) and Lemma 7.1,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(\|a_t r_\theta u\|_{a_t r_\theta x})^{\delta/2}} \leq e^{-k_3 t}\|u\|_x^{\delta/2}, \tag{28}$$

where $k_3 > 0$. We now claim that

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\|a_t r_\theta w\|^{\delta/2}} \le \frac{\kappa_2(x,t)}{\|w\|^{\delta/2}}, \tag{29}$$

where $\kappa_2(x,t)$ is bounded and

$$\kappa(x,t) \le Ce^{-\eta t}, \qquad \text{provided } \log u(x) < L_0 + \eta_0 t.$$

Assuming (29), we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{(\|a_t r_\theta v\|_{a_t r_\theta x})^{\delta/2}}\\
&\le \frac{1}{2\pi}\int_0^{2\pi}\min\left(\frac{1}{\lambda^{\delta/2}}, \frac{1}{(\|a_t r_\theta u\|_{a_t r_\theta x})^{\delta/2}}, \frac{1}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}}\right)d\theta && \text{by (27)}\\
&\le \min\left(\frac{1}{\lambda^{\delta/2}}, \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{(\|a_t r_\theta u\|_{a_t r_\theta x})^{\delta/2}}, \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}}\right)\\
&\le \min\left(\frac{1}{\lambda^{\delta/2}}, \frac{e^{-k_3 t}}{\|u\|_x^{\delta/2}}, \frac{\kappa_2(x,t)}{\|w\|_x^{\delta/2}}\right) && \text{by (28) and (29).}
\end{aligned}$$

Since we must have either $\lambda > \|v\|_x/3$, or $\|u\|_x > \|v\|_x/3$ or $\|w\|_x > \|v\|_x/3$, we have for all x, t,

$$\min\left(\frac{1}{\lambda^{\delta/2}}, \frac{e^{-k_3 t}}{\|u\|_x^{\delta/2}}, \frac{\kappa_2(x,t)}{\|w\|_x^{\delta/2}}\right) \le \frac{(3\max(1, e^{-k_3 t}, \kappa_2(x,t)))^{\delta/2}}{\|v\|_x^{\delta/2}} \le \frac{C_0}{\|v\|_x^{\delta/2}}.$$

where for the last estimate we used the fact that both k_3 and κ_2 are bounded functions. Also, we have $\|u+w\|_x = \psi_x(v)^{-1}\|v\|_x$, hence either $\|u\| \ge \psi_x(v)^{-1}\|v\|_x/2$ or $\|w\|_x \ge \psi_x(v)^{-1}\|v\|_x/2$, and therefore, for all x, t,

$$\min\left(\frac{1}{\lambda^{\delta/2}}, \frac{e^{-k_3 t}}{\|u\|_x^{\delta/2}}, \frac{\kappa_2(x,t)}{\|w\|_x^{\delta/2}}\right) \le \frac{(\psi_x(v)^{\delta/2}\max(e^{-k_3 t}, \kappa_2(x,t)))^{\delta/2}}{\|v\|_x^{\delta/2}} \equiv \frac{\psi_x(v)^{\delta/2}\kappa(x,t)}{\|v\|_x^{\delta/2}}.$$

Therefore, (23) holds. This completes the proof of the lemma, assuming (29).

It remains to prove (29). Let L_0 and η_0 be as in Theorem 4.4, and suppose $\log u(x) < L_0 + \eta_0 t$. We have

$$\int_0^{2\pi}\frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} = \int_{\mathcal{I}_1(t)}\frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} + \int_{\mathcal{I}_2(t)}\frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}}.$$

Using (16), (21) and Lemma 6.2 we get

$$\int_{\mathcal{I}_2(t)}\frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} \le \frac{e^{-\eta_1 t}e^{\delta t/2}}{\|v\|^{\delta/2}}.$$

Also,

$$\begin{aligned}
\int_{\mathcal{I}_1(t)} \frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} &\leq \int_{\mathcal{I}_1(t)} \frac{d\theta}{(\|\Re(a_t r_\theta w)\|_{a_t r_\theta x})^{\delta/2}} && \text{since } \|z\| \geq |\Re(z)\| \\
&= \int_{\mathcal{I}_1(t)} \frac{d\theta}{(\|e^t \Re(r_\theta w)\|_{a_t r_\theta x})^{\delta/2}} && \text{by (22)} \\
&\leq \int_{\mathcal{I}_1(t)} \frac{e^{-(1-\beta_t(r_\theta x))\delta t/2}}{\|\Re(r_\theta w)\|_x^{\delta/2}} && \text{by Lemma 6.2} \\
&\leq e^{-\eta_2 \delta t/2} \int_0^{2\pi} \frac{d\theta}{\|\Re(r_\theta w)\|_x^{\delta/2}} && \text{by (21)} \\
&= e^{-\eta_2 \delta t/2} \int_0^{2\pi} \frac{d\theta}{\|\cos\theta\, \Re(w) + \sin\theta\, \Im(w)\|_x^{\delta/2}} \\
&\leq \frac{C_2 e^{-\eta_2 \delta t/2}}{\|w\|^{\delta/2}}. && \text{since the integral converges.}
\end{aligned}$$

These estimates imply (29) for the case when $\log u(x) < L_0 + \eta_0 t$. If x is arbitrary, note that

$$\begin{aligned}
\|a_t r_\theta w\|_{a_t r_\theta x} &\geq \|\Re(a_t r_\theta w)\|_{a_t r_\theta x} && \text{since } \|z\| \geq \|\Re(z)\| \\
&= \|e^t(\cos\theta\, \Re(w) + \sin\theta\, \Im(w))\|_{a_t r_\theta x} && \text{by (22)} \\
&= e^t \|\cos\theta\, \Re(w) + \sin\theta\, \Im(w)\|_{a_t r_\theta x} \\
&\geq \|\cos\theta\, \Re(w) + \sin\theta\, \Im(w)\|_x && \text{by Lemma 6.2.}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} &\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\|\cos\theta\, \Re(w) + \sin\theta\, \Im(w)\|_x^{\delta/2}} \\
&\leq \frac{C_2}{\|w\|_x^{\delta/2}} && \text{since the integral converges}
\end{aligned}$$

This completes the proof of (29) for arbitrary x. □

7.2. **The Modified Hodge Norm.** For the application in §7.3, we will need to consider a modification of the Hodge norm in the thin part of moduli space.

The classes c_α, $*c_\alpha$. Let α be a homology class in $H_1(X, \mathbb{R})$. We can define the cohomology class $*c_\alpha \in H^1(X, \mathbb{R})$ so that for all $\omega \in H^1(X, \mathbb{R})$,

$$\int_\alpha \omega = \int_X \omega \wedge *c_\alpha.$$

Then,

$$\int_X *c_\alpha \wedge *c_\beta = I(\alpha, \beta),$$

where $I(\cdot,\cdot)$ denotes algebraic intersection number. We have, for any $\omega \in H^1(X,\mathbb{R})$,

$$\langle \omega, c_\alpha \rangle = \int_X \omega \wedge *c_\alpha = \int_\alpha \omega.$$

We note that $*c_\alpha$ is a purely topological construction which depends only on α, but c_α depends also on the complex structure of X.

Fix $\epsilon_* > 0$ (the *Margulis constant*) so that any two curves of hyperbolic length less than ϵ_* must be disjoint.

Let α be a simple closed curve on a Riemann surface X. Let $\ell_\alpha(\sigma)$ denote the length of the geodesic representative of α in the hyperbolic metric which is in the conformal class of X. We recall the following:

Theorem 7.3. [ABEM, Theorem 3.1] *For any constant $D > 1$ there exists a constant $c > 1$ such that for any simple closed curve α with $\ell_\alpha(\sigma) < D$,*

$$\frac{1}{c}\ell_\alpha(\sigma)^{1/2} \le \|c_\alpha\| < c\,\ell_\alpha(\sigma)^{1/2}. \tag{30}$$

Furthermore, if $\ell_\alpha(\sigma) < \epsilon_0$ and β is the shortest simple closed curve crossing α, then

$$\frac{1}{c}\ell_\alpha(\sigma)^{-1/2} \le \|c_\beta\| < c\,\ell_\alpha(\sigma)^{-1/2}.$$

Short bases. Suppose $(X,\omega) \in \mathcal{H}_1(\alpha)$. Fix $\epsilon_1 < \epsilon_0$ and let $\alpha_1, \dots, \alpha_k$ be the curves with hyperbolic length less than ϵ_1 on X. For $1 \le i \le k$, let β_i be the shortest curve in the flat metric defined by ω with $i(\alpha_i, \beta_i) = 1$. We can pick simple closed curves γ_r, $1 \le r \le 2g-2k$ on X so that the hyperbolic length of each γ_r is bounded by a constant L depending only on the genus, and so that the α_j, β_j and γ_j are a symplectic basis $\mathcal{S}$ for $H_1(X,\mathbb{R})$. We will call such a basis *short.*

We now define a modification of the Hodge norm, which is similar to the one used in [ABEM]. The modified norm is defined on the tangent space to the space of pairs (X,ω) where X is a Riemann surface and ω is a holomorphic 1-form on X. Unlike the Hodge norm, the modified Hodge norm will depend not only on the complex structure on X but also on the choice of a holomorphic 1-form ω on X. Let $\{\alpha_i, \beta_i, \gamma_r\}_{1\le i\le k, 1\le r\le 2g-2k}$ be a short basis for (X,ω).

We can write any $\theta \in H^1(X,\mathbb{R})$ as

$$\theta = \sum_{i=1}^{k} a_i(*c_{\alpha_i}) + \sum_{i=1}^{k} b_i \ell_{\alpha_i}(\sigma)^{1/2}(*c_{\beta_i}) + \sum_{r=1}^{2g-2k} u_i(*c_{\gamma_r}), \tag{31}$$

where σ denotes the hyperbolic metric in the conformal class of X, and for a curve α on X, $\ell_\alpha(\sigma)$ denotes the length of α in the metric σ. We then define

$$\|\theta\|'' = \|\theta\| + \left(\sum_{i=1}^{k} |a_i| + \sum_{i=1}^{k} |b_i| + \sum_{r=1}^{2g-2k} |u_r| \right). \tag{32}$$

From (32) we have for $1 \leq i \leq k$,

$$\| *c_{\alpha_i}\|'' \approx 1, \tag{33}$$

as long as α_i has no flat annulus in the metric defined by ω. Similarly, we have

$$\| *c_{\beta_i}\|'' \approx \| *c_{\beta_i}\| \approx \frac{1}{\ell_{\alpha_i}(\sigma)^{1/2}}. \tag{34}$$

In addition, in view of Theorem 7.3, if γ is any other moderate length curve on X, $\| *c_\gamma\|'' \approx \| *c_\gamma\| = O(1)$. Thus, if $\mathcal{B}$ is a short basis associated to ω, then for any $\gamma \in \mathcal{B}$,

$$\operatorname{Ext}_\gamma(\omega)^{1/2} \approx \|{*c_\gamma}\| \leq \|{*c_\gamma}\|'' \tag{35}$$

(By $\operatorname{Ext}_\gamma(\omega)$ we mean the extremal length of γ in X, the conformal structure defined by ω).

Remark. From the construction, we see that the modified Hodge norm is greater than the Hodge norm. Also, if the flat length of shortest curve in the flat metric defined by ω is greater than ϵ_1, then for any cohomology class λ, for some C depending on ϵ_1 and the genus,

$$\|\lambda\|'' \leq C\|\lambda\|, \tag{36}$$

i.e. the modified Hodge norm is within a multiplicative constant of the Hodge norm.

From the defintion, we have the following:

Lemma 7.4. *There exists a constant $C > 1$ depending only on the genus such that for any $t > 0$, any $x \in \mathcal{H}_1(\alpha)$ and any $\lambda \in H^1(M, \mathbb{R})$,*

$$C^{-1} e^{-2t} \|\lambda\|''_x \leq \|\lambda\|''_{a_t x} \leq C e^{2t} \|\lambda\|''_x.$$

Proof. From the definition of $\| \cdot \|''$,

$$C_1^{-1}\|\lambda\| \leq \|\lambda\|'' \leq C_1 \ell_{hyp}(x)^{-1/2} \|\lambda\|, \tag{37}$$

where C depends only on the genus, and $\ell_{hyp}(x)$ is the hyperbolic length of the shortest closed curve on x. It is well known that for very short curves, the hyperbolic length is comparable to the extremal length, see e.g. [Maskit]. It follows immediately from Kerckhoff's formula for the Teichmüller distance that

$$e^{-2t} \operatorname{Ext}_\gamma(x) \leq \operatorname{Ext}_\gamma(a_t x) \leq e^{2t} \operatorname{Ext}_\gamma(x).$$

Therefore,

$$C_2 e^{-2t}\ell_{hyp}(x) \le \ell_{hyp}(a_t x) \le C_2 e^{2t}\ell_{hyp}(x), \tag{38}$$

where C_2 depends only on the genus. Now the lemma follows immediately from (37), (38) and Lemma 6.2. $\square$

One annoying feature of our definition is that for a fixed absolute cohomology class λ, $\|\lambda\|''_x$ is not a continuous function of x, as x varies in a Teichmüller disk, due to the dependence on the choice of short basis. To remedy this, we pick a positive continuous $SO(2)$-bi-invariant function ϕ on $SL(2,\mathbb{R})$ supported on a neighborhood of the identity e such that $\int_{SL(2,\mathbb{R})}\phi(g)\,dg = 1$, and define

$$\|\lambda\|'_x = \|\lambda\|_x + \int_{SL(2,\mathbb{R})} \|\lambda\|''_{gx}\,\phi(g)\,dg.$$

Then, it follows from Lemma 7.4 that for a fixed λ, $\log\|\lambda\|'_x$ is uniformly continuous as x varies in a Teichmüller disk. In fact, there is a constant m_0 such that for all $x \in \mathcal{H}_1(\alpha)$, all $\lambda \in H^1(M,\mathbb{R})$ and all $t > 0$,

$$e^{-m_0 t}\|\lambda\|'_x \le \|\lambda\|'_{a_t x} \le e^{m_0 t}\|\lambda\|'_x. \tag{39}$$

7.3. **Relative cohomology.** For $c \in H^1(M,\Sigma,\mathbb{R})$ and $x = (M,\omega) \in \mathcal{H}_1(\alpha)$, let $\mathfrak{p}_x(c)$ denote the harmonic representative of $p(c)$, where $p : H^1(M,\Sigma,\mathbb{R}) \to H^1(M,\mathbb{R})$ is the natural map. We view $\mathfrak{p}_x(c)$ as an element of $H^1(M,\Sigma,\mathbb{R})$. Then, (similarly to [EMR]) we define the Hodge norm on $H^1(M,\Sigma,\mathbb{R})$ as

$$\|c\|'_x = \|p(c)\|'_x + \sum_{(z,z')\in\Sigma\times\Sigma}\left|\int_{\gamma_{z,z'}}(c - \mathfrak{p}_x(c))\right|,$$

where $\gamma_{z,z'}$ is any path connecting the zeroes z and z' of ω. Since $c - \mathfrak{p}_x(c)$ represents the zero class in absolute cohomology, the integral does not depend on the choice of $\gamma_{z,z'}$. Note that the $\|\cdot\|'$ norm on $H^1(M,\Sigma,\mathbb{R})$ is invariant under the action of $SO(2)$.

As above, we pick a positive continuous $SO(2)$-bi-invariant function ϕ on $SL(2,\mathbb{R})$ supported on a neighborhood of the identity e such that $\int_{SL(2,\mathbb{R})}\phi(g)\,dg = 1$, and define

$$\|\lambda\|_x = \int_{SL(2,\mathbb{R})} \|\lambda\|'_{gx}\,\phi(g)\,dg. \tag{40}$$

Then, the $\|\cdot\|$ norm on $H^1(M,\Sigma,\mathbb{R})$ is also invariant under the action of $SO(2)$.

Notational warning. If λ is an absolute cohomology class, then $\|\lambda\|_x$ denotes the Hodge norm of λ at x defined in §6. If, however λ is a relative cohomology class, then $\|\lambda\|_x$ is defined in (40). We hope the meaning will be clear from the context.

We will use the following crude version of Lemma 6.2 (much more accurate versions are possible, especially in compact sets, see e.g. [EMR]).

Lemma 7.5. *There exists a constant $m' > m_0 > 0$ such that for any $x \in \mathcal{H}_1(\alpha)$, any $\lambda \in H^1(M, \Sigma, \mathbb{R})$ and any $t > 0$,*

$$e^{-m't}\|\lambda\|_x \leq \|\lambda\|_{a_t x} \leq e^{m't}\|\lambda\|_x$$

Proof. We remark that this proof fails if we use the standard Hodge norm on absolute homology. It is enough to prove the statement assuming $0 \leq t \leq 1$, since the statement for arbitrary t then follows by iteration. It is also enough to check this for the case when $p(\lambda) = *c_\gamma$, where γ is an element of a short basis.

Let $\alpha_1, \ldots, \alpha_n$ be the curves with hyperbolic length less than ϵ_0. For $1 \leq k \leq n$, let β_k be the shortest curve with $i(\alpha_k, \beta_k) = 1$, where $i(\cdot, \cdot)$ denotes the geometric intersection number. Let γ_r, $1 \leq r \leq 2g - 2k$ be moderate length curves on X so that the α_j, β_j and γ_j are a symplectic basis $\mathcal{S}$ for $H_1(X, \mathbb{R})$. Then $\mathcal{S}$ is a short basis for $x = (M, \omega)$.

We now claim that for any curve $\gamma \in \mathcal{S}$, and any i, j

$$\left| \int_{\zeta_{ij}} *\gamma \right| \leq C\|\gamma\|''_x, \tag{41}$$

where C is a universal constant, and ζ_{ij} is the path connecting the zeroes z_i and z_j of ω and minimizing the hyperbolic distance. (Of course since $*\gamma$ is harmonic, only the homotopy class of ζ_{ij} matters in the integral on the right hand side of (41)).

It is enough to prove (41) for the α_k and the β_k (the estimate for other $\gamma \in \mathcal{S}$ follows from a compactness argument).

We can find a collar region around α_k as follows: take two annuli $\{z_k \ : \ 1 > |z_k| > |t_k|^{1/2}\}$ and $\{w_k \ : \ 1 > w_k > |t_k|^{1/2}\}$ and identify the inner boundaries via the map $w_k = t_k/z_k$. (This coordinate system is used in e.g. [Fa, Chapter 3], also [Mas1], [Fo], [Wo, §3] and elsewhere). The hyperbolic metric σ in the collar region is approximately $|dz|/(|z||\log|z||)$. Then $\ell_{\alpha_k}(\sigma) \approx 1/|\log t_k|$. By [Fa, Chapter 3] any holomorphic 1-form ω can be written in the collar region as

$$\left(a_0(z_k + t_k/z_k, t_k) + \frac{a_1(z_k + t_k/z_k, t_k)}{z_k} \right) dz_k,$$

where a_0 and a_1 are holomorphic in both variables. (We assume here that the limit surface on the boundary of Teichmüller space is fixed). This implies that as $t_k \to 0$,

$$\omega = \left(\frac{a}{z_k} + h(z_k) + O(t_k/z_k^2) \right) dz_k$$

where h is a holomorphic function which remains bounded as $t_k \to 0$, and the implied constant is bounded as $t_k \to 0$. (Note that when $|z_k| \geq |t_k|^{1/2}$, $|t_k/z_k^2| \leq 1$). Now from the condition $\int_{\alpha_k} *c_{\beta_k} = 1$ we see that on the collar of α_j,

$$c_{\beta_k} + i *c_{\beta_k} = \left(\frac{\delta_{kj}}{(2\pi) z_j} + h_{kj}(z_j) + O(t_j/z_j^2) \right) dz_j, \tag{42}$$

where the h_{kj} are holomorphic and bounded as $t_j \to 0$. (We use the notation $\delta_{kj} = 1$ if $k = j$ and zero otherwise). Also from the condition $\int_{\beta_k} *c_{\alpha_k} = 1$ we have

$$c_{\alpha_k} + i *c_{\alpha_k} = \frac{i}{|\log t_j|} \left(\frac{\delta_{kj}}{z_j} + s_{kj}(z_j) + O(t_j/z_k^2) \right) dz_j, \tag{43}$$

where s_{kj} also remains holomorphic and is bounded as $t_j \to 0$.

Then, on the collar of α_j,

$$*c_{\alpha_k} = \frac{\delta_{jk}}{|\log t_j|} d \log |z_j|^2 + \text{ bounded 1-form}$$

and thus,

$$\left| \int_{\zeta_{ij}} *c_{\alpha_k} \right| = O(1).$$

Also, on the collar of α_j,

$$*c_{\beta_k} = \frac{\delta_{jk}}{2\pi} d \arg |z_j| + \text{ bounded 1-form}$$

and so

$$\left| \int_{\zeta_{ij}} *c_{\beta_k} \right| = O(1).$$

By Theorem 7.3,

$$\| * c_{\alpha_k} \|'' \approx O(1) \quad \text{ and } \quad \| * c_{\beta_k} \|'' \approx \| * c_{\beta_k} \| \approx \ell_{\alpha_k}(\sigma)^{1/2} \gg 1.$$

Thus, (41) holds for $*c_{\beta_k}$ and $*c_{\alpha_k}$, and therefore for any $\gamma \in \mathcal{S}$. By the definition of $\| \cdot \|''$, (41) holds for any $\lambda \in H^1(M, \Sigma, \mathbb{R})$. For $0 \leq t \leq 1$, let θ_t denote the harmonic representative of $p(\lambda)$ on $g_t x$. Then, for $0 \leq t < 1$,

$$\begin{aligned} \|\lambda\|'_{g_t x} &= \|p(\lambda)\|'_{g_t x} + \sum_{i,j} \left| \int_{z_i}^{z_j} (\lambda - \theta_t) \right| \\ &\leq C\|p(\lambda)\|'_x + \sum_{i,j} \left| \int_{z_i}^{z_j} (\lambda - \theta_0) \right| + \sum_{i,j} \left| \int_{z_i}^{z_j} (\theta_t - \theta_0) \right| && \text{by (39)} \\ &\leq C\|\lambda\|'_x + \sum_{i,j} \left| \int_{\gamma_{ij}} (\theta_t - \theta_0) \right| \\ &\leq C\|\lambda\|'_x + C\sum_{i,j} (\|p(\lambda)\|'_{g_t x} + \|p(\lambda)\|'_x) && \text{by (41)} \\ &\leq C'\|\lambda\|'_x && \text{by (39)} \end{aligned}$$

Therefore, there exists m' such that for $0 \leq t \leq 1$ and any $\lambda \in H^1(M, \Sigma, \mathbb{R})$,

$$\|\lambda\|_{g_t x} \leq e^{m't}\|\lambda\|_x.$$

This implies the lemma for all t. $\square$

In the sequel we will need to have a control of the matrix coefficients of the cocycle. Let $x \in \mathcal{H}_1(\alpha)$ and $t \in \mathbb{R}$ we let $A(x,t) \equiv A(x, a_t)$ denote the cocycle. Using the map p above we may write

$$A(x,t) = \begin{pmatrix} I & U(x,t) \\ 0 & S(x,t) \end{pmatrix} \tag{44}$$

(Note that the action of the cocycle on $\ker p$ is trivial.)

The following is an immediate corollary of Lemma 7.5:

Lemma 7.6. *There is some $m' \in \mathbb{N}$ such that for all $x \in \mathcal{H}_1(\alpha)$ and all $g \in \mathrm{SL}(2,\mathbb{R})$ we have*

$$\|U(x,t)\| \leq e^{m'|t|},$$

where

$$\|U(x,t)\| \equiv \sup_{c \in H^1(M,\Sigma,\mathbb{R})} \frac{\|\mathfrak{p}_x(c) - \mathfrak{p}_{a_t x}(c)\|}{\|p(c)\|'_x}. \tag{45}$$

Note that since $\mathfrak{p}_x(c) - \mathfrak{p}_{a_t x}(c) \in \ker p$, $\|\mathfrak{p}_x(c) - \mathfrak{p}_{a_t x}(c)\|_y$ is independent of y.

Suppose $L \subset H^1(M, \Sigma, \mathbb{R})$ is a subspace such that $p(L) \subset H^1(M, \mathbb{R})$ is symplectic (in the sense that the intersection form restricted to $p(L)$ is non-degenerate). Let $p(L)^\perp$ denote the symplectic complement of $p(L)$ in $H^1(M, \mathbb{R})$. Suppose $x \in \mathcal{H}_1(\alpha)$. For any $c \in H^1(M, \Sigma, \mathbb{R})$ we may write

$$c = h + c' + v,$$

where h is harmonic with $p(h) \in p(L)^\perp$, $v \in L$ and $c' \in \ker p$. This decomposition is not unique since for $u \in L \cap \ker p$, we can replace c' by $c'+u$ and v by $v-u$. We denote the c' with smallest possible $\|\cdot\|_x$ norm by c'_L. Thus, we have the decomposition

$$c = \mathfrak{p}_{x,L}(c) + c'_L + v, \tag{46}$$

where $\mathfrak{p}_{x,L}(c)$ is the harmonic representative at x of $p_L(c) \equiv \pi_{L^\perp}(p(c))$, $c'_L \in \ker p$, $v \in L$, and c'_L has minimal norm.

Define $\nu_{x,L} : H^1(M, \Sigma, \mathbb{R}) \to \mathbb{R}$ by

$$\nu_{x,L}(c) = \begin{cases} \max\{\|c'_L\|_x, (\|p_L(c)\|'_x)^{1/2}\} & \text{if } \max\{\|c'_L\|_x, \|p_L(c)\|'_x\} \le 1 \\ 1 & \text{otherwise.} \end{cases}$$

We record (without proof) some simple properties of $\nu_{x,L}$.

Lemma 7.7. *We have*

(a) $\nu_{x,L}(c) = 0$ *if an only if* $c \in L$.
(b) *For* $v \in L$, $\nu_{x,L}(c+v) = \nu_{x,L}(v)$.
(c) *For* $v' \in \ker p$, $\nu_{x,L}(c) - \|v'\|_x \le \nu_{x,L}(c+v') \le \nu_{x,L}(c) + \|v'\|_x$.

In view of Lemma 7.7, for an affine subspace $\mathcal{L} = v_0 + L$ of $H^1(M, \Sigma, \mathbb{R})$, we can define $\nu_{x,\mathcal{L}}(c)$ to be $\nu_{x,L}(c - v_0)$.

Extend $\nu_{x,\mathcal{L}}$ to $H^1(M, \Sigma, \mathbb{C})$ by

$$\nu_{x,\mathcal{L}}(c_1 + ic_2) = \max\{\nu_{x,\mathcal{L}}(c_1), \nu_{x,\mathcal{L}}(c_2)\}.$$

For an affine subspace $\mathcal{L} \subset H^1(M, \Sigma, \mathbb{R})$, let $\mathcal{L}_\mathbb{C} \subset H^1(M, \Sigma, \mathbb{C})$ denote the complexification $\mathbb{C} \otimes \mathcal{L}$. We use the notation (here we are working in period coordinates)

$$d'(x, \mathcal{L}_\mathbb{C}) = \nu_{x,\mathcal{L}}(x - v)$$

where v is any vector in $\mathcal{L}_\mathbb{C}$ (and the choice of v does not matter by Lemma 7.7 (b)). Note that $d'(\cdot, \mathcal{L})$ is defined only if $\mathcal{L} = v_0 + L$ where $p(L)$ is symplectic. We think of $d'(x, \mathcal{L})$ as measuring the distance between x and $\mathcal{L}_\mathbb{C} \subset H^1(M, \mathbb{C})$. In view of Lemma 7.5, we have for all $t > 0$

$$e^{-m't} d'(x, \mathcal{L}) \le d'(a_t x, a_t \mathcal{L}) \le e^{m't} d'(x, \mathcal{L}). \tag{47}$$

Lemma 7.8. *Let the notation be as above. Then, there exists constants* $C_0 > 0$, $L_0 > 0$, $\eta'_0 > 0$, $\eta_3 > 0$, $t'_0 > 0$ *and continuous functions* $\kappa_1 : \mathcal{H}_1(\alpha) \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $b : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

- $\kappa_1(x,t) \le C_0 e^{m'\delta t}$ *for all* $x \in \mathcal{H}_1(\alpha)$ *and all* $t > 0$
- $\kappa_1(x,t) \le e^{-\eta_3 t}$ *for all* $x \in \mathcal{H}_1(\alpha)$ *and* $t > t'_0$ *with* $\log u(x) < L_0 + \eta'_0 t$,

so that for any affine subspace $\mathcal{L} \subset H^1(M, \Sigma, \mathbb{R})$ *such that for all* $x = (M, \omega) \in \mathcal{L}$, $E(x) \equiv \operatorname{span}(\Re(\omega), \Im(\omega))$ *is contained in the linear part of* $\mathcal{L}$*, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{d'(a_t r_\theta x, a_t r_\theta \mathcal{L}_{\mathbb{C}})^\delta} \leq \frac{\kappa_1(x,t)}{d'(x, \mathcal{L}_{\mathbb{C}})^\delta} + b(t) \tag{48}$$

Proof. We may assume that

$$d'(x, \mathcal{L}_{\mathbb{C}}) = \nu_{x,\mathcal{L}}(v) = \max(\|v'\|_x, (\|p(v)\|'_x)^{1/2}), \tag{49}$$

where

$$v = \mathfrak{p}_x(v) + v', \qquad p(v) \in p(L_{\mathbb{C}})^\perp, \ L \text{ is the linear part of } \mathcal{L}, \text{ and } v' \in \ker p. \tag{50}$$

Suppose $d'(x, \mathcal{L}) = \nu_{x,\mathcal{L}}(v) \geq \frac{1}{2}e^{-3m't}$. Then we have the crude estimate

$$d'(a_t r_\theta x, a_t r_\theta \mathcal{L})^{-\delta} \leq d'(a_t r_\theta x, a_t r_\theta \mathcal{L})^{-1} \leq 2e^{5m't}$$

and thus (48) holds with $b(t) = 2e^{5m't}$. Hence, we may assume that $\nu_{x,\mathcal{L}}(v) < \frac{1}{2}e^{-3m't}$. Then,

$$\begin{aligned}
e^{2m't}(\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2} &\leq e^{(2m'+0.5)t}(\|p(v)\|'_{a_t r_\theta x})^{1/2} && \text{by (22)}\\
&\leq e^{(2m'+0.5+0.5m_0)t}(\|p(v)\|'_x)^{1/2} && \text{by (39)}\\
&\leq e^{3m't}\nu_{x,\mathcal{L}}(v) && \text{since } m' > m_0 > 1\\
&\leq \tfrac{1}{2}. && (51)
\end{aligned}$$

Let us introduce the notation, for $u \in \ker p$,

$$\|u\|_{\mathcal{L}} = \inf\{\|u - w\|, \ : \ w \in \mathcal{L} \cap \ker p\}.$$

Then

$$d'(a_t r_\theta x, a_t r_\theta \mathcal{L}_{\mathbb{C}}) = \max((\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2}, \|a_t r_\theta v - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}}). \tag{52}$$

But,

$$\begin{aligned}
&\|a_t r_\theta v - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}} =\\
&\quad = \|a_t r_\theta(v' + \mathfrak{p}_x(v)) - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}} && \text{by (50)}\\
&\quad = \|a_t r_\theta v' + \mathfrak{p}_x(a_t r_\theta v) - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}}\\
&\quad \geq \|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} - \|\mathfrak{p}_x(a_t r_\theta v) - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\| && \text{by the reverse triangle inequality}\\
&\quad \geq \|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} - \|U(r_\theta x, t)\|\|p(a_t r_\theta v)\|'_x && \text{by (45)}\\
&\quad \geq \|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} - e^{2m't}\|p(a_t r_\theta v)\|'_{a_t r_\theta x} && \text{by Lemma 7.6.} \quad (53)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&d'(a_t r_\theta x, a_t r_\theta \mathcal{L}_{\mathbb{C}}) = \\
&= \max\left((\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2}, \|a_t r_\theta v - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}}\right) && \text{by (52)}\\
&\geq \frac{1}{2}\left(\|a_t r_\theta v - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}} + (\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2}\right)\\
&\geq \frac{1}{2}\left(\|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} - e^{2m't}\|p(a_t r_\theta v)\|'_{a_t r_\theta x} + (\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2}\right) && \text{by (53)}\\
&= \frac{1}{2}\left(\|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} + (\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2}(1 - e^{2m't}(\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2})\right)\\
&\geq \frac{1}{2}\left(\|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} + \frac{1}{2}(\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2}\right) && \text{by (51)}\\
&\geq \frac{1}{2}\left(\|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} + \frac{1}{2}(\|p(a_t r_\theta v)\|_{a_t r_\theta x})^{1/2}\right) && \text{since } \|\cdot\|' \geq \|\cdot\|
\end{aligned}
$$

However, since the action of the cocycle on $\ker p$ is trivial, $v' \in \ker p$ and $\mathcal{L}$ is invariant,

$$\|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} = \|a_t r_\theta v'\|.$$

Then,

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{d'(a_t r_\theta x, a_t r_\theta \mathcal{L}_{\mathbb{C}})^\delta} \leq \\
&\qquad \leq \frac{1}{2\pi}\int_0^{2\pi} 4\min\left(\frac{1}{\|a_t r_\theta v'\|^\delta}, \frac{1}{\|a_t r_\theta p(v)\|^{\delta/2}}\right) d\theta\\
&\qquad \leq 4\min\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\|a_t r_\theta v'\|^\delta}, \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\|a_t r_\theta p(v)\|^{\delta/2}}\right) d\theta\\
&\qquad \leq 4\min\left(\frac{e^{-k_2(\delta)t}}{\|v'\|_x^\delta}, \frac{\min(C_0, \psi_x(p(v))^{\delta/2}\kappa(x,t))}{\|p(v)\|_x^{\delta/2}}\right) \text{by Lemma 7.1 and Lemma 7.2}
\end{aligned}
$$

Let $\eta_0' > 0$ be a constant to be chosen later. Suppose $\log u(x) < L_0 + \eta_0' t$. By Theorem 4.3 there exists $\theta \in [0, 2\pi]$ and $\tau \leq m'' \log u(x)$ such that $x' \equiv a_\tau r_\theta x \in K_\rho$. Then,

$$\tau \leq m'' L_0 + m'' \eta_0' t.$$

Then, for any v,

$$\|p(v)\|'_x \leq e^{m_0\tau}\|p(v)\|'_{x'} \leq C_0 e^{m_0\tau}\|p(v)\|_{x'} \leq C_0 e^{(m_0+2)\tau}\|p(v)\|_x$$

Therefore, by Lemma 7.2 (b),

$$\frac{\kappa(x,t)}{\|p(v)\|_x^{\delta/2}} \leq e^{-\eta t} C_0 e^{(\delta/2)(m_0+2)(m'' L_0 + m''\eta_0' t)}(\|p(v)\|'_x)^{-\delta/2} \leq e^{-(\eta/2)t}(\|p(v)\|'_x)^{-\delta/2},$$

provided $(\delta/2)m''\eta_0' < \eta/2$ and t_0' is sufficiently large.

Let v be as defined in (49). Note that $x+v \in \mathcal{L}_{\mathbb{C}}$, and $p(v)$ is (symplectically) orthogonal to $p(\mathcal{L}_{\mathbb{C}})$. Let $w = a_\tau r_\theta v$. Then, since $\mathcal{L}$ is invariant, $p(w)$ is sympectically orthogonal to $p(\mathcal{L}_{\mathbb{C}})$. Therefore, $\psi_{x'+w}(p(w)) = 1$. Also, by definition, the subspace $E(x')$ varies continuously with x', hence for any $y \in \mathcal{L}_{\mathbb{C}}$,

$$\lim_{x' \to y} \psi_{x'}(p(w)) = 1.$$

Since we are assuming that $d'(x', \mathcal{L}_{\mathbb{C}})$ is small (in fact $d'(x, \mathcal{L}_{\mathbb{C}}) \leq \frac{1}{2}e^{-m't}$ and $\tau \ll t$), we conclude that $\psi_{x'}(p(w))$ is uniformly bounded. Therefore,

$$\psi_x(p(v))^{\delta/2} \leq e^{C\eta_0'(\delta/2)2\tau} \leq e^{(\eta/4)t}$$

provided η_0' is small enough. Thus, we get, for $t > t_0'$ and $x \in \mathcal{H}_1(\alpha)$ so that $\log u(x) < L_0 + \eta_0' t$,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{d'(a_t r_\theta x, a_t r_\theta \mathcal{L}_{\mathbb{C}})^\delta} \leq 4 \min\left(\frac{e^{-k_2(\delta)t}}{\|v'\|_x^\delta}, \frac{e^{-(\eta/4)t}}{(\|p(v)\|_x')^{\delta/2}} \right)$$

The estimate (48) now follows. □

8. The sets $J_{k,\mathcal{M}}$

Fix $\rho > 0$ so that Theorem 4.2 and Theorem 4.3 hold. Let K_ρ be as in Theorem 4.2. and let $K' = \{x + v \;:\; x \in K_{0.01} \text{ and } \|v\|_x \leq 1\}$. Then, K' is a compact subset of $\mathcal{H}_1(\alpha)$. We lift K' to a compact subset of Teichmüller space , which we also denote by K'.

Definition 8.1 (Complexity)**.** For an affine invariant submanifold $\mathcal{M} \subset \mathcal{H}_1(\alpha)$, let $n(\mathcal{M})$ denote the smallest integer such that $\mathcal{M} \cap K'$ is contained in a union of at most $n(\mathcal{M})$ affine subspaces. We call $n(\mathcal{M})$ the "complexity" of $\mathcal{M}$.

Since $\mathcal{M}$ is closed and K' is compact, $n(\mathcal{M})$ is always finite. Clearly $n(\mathcal{M})$ depends also on the choice of K', but since K' is fixed once and for all, we drop this dependence from the notation.

Lemma 8.2. *Let $\mathcal{M}$ be an affine manifold, and let $\tilde{\mathcal{M}}$ be the lift of $\mathcal{M}$ to Teichmüller space. Let*

$$J_{k,\mathcal{M}}(x) = \{\mathcal{L} \;:\; d'(\mathcal{L}, x) \leq u(x)^{-k}, \quad \mathcal{L} \textit{ is an affine subspace tangent to } \tilde{\mathcal{M}} \;\}.$$

Then, there exists $k > 0$, depending only on α such that for any affine manifold $\mathcal{M} \subset \mathcal{H}_1(\alpha)$,

$$|J_{k,\mathcal{M}}(x)| \leq n(\mathcal{M})$$

where $|J_{k,\mathcal{M}}(x)|$ denotes the cardinality of $J_{k,\mathcal{M}}(x)$, and $n(\mathcal{M})$ is as in Definition 8.1.

Proof. Let

$$B'(x,r) = \{x + h + v \;:\; h \text{ harmonic}, v \in \ker p, \max(\|h\|_x^{1/2}, \|v\|_x) \le r\}$$

For every $x \in \mathcal{H}_1(\alpha)$, there exists $r(x) > 0$ such that $B'(x, r(x))$ is embedded (in the sense that the projection from Teichmüller space to Moduli space, restricted to $B'(x, r(x))$ is injective). Let $r_0 = \inf_{x \in K_\rho} r(x)$. By compactness of K_ρ, $r_0 > 0$. Then, choose k_0 so that

$$2^{m''m'-k_0} < r_0. \tag{54}$$

where m'' be as in Theorem 4.3, and m' is as in (47).

We now claim that for any $k > k_0$ and any $x \in \mathcal{H}_1(\alpha)$, $B'(x, u(x)^{-k_0})$ is embedded. Suppose not, then there exist $x \in \mathcal{H}_1(\alpha)$ and $x_1, x_2 \in B'(x, u(x)^{-k_0})$ such that $x_2 = \gamma x_1$ for some γ in the mapping class group. Write

$$x_i = x + h_i + v_i, \quad h_i \text{ harmonic}, v_i \in \ker p, \max(\|h_i\|_x^{1/2}, \|v_i\|_x) \le u(x)^{-k_0}$$

By Theorem 4.3 there exists $\theta \in [0, 2\pi]$ and $\tau \le m'' \log u(x)$ such that $x' \equiv a_\tau r_\theta x \in K_\rho$.

Let $x_i' = a_\tau t_\theta x_i$. Then, by Lemma 7.5 we have

$$\max(\|h_i\|_{x'}^{1/2}, \|v_i\|_{x'}) \le e^{-m'\tau} u(x)^{-k_0} \le u(x)^{m'm''-k_0} \le 2^{mm'-k_0} \le r_0$$

where for the last estimate we used (54) and the fact that $u(x) \ge 2$. Thus, both x_1' and x_2' belong to $B'(x', r_0)$, which is embedded by construction, contradicting the fact that $x_2' = \gamma x_1'$. Thus, $B'(x, u(x)^{-k})$ is embedded.

Now suppose $\mathcal{L} \in J_{k,\mathcal{M}}(x)$, so that

$$d'(x, \mathcal{L}) \le u(x)^{-k}.$$

Write $\mathcal{L}' = a_\tau r_\theta \mathcal{L}$. Then, by (47),

$$d'(x', \mathcal{L}') \le e^{m'\tau} u(x)^{-k} \le u(x)^{m''m'} u(x)^{-k} < r_0,$$

Hence, $\mathcal{L}'$ intersects $B'(x', r_0)$. Furthermore, since $B'(x', r_0)$ and $B'(x, u(x)^{-k})$ are embedded, there is a one-to-one map between subspaces contained in $J_{k,\mathcal{M}}(x)$ and subspaces intersecting $B'(x', r_0)$.

Since $x' \in K_\rho$, and $r_0 < 1$, $B'(x', r_0) \subset K'$. Hence, there are at most $n(\mathcal{M})$ possibilities for $\mathcal{L}'$, and hence at most $n(\mathcal{M})$ possibilities for $\mathcal{L}$. $\square$

9. Standard Recurrence Lemmas

Lemma 9.1. *For every $\sigma > 1$ there exists a constant $c_0 = c_0(\sigma) > 0$ such that the following holds: Suppose X is a space on which $SL(2,\mathbb{R})$ acts, and suppose $f : X \to [2, \infty]$ is an $SO(2)$-invariant function with the following properties:*

(a) *There exist* $\sigma > 1$ *such that for all* $0 \le t \le 1$ *and all* $x \in X$,

$$\sigma^{-1} f(x) \le f(a_t x) \le \sigma f(x). \tag{55}$$

(b) *There exists* $\tau > 0$ *and* $b = b(\tau) > 0$ *such that for all* $x \in X$,

$$A_\tau f(x) \le c_0 f(x) + b_0.$$

Then,

(i) *For all* $c < 1$ *there exists* $t_0 > 0$ *(depending on* σ*, and* c*) and* $b > 0$ *(depending only on* b_0*,* c_0 *and* σ*) such that for all* $t > t_0$ *and all* $x \in X$,

$$(A_t f)(x) \le c f(x) + b.$$

(ii) *There exists* $B > 0$ *(depending only on* c_0*,* b_0 *and* σ*) such that for all* $x \in X$*, there exits* $T_0 = T_0(x, c_0, b_0, \sigma)$ *such that for all* $t > T_0$,

$$(A_t f)(x) \le B.$$

For completeness, we include the proof of this lemma. It is essentially taken from [EMM, §5.3], specialized to the case $G = SL(2, \mathbb{R})$. The basic observation is the following standard fact from hyperbolic geometry:

Lemma 9.2. *There exist absolute constants* $0 < \delta' < 1$ *and* $\delta > 0$ *such that for any* $t > 0$*, any* $\tau > 0$ *and any* $z \in \mathbb{H}$*, for at least* δ'*-fraction of* $\phi \in [0, 2\pi]$,

$$t + s - \delta \le d(a_t r_\phi a_s z, z) \le t + s, \tag{56}$$

where $d(\cdot,\cdot)$ *is the hyperbolic distance in* $\mathbb{H}$*, normalized so that* $d(a_t r_\theta z, z) = t$.

Corollary 9.3. *Suppose* $f : X \to [1, \infty]$ *satisfies (55). Then, there exists* $\sigma' > 1$ *depending only on* σ *such that for any* $t > 0$*,* $s > 0$ *and any* $x \in X$,

$$(A_{t+s} f)(x) \le \sigma' (A_t A_s f)(x). \tag{57}$$

Outline of proof. Fix $x \in \mathcal{H}_1(\alpha)$. For $g \in SL(2, \mathbb{R})$, let $f_x(g) = f(gx)$, and let

$$\tilde{f}_x(g) = \int_0^{2\pi} f(g r_\theta x)\, d\theta.$$

Then, $\tilde{f}_x : \mathbb{H} \to [2, \infty]$ is a spherically symmetric function, i.e. $\tilde{f}_x(g)$ depends only on $d(g \cdot o, o)$ where o is the origin of $\mathbb{H}$.

We have

$$(A_t A_s f)(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} f(a_t r_\phi a_s r_\theta x)\, d\phi\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}_x(a_t r_\phi a_s). \tag{58}$$

By Lemma 9.2, for at least δ'-fraction of $\phi \in [0, 2\pi]$, (56) holds. Then, by (55), for at least δ'-fraction of $\phi \in [0, 2\pi]$,

$$\tilde{f}_x(a_t r_\phi a_s) \ge \sigma_1^{-1} \tilde{f}_x(a_{t+s})$$

where $\sigma_1 = \sigma_1(\sigma, \delta) > 1$. Plugging in to (58), we get

$$(A_t A_s f)(x) \geq (\delta' \sigma_1^{-1}) \tilde{f}_x(a_{t+s}) = (\delta' \sigma_1^{-1})(A_{t+s} f)(x),$$

as required. □

Proof of Lemma 9.1. Let $c_0(\sigma)$ be such that $\kappa \equiv c_0 \sigma' < 1$, where σ' is as in Corollary 9.3. Then, for any $s \in \mathbb{R}$ and for all x,

$$\begin{aligned}
(A_{s+\tau} f)(x) &\leq \sigma' A_s (A_\tau f)(x) && \text{by (57)} \\
&\leq \sigma' A_s (c_0 f(x) + b_0) && \text{by condition (b)} \\
&= \kappa (A_s f)(x) + \sigma' b_0 && \text{since } \sigma' c_0 = \kappa.
\end{aligned}$$

Iterating this we get, for $n \in \mathbb{N}$

$$(A_{n\tau} f)(x) \leq \kappa^n f(x) + \sigma' b_0 + \kappa \sigma' b_0 + \cdots + \kappa^{n-1} \sigma' b_0 \leq \kappa^n f(x) + B,$$

where $B = \frac{\sigma' b_0}{1-\kappa}$. Since $\kappa < 1$, $\kappa^n f(x) \to 0$ as $n \to \infty$. Therefore both (i) and (ii) follow for $t \in \tau\mathbb{N}$. The general case of both (i) and (ii) then follows by applying again condition (a). □

10. Construction of the function

Note that by Jensen's inequality, for $0 < \epsilon < 1$,

$$A_t(f^\epsilon) \leq (A_t f)^\epsilon \tag{59}$$

Also, we will repeatedly use inequality

$$(a+b)^\epsilon \leq a^\epsilon + b^\epsilon \tag{60}$$

valid for $\epsilon < 1$, $a \geq 0$, $b \geq 0$.

Fix an affine invariant submanifold $\mathcal{M}$, and let k be as in Lemma 8.2. For $\epsilon > 0$, let

$$s_{\mathcal{M},\epsilon}(x) = \begin{cases} \sum\limits_{\mathcal{L} \in J_{k,\mathcal{M}}(x)} d'(x, \mathcal{L})^{-\epsilon\delta}, & \text{if } J_{k,\mathcal{M}}(x) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

where $\delta > 0$ is as in Lemma 7.8.

Proposition 10.1. *Suppose $\mathcal{M} \subset \mathcal{H}_1(\alpha)$ is an affine manifold and $0 < c < 1$. For $\epsilon > 0$ and $\lambda > 0$, let*

$$f_{\mathcal{M}}(x) = s_{\mathcal{M},\epsilon}(x) u(x)^{1/2} + \lambda u(x).$$

Then, $f_{\mathcal{M}}$ is $SO(2)$-invariant, and $f(x) = +\infty$ if and only if $x \in \mathcal{M}$. Also, if ϵ is sufficiently small (depending on α) and λ is sufficiently large (depending on α, c and $n(\mathcal{M})$), there exists $t_1 > 0$ (depending on $n(\mathcal{M})$ and c) such that for all $t \geq t_1$ we have

$$A_t f_{\mathcal{M}}(x) < c f_{\mathcal{M}}(x) + b, \tag{61}$$

where $b = b(\alpha, n(\mathcal{M}))$.

The proof of Proposition 10.1 will use Lemma 9.1. Thus, in order to prove Proposition 10.1, it is enough to show that $f_{\mathcal{M}}$ satisfies conditions (a) and (b) of Lemma 9.1. We start with the following:

Claim 10.2. *For $\epsilon > 0$ sufficiently small, and $\lambda > 0$ sufficiently large, $f_{\mathcal{M}}$ satisfies condition (a) of Lemma 9.1, with $\sigma = \sigma(k, m, m')$.*

Proof of Claim 10.2. We will choose $\epsilon < 1/(2k\delta)$. Suppose $x \in \mathcal{H}_1(\alpha)$ and $0 \le t < 1$. We consider three sets of subspaces:

$$\Delta_1 = \{\mathcal{L} \in J_{k,\mathcal{M}}(x) \;\;:\;\; a_t\mathcal{L} \in J_{k,\mathcal{M}}(a_t x)\},$$

$$\Delta_2 = \{\mathcal{L} \in J_{k,\mathcal{M}}(x) \;\;:\;\; a_t\mathcal{L} \notin J_{k,\mathcal{M}}(a_t x)\},$$

$$\Delta_3 = \{\mathcal{L} \notin J_{k,\mathcal{M}}(x) \;\;:\;\; a_t\mathcal{L} \in J_{k,\mathcal{M}}(a_t x)\}.$$

Let

$$S_i(x) = \sum_{\mathcal{L} \in \Delta_i} d'(x, \mathcal{L})^{-\epsilon\delta}.$$

Then,

$$s_{\mathcal{M},\epsilon}(x) = S_1(x) + S_2(x) \qquad s_{\mathcal{M},\epsilon}(a_t x) = S_1(a_t x) + S_3(a_t x).$$

For $\mathcal{L} \in \Delta_1$, by (47),

$$e^{-m'\epsilon\delta} d'(x, \mathcal{L})^{-\epsilon\delta} \le d'(a_t x, a_t\mathcal{L})^{-\epsilon\delta} \le e^{m'\epsilon\delta} d'(x, \mathcal{L})^{-\epsilon\delta},$$

and thus

$$e^{-m'\epsilon\delta} S_1(x) \le S_1(a_t x) \le e^{m'\epsilon\delta} S_1(x)$$

Then, using (14),

$$e^{-m'\epsilon\delta - m/2} S_1(x) u(x)^{1/2} \le S_1(a_t x) u(a_t x)^{1/2} \le e^{m'\epsilon\delta + m/2} S_1(x) u(x)^{1/2}.$$

Suppose $\mathcal{L} \in \Delta_2 \cup \Delta_3$. Then, by (14) and (47),

$$d'(x, \mathcal{L}) \ge C u(x)^{-k},$$

where $C = O(1)$ (depending only on k, m and m'), and thus, for $i = 2, 3$, and using Lemma 8.2,

$$S_i(a_t x) \le C n(\mathcal{M}) u(x)^{-\epsilon\delta k} \quad \text{and} \quad S_i(x) \le C n(\mathcal{M}) u(a_t x)^{-\epsilon\delta k}, \qquad i = 2, 3$$

Now choose $\epsilon > 0$ so that $k\epsilon\delta < 1/2$ and $\lambda > 0$ so that $\lambda > 10 C e^m n(\mathcal{M})$. Then,

$$S_i(a_t x) u(a_t x)^{1/2} \le (0.1)\lambda u(x) \quad \text{and} \quad S_i(x) u(x)^{1/2} \le (0.1)\lambda u(a_t x), \qquad i = 2, 3$$

Then,

$$
\begin{aligned}
f_{\mathcal{M}}(a_t x) &= S_1(a_t x)u(a_t x)^{1/2} + S_3(a_t x)u(a_t x)^{1/2} + \lambda u(a_t x) \\
&\le e^{m'\epsilon\delta + m/2} S_1(x)u(x)^{1/2} + (0.1)\lambda u(x) + e^m \lambda u(x) \qquad \text{by (14) and (47)} \\
&\le (e^{m'\epsilon\delta+m/2} + (0.1) + e^m)(S_1(x)u(x)^{1/2} + \lambda u(x)) \\
&\le (e^{m'\epsilon+\delta m/2} + (0.1) + e^m) f_{\mathcal{M}}(x).
\end{aligned}
$$

In the same way,

$$
\begin{aligned}
f_{\mathcal{M}}(x) &= S_1(x)u(x)^{1/2} + S_2(x)u(x)^{1/2} + \lambda u(x) \\
&\le e^{m'\epsilon\delta + m/2} S_1(a_t x)u(a_t x)^{1/2} + (0.1)\lambda u(a_t x) + e^m \lambda u(a_t x) \qquad \text{by (14) and (47)} \\
&\le (e^{m'\epsilon\delta+m/2} + (0.1) + e^m)(S_1(a_t x)u(a_t x)^{1/2} + \lambda u(a_t x)) \\
&\le (e^{m'\epsilon+\delta m/2} + (0.1) + e^m) f_{\mathcal{M}}(a_t x).
\end{aligned}
$$

□

We now begin the verification of condition (b) of Lemma 9.1. The first step is the following:

Claim 10.3. *Suppose ϵ is sufficiently small (depending on k, δ). Then there exist $t_2 > 0$ and $\tilde{b} > 0$ such that for all $x \in \mathcal{H}_1(\alpha)$ and all $t > t_2$,*

$$
\begin{aligned}
A_t(s_{\mathcal{M},\epsilon} u^{1/2})(x) \le \kappa_1(x,t)^\epsilon \tilde{c}^{1/2} s_{\mathcal{M},\epsilon}(x) u(x)^{1/2} + \kappa_1(x,t)^\epsilon \tilde{b}^{1/2} s_{\mathcal{M},\epsilon}(x) + \\
+ b_3(t) n(\mathcal{M}) u(x),
\end{aligned}
\tag{62}
$$

where $\tilde{c} = e^{-\tilde{\eta} t}$ and $\kappa_1(x,t)$ is as in Lemma 7.8.

Proof of Claim 10.3. In this proof, the $b_i(t)$ denote constants depending on t. Choose $\epsilon > 0$ so that $2k\epsilon\delta \le 1$. Suppose $t > 0$ is fixed. Let $J'(x) \subset J_{k,\mathcal{M}}(x)$ be the subset

$$J'(x) = \{\mathcal{L} \;:\; a_t r_\theta \mathcal{L} \in J_{k,\mathcal{M}}(a_t r_\theta x) \text{ for all } 0 \le \theta \le 2\pi\}.$$

Suppose $\mathcal{L} \subset J'(x)$. For $0 \le \tau \le t$ and $0 \le \theta \le 2\pi$, let

$$\ell_{\mathcal{L}}(a_\tau r_\theta x) = d'(a_\tau r_\theta \mathcal{L}, a_\tau r_\theta x)^{-\delta}.$$

Then,

$$
\begin{aligned}
A_t(\ell_{\mathcal{L}}^{2\epsilon})(x) &\le (A_t \ell_{\mathcal{L}})^{2\epsilon}(x) && \text{by (59)} \\
&\le (\kappa_1(x,t)\ell_{\mathcal{L}}(x) + b(t))^{2\epsilon} && \text{by Lemma 7.8} \\
&\le \kappa_1(x,t)^{2\epsilon}\ell_{\mathcal{L}}(x)^{2\epsilon} + b(t)^{2\epsilon} && \text{by (60)} \qquad (63)
\end{aligned}
$$

We have, at the point x,

$$\begin{aligned}
A_t(\ell_{\mathcal{L}}^{\epsilon} u^{1/2}) &\le (A_t \ell_{\mathcal{L}}^{2\epsilon})^{1/2} (A_t u)^{1/2} && \text{by Cauchy-Schwartz} \\
&\le [(\kappa_1(x,t)^{2\epsilon} \ell_{\mathcal{L}}(x)^{2\epsilon} + b_1(t) u(x)]^{1/2} (\tilde{c} u(x) + \tilde{b})^{1/2} && \text{by (63) and (15)} \\
&\le [\kappa_1(x,t)^{\epsilon} \ell_{\mathcal{L}}(x)^{\epsilon} + b_1(t)^{1/2} u(x)^{1/2}] (\tilde{c}^{1/2} u(x)^{1/2} + \tilde{b}^{1/2}) && \text{by (60)} \\
&= \kappa_1(x,t)^{\epsilon} \ell_{\mathcal{L}}(x)^{\epsilon} (\tilde{c}^{1/2} u(x)^{1/2} + \tilde{b}^{1/2}) \\
&\qquad + b_1(t)^{1/2} \tilde{c}^{1/2} u(x) + b_1(t)^{1/2} \tilde{b}^{1/2} u(x)^{1/2} \\
&\le \kappa_1(x,t)^{\epsilon} \ell_{\mathcal{L}}(x)^{\epsilon} (\tilde{c}^{1/2} u(x)^{1/2} + \tilde{b}^{1/2}) + \\
&\qquad + b_1(t)^{1/2} (\tilde{c}^{1/2} + \tilde{b}^{1/2}) u(x) && \text{since } u(x) \ge 1
\end{aligned}$$

$$\begin{aligned}
&= \kappa_1(x,t)^{\epsilon} \tilde{c}^{1/2} \ell_{\mathcal{L}}(x)^{\epsilon} u(x)^{1/2} + \kappa_1(x,t)^{\epsilon} \tilde{b}^{1/2} \ell_{\mathcal{L}}(x)^{\epsilon} + \\
&\qquad + b_2(t) u(x).
\end{aligned} \tag{64}$$

For $0 \le \tau \le t$ and $0 \le \theta \le 2\pi$, let

$$h(a_\tau r_\theta x) = \sum_{\mathcal{L} \in J'(x)} d'(a_\tau r_\theta \mathcal{L}, a_\tau r_\theta x)^{-\epsilon\delta} = \sum_{\mathcal{L} \in J'(x)} \ell_{\mathcal{L}}(a_\tau r_\theta x)^{\epsilon}.$$

Then, $h(a_\tau r_\theta x) \le s_{\mathcal{M},\epsilon}(a_\tau r_\theta x)$. Summing (64) over $\mathcal{L} \in J'(x)$ and using Lemma 8.2 we get

$$A_t(h u^{1/2})(x) \le \kappa_1(x,t)^{\epsilon} \tilde{c}^{1/2} h(x) u(x)^{1/2} + \kappa_1(x,t)^{\epsilon} \tilde{b}^{1/2} h(x) + b_2(t) n(\mathcal{M}) u(x) \tag{65}$$

We now need to estimate the contribution of subspaces not in $J'(x)$. Suppose $0 \le \theta \le 2\pi$, and suppose

$$a_t r_\theta \mathcal{L} \in J_{k,\mathcal{M}}(a_\tau r_\theta x), \quad \text{but } \mathcal{L} \notin J'(x).$$

Then, either $\mathcal{L} \notin J_{k,\mathcal{M}}(x)$ or for some $0 \le \theta' \le 2\pi$, $a_t r_{\theta'} \mathcal{L} \notin J_{k,\mathcal{M}}(a_t r_{\theta'} x)$. Then in either case, for some $\tau' \in \{0, t\}$ and some $0 \le \theta' \le 2\pi$, $a_{\tau'} r_{\theta'} \mathcal{L} \notin J_{k,\mathcal{M}}(a_{\tau'} r_{\theta'} x)$. Hence

$$d'(a_{\tau'} r_{\theta'} x, a_{\tau'} r_{\theta'} \mathcal{L}) \ge u(a_{\tau'} r_{\theta'} x)^{-k}$$

Then, by (47) and (14),

$$d'(x, \mathcal{L}) \ge b_0(\tau')^{-1} u(x)^{-k} \ge b_0(t)^{-1} u(x)^{-k}$$

and thus, for all $\theta \in [0, 2\pi]$, by (14) and (47),

$$d'(a_t r_\theta x, a_t r_\theta \mathcal{L}) \ge b_0(t)^{-2} u(x)^{-k}.$$

Hence, using (14) again,

$$d'(a_t r_\theta x, a_t r_\theta \mathcal{L})^{-\epsilon\delta} u(a_t r_\theta x)^{1/2} \le b_1(t) u(x)^{k\epsilon\delta + 1/2} \le b_1(t) u(x), \tag{66}$$

where for the last estimate we used $k\epsilon\delta \le 1/2$. Thus, for all $0 \le \theta \le 2\pi$,

$$\begin{aligned} s_{\mathcal{M},\epsilon}(a_t r_\theta x)u(a_t r_\theta x)^{1/2} &\le h(a_t r_\theta x)u(a_t r_\theta x)^{1/2} + |J(a_\tau r_\theta x)| b_1(t) u(x) && \text{using (66)} \\ &\le h(a_t r_\theta x)u(a_t r_\theta x)^{1/2} + b_1(t) n(\mathcal{M}) u(x) && \text{using Lemma 8.2.} \end{aligned}$$

Hence,

$$\begin{aligned} &A_t(s_{\mathcal{M},\epsilon}u^{1/2})(x) \le A_t(hu^{1/2})(x) + b_1(t)n(\mathcal{M})u(x) \\ &\quad \le \kappa_1(x,t)^\epsilon \tilde{c}^{1/2} h(x)u(x)^{1/2} + \kappa_1(x,t)^\epsilon \tilde{b}^{1/2} h(x) + b_3(t)n(\mathcal{M})u(x) && \text{using (65)} \\ &\quad \le \kappa_1(x,t)^\epsilon \tilde{c}^{1/2} s_{\mathcal{M},\epsilon}(x)u(x)^{1/2} + \kappa_1(x,t)^\epsilon \tilde{b}^{1/2} s_{\mathcal{M},\epsilon}(x) + b_3(t)n(\mathcal{M})u(x) && \text{since } h \le s_{\mathcal{M},\epsilon} \end{aligned}$$

□

Proof of Proposition 10.1. Let σ be as in Claim 10.2, and let $c_0 = c_0(\sigma)$ be as in Lemma 9.1. Let L_0, η_0', η_3, m', δ be as in Lemma 7.8. Suppose $\epsilon > 0$ is small enough so that

$$\epsilon m'\delta < \frac{1}{2}\tilde{\eta}, \tag{67}$$

where $\tilde{\eta}$ is as in Theorem 4.1. We also assume that $\epsilon > 0$ is small enough so that

$$\epsilon m'\delta < \frac{1}{2}\min(\eta_3, \eta_0') \tag{68}$$

where η_3 is in Lemma 7.8. Choose $t_0 > 0$ so that Theorem 4.1 holds for $t > t_0$, and so that $e^{-\tilde{\eta}t_0} < (0.1)c_0$. Since $\kappa_1(x,t) < e^{m'\delta t}$, we can also, in view of (67) make sure that for $t > t_0$,

$$\kappa_1(x,t)^\epsilon e^{-\tilde{\eta}t/2} \le (0.1)c_0 \tag{69}$$

Let $t_2 > 0$ be such that Claim 10.3 holds. By (68), there exists $t_3 > 0$ so that for $t > t_3$,

$$\kappa_1(x,t)^\epsilon \tilde{b}^{1/2} \le e^{m'\delta\epsilon t}\tilde{b}^{1/2} \le (0.1)c_0 e^{\eta_0' t/2} \tag{70}$$

By Lemma 7.8 there exists $\tau > \max(t_0, t_2, t_3)$ such that for all x with $\log u(x) < L_0 + \eta_0'\tau$,

$$\kappa_1(x,\tau)^\epsilon \tilde{b}^{1/2} \le (0.1)c_0 \le (0.1)c_0 u(x)^{1/2}.$$

If $\log u(x) \ge L_0 + \eta_0'\tau$, then $u(x)^{1/2} \ge e^{(\eta_0'/2)\tau}$, and therefore, since $\tau > t_3$, by (70),

$$\kappa_1(x,\tau)^\epsilon \tilde{b}^{1/2} \le e^{m'\delta\epsilon\tau}\tilde{b}^{1/2} \le (0.1)c_0 e^{\eta_0'\tau/2} \le (0.1)c_0 u(x)^{1/2}.$$

Thus, for all $x \in \mathcal{H}_1(\alpha)$,

$$\kappa_1(x,\tau)^\epsilon \tilde{b}^{1/2} \le (0.1)c_0 u(x)^{1/2}. \tag{71}$$

Thus, substituting (69) and (71) into (62), we get, for all $x \in \mathcal{H}_1(\alpha)$,

$$A_\tau(s_{\mathcal{M},\epsilon}u^{1/2})(x) \le (0.2)c_0\, s_{\mathcal{M},\epsilon}(x)u(x)^{1/2} + b_3(\tau)n(\mathcal{M})u(x). \tag{72}$$

Choose

$$\lambda > 10 b_3(\tau) n(\mathcal{M})/c_0.$$

Then, in view of (72), we have

$$A_\tau(s_{\mathcal{M},\epsilon} u^{1/2})(x) \le (0.2) c_0\, s_{\mathcal{M},\epsilon}(x) u^{1/2} + (0.1) c_0 \lambda u(x). \tag{73}$$

Finally, since $\tilde{c} \le (0.1)c_0$, we have

$$\begin{aligned} A_\tau(f_{\mathcal{M}})(x) &= A_\tau(s_{\mathcal{M},\epsilon} u^{1/2})(x) + A_\tau(\lambda u)(x) \\ &\le [(0.2)c_0 s_{\mathcal{M},\epsilon}(x) u^{1/2} + (0.1)c_0\lambda u(x)] + (0.1)c_0 \lambda u(x) + \lambda \tilde{b} && \text{by (73) and (15)} \\ &\le (0.2) c_0 f_{\mathcal{M}}(x) + b_{\mathcal{M}} && \text{where } b_{\mathcal{M}} = \lambda \tilde{b}. \end{aligned}$$

Thus, condition (b) of Lemma 9.1 holds for $f_{\mathcal{M}}$. In view of Lemma 9.1 this completes the proof of Proposition 10.1. □

11. Countability

The following lemma is standard:

Lemma 11.1. *Suppose $SL(2,\mathbb{R})$ acts on a space X, and suppose there exists a proper function $f : X \to [1,\infty]$ such that for all $c < 1$ there exists $t_0 > 0$ and $b > 0$ such that for all $t > t_0$ and all $x \in X$,*

$$A_t f(x) \le c f(x) + b,$$

and also for some $\sigma > 1$ all $0 \le t \le 1$ and all $x \in X$,

$$\sigma^{-1} f(x) \le f(a_t x) \le \sigma f(x).$$

Suppose μ is an ergodic $SL(2,\mathbb{R})$-invariant measure on X, such that $\mu(\{f < \infty\}) > 0$. Then,

$$\int_X f \, d\mu \le B, \tag{74}$$

where B depends only on b, c and σ.

Proof. For $n \in \mathbb{N}$, let $f_n = \min(f, n)$. By the Moore ergodicity theorem, the action of $A \equiv \{a_t \;:\; t \in \mathbb{R}\}$ on X is ergodic. Then, by the Birkhoff ergodic theorem, there exists a point $x_0 \in X$ such that for almost all $\theta \in [0, 2\pi]$ and all $n \in \mathbb{N}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f_n(a_t r_\theta x_0)\, dt = \int_X f_n \, d\nu \tag{75}$$

Therefore for each n there exists a subset $E_n \subset [0, 2\pi]$ such that the convergence in (75) is uniform over $\theta \in E_n$. Then there exists $T_n > 0$ such that for all $T > T_n$,

$$\frac{1}{T} \int_0^T f_n(a_t r_\theta x_0)\, dt \ge \frac{1}{2} \int_X f_n \, d\nu \quad \text{for } \theta \in E_n. \tag{76}$$

We integrate (76) over $\theta \in [0, 2\pi]$. Then for all $T > T_n$,

$$\frac{1}{T}\int_0^T \left(\int_0^{2\pi} f_n(a_t r_\theta x_0)\, d\theta\right) dt \ge \frac{1}{4}\int_X f_n\, d\nu \tag{77}$$

But, by Lemma 9.1 (ii), for sufficiently large T, the integral in parenthesis on the left hand side of (77) is bounded above by $B' = B'(c, b, \sigma)$. Therefore, for all n,

$$\int_X f_n \le 4B'$$

Taking the limit as $n \to \infty$ we get that $f \in L^1(X, \mu)$ and (74) holds. □

Proof of Proposition 2.16. Let $X_d(\alpha)$ denote the set of affine manifolds of dimension d. It enough to show that each $X_d(\alpha)$ is countable.

For an affine subspace $\mathcal{L} \subset H^1(M, \Sigma, \mathbb{R})$ whose linear part is L, let $H_{\mathcal{L}} : p(L) \to \ker p/(L \cap \ker p)$ denote the linear map such that for $v \in p(L)$, $v + H_{\mathcal{L}}(v) \in L$. For an affine manifold $\mathcal{M}$, let

$$H(\mathcal{M}) = \sup_{x \in \mathcal{M} \cap K'} \|H_{\mathcal{M}_x}\|_x$$

where we use the notation $\mathcal{M}_x$ for the affine subspace tangent to $\mathcal{M}$ at x.

For an integer $R > 0$, let

$$X_{d,R}(\alpha) = \{\mathcal{M} \in X_d(\alpha) \ : \ n(\mathcal{M}) \le R \text{ and } H(\mathcal{M}) \le R\}.$$

Since $X_d(\alpha) = \bigcup_{R=1}^{\infty} X_{d,R}(\alpha)$, it is enough to show that each $X_{d,R}(\alpha)$ is finite.

Let K' be as in the definition Definition 8.1 of $n(\cdot)$ and let $L_R(K')$ denote the set of (unordered) $\le R$-tuples of d dimensional affine subspaces intersecting K'. Then $L_R(K')$ is compact, and we have the map $\phi : X_{d,R} \to L_R(K')$ which takes the affine manifold $\mathcal{M}$ to the (minimal) set of affine subspaces containing $\mathcal{M} \cap K'$.

Suppose $\mathcal{M}_j \in X_{d,R}(\alpha)$ is an infinite sequence, with $\mathcal{M}_j \ne \mathcal{M}_k$ for $j \ne k$. Then, $\mathcal{M}_j \cap K' \ne \mathcal{M}_k \cap K'$ for $j \ne k$. (If $\mathcal{M}_j \cap K' = \mathcal{M}_k \cap K'$ then by the ergodicity of the $SL(2, \mathbb{R})$ action, $\mathcal{M}_j = \mathcal{M}_k$).

Since $L_R(K')$ is compact, after passing to a subsequence, we may assume that $\phi(\mathcal{M}_j)$ converges. Therefore,

$$hd(\mathcal{M}_j \cap K', \mathcal{M}_{j+1} \cap K') \to 0 \quad \text{as } j \to \infty, \tag{78}$$

where $hd(\cdot, \cdot)$ denotes the Hausdorff distance. Then, because of (78) and the bound on $H(\mathcal{M})$, for all $x \in \mathcal{M}_{j+1} \cap K$, $d'(x, \mathcal{M}_j) \to 0$. Therefore, there exists a sequence $T_j \to \infty$ such that we have

$$f_{\mathcal{M}_{j+1}}(x) \ge T_j \quad \text{for all } x \in \mathcal{M}_j \cap K' \tag{79}$$

Let μ_j be the affine $SL(2,\mathbb{R})$-invariant probability measure whose support is $\mathcal{M}_j$. Then, by Proposition 10.1 and Lemma 11.1, we have for all j,

$$\int_{\mathcal{H}_1(\alpha)} f_{\mathcal{M}_{j+1}}\, d\mu_j \leq B,$$

where B is independent of j. But, since $\mu_j(\mathcal{M}_j \cap K') \geq \rho \geq 1/2$, this is a contradiction to (79). Therefore, $X_{d,R}(\alpha)$ is finite. □

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