Simultaneous Diophantine Approximation in Non-degenerate p-adic Manifolds.

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Abstract

S-arithmetic Khintchine-type theorem for products of non-degenerate analytic p-adic manifolds is proved for the convergence case. In the p-adic case the divergence part is also obtained. [1]

1 Introduction

Metric Diophantine approximation The metric theory of Diophantine approximation studies the interplay between the precision of approximation of real or p-adic numbers by rationals and the measure of the approximated set within certain prescribed precision. The "finer" is the precision the "smaller" is the approximated set. The theory was initiated by A. Khintchine, who in [Kh24] proved an "almost every" vs "almost no" dichotomy for $\mathbb{R}$. Let us state Khintchine's result. Given a decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ we define the notion of a ψ-*approximable* number as follows; A real number ξ is called ψ-*approximable* if for infinitely many integers p and q, one has $|q\xi - p| < \psi(|q|)$. It is called *very well approximable* (VWA) if it is ψ_ε-A where $\psi_\varepsilon = 1/q^{1+\varepsilon}$ for some positive ε. A. Khintchine showed that Lebesgue almost every (resp. almost no) real number is ψ-A if $\sum_{q=1}^{\infty} \psi(q)$ diverges (resp. converges). We refer to [St80, Chapter IV, Section 5], [BD99, Chap. 1] and [Ca57, Chap. 5] for an account on this and further historical remarks.

The metric theory of Diophantine approximation on manifolds was considered as early as 1932 when K. Mahler [Ma32] conjectured that almost no point of the Veronese curve, $(x, x^2, \cdots, x^n)$, is VWA. This conjecture drew considerable amount of attention and was finally settled affirmatively by V. G. Sprindžuk in [Sp64, Sp69]. Sprindžuk's idea (the so called essential and non-essential domains) has been applied by many people to attack many problems stated by him in both the real and the p-adic setting (the definition of ψ-A in the p-adic setting is given bellow). One should mention several works conducted on this

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issue by V. Beresnevich, V. Bernik, M. M. Dodson, E. Kovalevskaya and others. See for example [Ber00b, Ber02, BBK05, BK03, Ko00].

In 1985 S. G. Dani observed a nice relationship between flows on homogeneous spaces and Diophantine approximation. This point of view was taken on and pushed much further in later works of D. Kleinbock and G. A. Margulis. In [KM98] Kleinbock and Margulis introduced a beautiful dynamical approach to the metric theory of Diophantine approximation and settled a multiplicative version of Sprindžuk 's conjecture; we refer to their paper for the formulation and further comments. The "almost every" vs "almost no" dichotomy was also completed within a few years from then, see in [BKM01, BBKM02]. It is worth mentioning that philosophically speaking the dynamical approach in [KM98] and the idea of essential and non-essential domains of Sprindžuk are both based on a delicate covering argument.

One expects, from the nature of the dynamical approach, that this approach would work just as well in the S-arithmetic setting. This was started by D. Kleinbock and G. Tomanov [KT07]. They defined the notion of VWA and showed an analogue of the Sprindžuk's conjecture in the S-arithmetic setting. This general philosophy was taken on in [MS08] where we proved a Khintchine type theorem in S-arithmetic setting. In that paper the assumption was that the finite set of places, S, contains the infinite place. We postponed the completion of the picture to this paper.

Let us fix some notations and conventions which are needed in order to state the main results of this paper, these notations will be used throughout the paper. We will not define the technical terms in here but rather refer the reader to the corresponding section for the precise definitions and remarks.

We let S be a finite set of places of $\mathbb{Q}$ whose cardinality will be denoted by κ throughout. We will always assume that S does not contain the infinite place and let $\tilde{S} = S \cup \{\infty\}$. Define $\mathbb{Q}_S = \prod_{\nu\in S}\mathbb{Q}_\nu$ and correspondingly $\mathbb{Q}_{\tilde{S}} = \prod_{\nu\in\tilde{S}}\mathbb{Q}_\nu$.

We let $\Psi : \mathbb{Z}^{n+1}\setminus\{0\} \to \mathbb{R}^+$ be a map. A vector $\xi \in \mathbb{Q}_S^n$ is said to be Ψ-A if

$$\text{For infinitely many } \tilde{\mathbf{q}} = (\mathbf{q}, q_0) \in \mathbb{Z}^n \times \mathbb{Z} \text{ one has } |\mathbf{q}\cdot\xi + q_0|_S^\kappa \le \Psi(\tilde{\mathbf{q}}).$$

For all $\nu \in S$, we fix once and for all an open bounded ball U_ν in $\mathbb{Q}_\nu^{d_\nu}$. Let

$$f_\nu = (f_\nu^{(1)}, \dots, f_\nu^{(n)}) : U_\nu \to \mathbb{Q}_\nu^n$$

be an analytic non-degenerate map i.e. f_ν's are analytic and the restrictions of $1, f_\nu^{(1)}, \cdots, f_\nu^{(n)}$ to any open ball of U_ν are linearly independent over $\mathbb{Q}_\nu$.

Define $\mathbf{U} = \prod_{\nu\in S} U_\nu$ and let $\mathbf{f}(\mathbf{x}) = (f_\nu(x_\nu))_{\nu\in S}$, where $\mathbf{x} = (x_\nu)_{\nu\in S} \in \mathbf{U}$. Since $\mathbf{U}$ is compact we may replace $\mathbf{f}$ by $\mathbf{f}/M$, for a suitable $\tilde{S}$-integer M, and assume that $\|f_\nu(x_\nu)\|_\nu \le 1$, $\|\nabla f_\nu(x_\nu)\|_\nu \le 1$, and $L \le 1$ where

$$L = \sup \bigcup_{|\beta|=2,\nu\in S} \{2|\Phi_\beta f_\nu(x)|_\nu \mid x \in U_\nu \times U_\nu \times U_\nu\},$$

for any $\nu \in S$. The functions $\Phi_\beta f_\nu$ here are certain two fold difference quotients of f_ν, see section 2 for the exact definition.

We can now state the theorems.

Theorem 1.1. *Let* $\mathbf{U}$ *and* $\mathbf{f}$ *be as above. Further assume that* $\Psi : \mathbb{Z}^{n+1} \setminus \{0\} \to \mathbb{R}_+$, *is a function which satisfies*

(i) Ψ *is norm-decreasing, i.e.* $\Psi(\mathbf{q}) \geq \Psi(\mathbf{q}')$ *for* $|\mathbf{q}|_\infty \leq |\mathbf{q}'|_\infty$,

(ii) $\sum_{\mathbf{q} \in \mathbb{Z}^{n+1} \setminus \{0\}} \Psi(\mathbf{q}) < \infty$.

then the set

$$\mathcal{W}_{\mathbf{f},\Psi} = \{\mathbf{x} \in \mathbf{U} |\ \mathbf{f}(\mathbf{x}) \text{ is } \Psi\text{-A}\}$$

has measure zero.

Our result in the divergent part is somewhat more restrictive. It is only the p-adic case that is proved here i.e. S consists of only one valuation. Note also the function Ψ in theorem 1.2 is more specific.

Theorem 1.2. *Let* U *be an open subset of* $\mathbb{Q}_\nu^d$ *and let* $f : U \to \mathbb{Q}_\nu^n$ *be a non-degenerate analytic map (in the above sense). Let* $\psi : \mathbb{Z} \to \mathbb{R}_+$ *be a non-increasing function for which* $\sum_k \psi(k)$ *diverges. Define the function* Ψ *by*

$$\Psi(\tilde{q}) = \|\tilde{q}\|_\infty^{-n} \psi(\|\tilde{q}\|_\infty) \quad \textit{for any} \quad \tilde{q} \in \mathbb{Z}^{n+1} \setminus \{0\}$$

Then the set

$$\mathcal{W}_{f,\Psi} = \{x \in U |\ f(x) \text{ is } \Psi\text{-A}\}$$

has full measure.

Remark 1.3.

1. It is clear that Theorem 1.1 and Theorem 1.2 are valid for any non-degenerate analytic $\mathbb{Q}_S$-manifold. The general case is a direct consequence of the above theorems, and the posed conditions are merely for our convenience in the course of the proof.
2. D. Kleinbock and G. Tomanov [KT07] proved Theorem 1.1 when $\Psi(\tilde{\mathbf{q}}) = \|\tilde{\mathbf{q}}\|_S^{-(1+n)(1+\varepsilon)}$ and the above theorems answers their question.
3. V. Beresnevich, V. Bernik, E. Kovalevskaya [BBK05] proved Theorem 1.1 and Theorem 1.2 for the the Veronese curve, i.e. $f(x) = (x, x^2, \ldots, x^n)$, and $S = \{\nu\}$.
4. Although our result in the divergent case is more restrictive than that of the convergence case, it actually is the formulation which has been historically considered. For example Mahler's conjecture for the Veronese curve is formulated in the same setting as in Theorem 1.2. However it is interesting to prove a multiplicative version of the divergence part, this is not known in the real case either.

5. Here we just look at the simultaneous approximation in non-Archimedean places by integers. As was mentioned above, in [MS08], we proved a convergence Khintchine-type theorem for $\mathbb{Q}_S$-manifolds, where S contains the infinite place. Indeed, in that case, we considered the field of rational numbers as the quotient field of $\mathcal{R}$ a finitely generated subring of $\mathbb{Q}$ and asked how "large" the denominator should be to get a good approximation. More precisely, we carefully defined the notion of $(\Psi, \mathcal{R})$-A for any $\mathcal{R}$ a finitely generated subring of $\mathbb{Q}$ and Ψ a function from $\mathcal{R}^n$ to $\mathbb{R}^+$, and proved a convergence $\mathcal{R}$-Khintchine-type theorem (we refer the reader to [MS08] for the definition and the exact statement of the mentioned theorem). Unfortunately, we could not come up with a good definition of such a notion in the absence of the Archimedean place. That is why, in this article, we only approximate $\prod_{\nu \in S} \mathbb{Z}_\nu$-points by integral points instead of approximating them by $\mathbb{Z}_{S'}$-points, where S and S' are two disjoint finite sets of finite places.
6. Most of our proof in the convergence part works for a function Ψ which is only decreasing in the S-norm of the coordinates, and not necessarily in the norm of the vector. In fact, in [MS08], we work with this more general class of Ψ's. It is interesting to see if the norm-decreasing condition can be relaxed.

The proof of the convergence part is very similar to the proof in [MS08]. In particular the main technical difficulties that arise in carrying out the strategies developed in [BKM01] to the S-arithmetic setting have the same nature in these two papers. As similar as these two papers are, there are many differences in details in both the "calculus lemma" and the "dynamics part". This and the fact that the divergence case is also treated here demanded a coherent separate paper in the p-adic setting.

Theorem 1.1 will be proved with the aid of the following two theorems and applying Borel-Cantelli lemma. In what follows, let $\widetilde{f_\nu} = (f_\nu, 1)$ and $\widetilde{\mathbf{f}}(\mathbf{x}) = (\widetilde{f_\nu}(x_\nu))_{\nu \in S}$. The following is what we refered to as the calculus lemma.

Theorem 1.4. *Let $\mathbf{U}$, $\mathbf{f}$ be as in Theorem 1.1 and $0 < \epsilon < \frac{1}{2\kappa}$. For $\delta > 0$ and any ball $\mathbf{B} \subset \mathbf{U}$ let*

$$\mathcal{A} = \left\{ \mathbf{x} \in \mathbf{B} | \ \exists \tilde{\mathbf{q}} \in \mathbb{Z}^{n+1} \setminus \{0\}, \ |\tilde{\mathbf{q}}|_\infty < T, \quad \begin{array}{l} |\tilde{\mathbf{q}} \cdot \widetilde{\mathbf{f}}(\mathbf{x})|_S^\kappa \ < \delta T^{-n-1} \\ \|\mathbf{q} \nabla \mathbf{f}(\mathbf{x})\|_\nu > |\tilde{\mathbf{q}}|_\infty^{-\epsilon} \text{ for all } \nu \in S \end{array} \right\}.$$

Then we have $|\mathcal{A}| < C\delta \, |\mathbf{B}|$, for some universal constant C.

The remaining set will be controlled using the following theorem. The proof of this theorm has dynamical nature.

Theorem 1.5. *Let $\mathbf{U}$ and $\mathbf{f}$ be as in theorem 1.1. For any $\mathbf{x} = (x_\nu)_{\nu \in S} \in \mathbf{U}$, one can find a neighborhood $\mathbf{V} = \prod_{\nu \in S} V_\nu \subseteq \mathbf{U}$ of $\mathbf{x}$ and $\alpha > 0$ with the following property: If $\mathbf{B} \subseteq \mathbf{V}$ is any ball, then there exists $E > 0$ such that for any choice of $T_0, \cdots, T_n \geq 1$, $K_\nu > 0$ and $0 < \delta^\kappa \leq \min \frac{1}{T_i}$ where $\delta^\kappa T_0 \cdots T_n \prod_{\nu \in S} K_\nu \leq 1$ one has*

$$\left| \left\{ \mathbf{x} \in \mathbf{B} | \ \exists \tilde{\mathbf{q}} \in \mathbb{Z}^{n+1} \setminus \{0\} : \ \begin{array}{l} |\tilde{\mathbf{q}} \cdot \widetilde{\mathbf{f}}(\mathbf{x})|_S < \delta \\ \|\nabla q f_\nu(x)\|_\nu < K_\nu \\ |q_i|_\infty < T_i \end{array} \right\} \right| \leq E \, \varepsilon^\alpha |\mathbf{B}|, \tag{1.5}$$

where $\varepsilon = \max\{\delta, (\delta^\kappa T_0 \dots T_n \prod_{\nu\in S} K_\nu)^{\frac{1}{n+1}})\}$.

The main idea in the proof of Theorem 1.2 is based on the method of regular systems. This method was introduced by Baker and Schmidt [BS70] and was applied in [Ber99] in dimension one and later in [BBKM02] it was generalized to any dimension. In our proof we will use the estimates obtained in theorems 1.4 and 1.5, and get the following theorem, which provides us with a suitable *regular system of resonant sets*. Then a general result on regular systems (cf. Theorem 7.3) will kick in and the proof will be concluded. See section 7 for the definitions.

Theorem 1.6. *Let f and U be as in Theorem 1.2. Given $\tilde{q} = (q, q_0) \in \mathbb{Z}^n \times \mathbb{Z}$, we let*

$$R_{q,q_0} = \{x \in U \mid q \cdot f(x) + q_0 = 0\}.$$

Define the following set

$$\mathcal{R}_f = \{R_{q,q_0} \mid (q, q_0) \in \mathbb{Z}^n \times \mathbb{Z}\}$$

and the function $N(R_{q,q_0}) = \|\tilde{q}\|_\infty^{n+1}$. Then, for almost every $x_0 \in U$, there is a ball $B_0 \subset U$ centered at x_0 such that $(\mathcal{R}, N, d-1)$ is a regular system in B_0.

Structure of the paper. In section 2 we recall some basic geometric and analytic facts about p-adic and S-arithmetic spaces. Theorem 1.4 is proved in section 3. Section 4 is devoted to the notion of good functions. This section involves only statements of several technical ingredients needed later in the paper. Most of the proofs can be found in [MS08]. Section 5 is devoted to the proof of theorem 1.5. This is actually done modulo theorem 5.1 which provides a translation of theorem 1.5 into a problem with dynamical nature. This dynamical problem is then solved in section 6 using an S-arithmetic version of a theorem in [KM98] which was proved in [KT07]. We recall the notion of regular systems in section 7 and prove theorem 1.6 in this section. The proof of the main theorems will be completed in section 8. The final section contains some concluding remarks and open problems.

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2 Notations and Preliminaries

Calculus of functions on local fields. The terminology recalled here is from [Sf84]. Let F be a local field and let f be an F-valued function defined on an open subset U of F. Let

$$\nabla^k U := \{(x_1, \cdots, x_k) \in U^k \mid x_i \neq x_j \text{ for } i \neq j\},$$

and define the k^{th} order difference quotient $\Phi^k f : \nabla^{k+1}U \to F$ of f inductively by $\Phi^0 f = f$ and

$$\Phi^k f(x_1, x_2, \cdots, x_{k+1}) := \frac{\Phi^{k-1} f(x_1, x_3, \cdots, x_{k+1}) - \Phi^{k-1} f(x_2, x_3, \cdots, x_{k+1})}{x_1 - x_2}.$$

As one sees readily $\Phi^k f$ is a symmetric function of its $k+1$ variables. The function f, is then called C^k at $a \in U$ if the following limit exits

$$\lim_{(x_1, \cdots, x_{k+1}) \to (a, \cdots, a)} \Phi^k f(x_1, \cdots, x_{k+1}),$$

and f is called C^k on U if it is C^k at every point $a \in U$. This is equivalent to $\Phi^k f$ being extendable to $\bar{\Phi}^k f : U^{k+1} \to F$. This extension, if exists, is indeed unique. The C^k functions are k times differentiable, and

$$f^{(k)}(x) = k! \bar{\Phi}^k(x, \cdots, x).$$

Note that, $f \in C^k$ implies $f^{(k)}$ is continuous but the converse fails. Also $C^\infty(U)$ is defined to be the class of functions which are C^k on U, for any k. Analytic functions are indeed C^∞.

Let now f be an F-valued function of several variables. Denote by $\Phi_i^k f$ the k^{th} order difference quotient of f with respect to the i^{th} coordinate. Then for any multi-index $\beta = (i_1, \cdots, i_d)$ let

$$\Phi_\beta f := \Phi_1^{i_1} \circ \cdots \circ \Phi_d^{i_d} f.$$

One defines the notion of C^k functions correspondingly.

S-Arithmetic spaces: Let ν be any place of $\mathbb{Q}$ we denote by $\mathbb{Q}_\nu$ the completion of $\mathbb{Q}$ with respect to ν and let $\mathbb{Q}_S = \prod_{\nu \in S} \mathbb{Q}_\nu$. If ν is a finite place we let p_ν be the uniformizer and $\mathbb{Z}_\nu$ the ring of ν-integers. Given a $\mathbb{Q}_\nu$-vector space $\mathcal{V}$ and a basis $\mathfrak{B}$ we let $\| \, \|_{\mathfrak{B}}$ denote the max norm with respect to this basis and we drop the index $\mathfrak{B}$ from the notation if there is no confusion. This naturally extends to a norm on $\bigwedge \mathcal{V}$. If $\mathcal{R}$ is any ring and $x, y \in \mathcal{R}^n$ we let $x \cdot y = \sum_{i=1}^n x^{(i)} y^{(i)}$. The following is the definition of "orthogonality" which will be useful in the sequel.

Definition 2.1. Let ν be a finite place of $\mathbb{Q}$. A set of vectors $x_1, \cdots, x_n$ in $\mathbb{Q}_\nu^m$, is called orthonormal if $\|x_1\| = \|x_2\| = \cdots = \|x_n\| = \|x_1 \wedge \cdots \wedge x_n\| = 1$, or equivalently when it can be extended to a $\mathbb{Z}_\nu$-base of $\mathbb{Z}_\nu^m$.

Recall that $\mathbb{Z}_{\tilde{S}} = \mathbb{Q} \cap \mathbb{Q}_{\tilde{S}} \cdot \prod_{\nu \notin S} \mathbb{Z}_\nu$ is a co-compact lattice in $\mathbb{Q}_{\tilde{S}}$, where $\mathbb{Q}$ is embedded diagonally. We normalize the Haar measure so that $\mu_\nu(\mathbb{Z}_\nu) = 1$ for all finite places and $\mu_\infty([0,1]) = 1$ for the infinite place, and we let μ be the product measure on $\mathbb{Q}_{\tilde{S}}$. With this normalization $\mathbb{Z}_{\tilde{S}}$ has co-volume one in $\mathbb{Q}_{\tilde{S}}$. For any $\mathbf{x} \in \prod_{\nu \in S} \mathbb{Q}_\nu^{m_\nu}$, let $c(\mathbf{x}) = \prod_{\nu \in S} \|x_\nu\|_\nu$. One clearly has $c(\mathbf{x}) \leq \|\mathbf{x}\|_S^\kappa$. The following gives the description of of discrete $\mathbb{Z}_{\tilde{S}}$-modules in $\prod_{\nu \in \tilde{S}} \mathbb{Q}_\nu^{m_\nu}$.

Lemma 2.2. *(cf. [KT07, Proposition 7.2]) If Δ is a discrete $\mathbb{Z}_{\tilde{S}}$-submodule of $\prod_{\nu \in \tilde{S}} \mathbb{Q}_\nu^{m_\nu}$, then there are $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(r)}$ in $\prod_{\nu \in \tilde{S}} \mathbb{Q}_\nu^{m_\nu}$ so that $\Delta = \mathbb{Z}_{\tilde{S}}\mathbf{x}^{(1)} \oplus \cdots \oplus \mathbb{Z}_{\tilde{S}}\mathbf{x}^{(r)}$. Moreover $x_\nu^{(1)}, \cdots, x_\nu^{(r)}$ are linearly independent over $\mathbb{Q}_\nu$ for any place $\nu \in \tilde{S}$*

Definition 2.3. Let Γ be a discrete $\mathbb{Z}_{\tilde{S}}$-submodule of $\prod_{\nu \in \tilde{S}} \mathbb{Q}_\nu^{m_\nu}$ then a submodule Δ of Γ is called a "*primitive submodule*" if $\Delta = \Delta_{\mathbb{Q}_{\tilde{S}}} \cap \Gamma$, where $\Delta_{\mathbb{Q}_{\tilde{S}}}$ is the $\mathbb{Q}_{\tilde{S}}$-span of Δ.

Remark 2.4. Let Γ and Δ be as in definition 2.3; then Δ is a primitive submodule of Γ, if and only if there exists a complementary $\mathbb{Z}_{\tilde{S}}$-submodule $\Delta' \subseteq \Gamma$, i.e. $\Delta \cap \Delta' = 0$ and $\Delta + \Delta' = \Gamma$.

3 Proof of Theorem 1.4

Fix $\mathbf{q} = (q_1, \ldots, q_n)$ with $|\mathbf{q}|_\infty < T$ and, as in the introduction, let $\tilde{\mathbf{q}} = (\mathbf{q}, q_0)$. Let

$$\mathcal{A}_{\mathbf{q}} = \left\{ \mathbf{x} \in \mathcal{A} | \exists\, q_0,\ |q_0|_\infty < T, \quad \begin{array}{l} |\mathbf{q} \cdot \mathbf{f}(\mathbf{x}) + q_0|_S^\kappa \ < \delta T^{-n-1} \\ \|\mathbf{q}\nabla \mathbf{f}(\mathbf{x})\|_\nu > |\tilde{\mathbf{q}}|_\infty^{-\epsilon} \text{ for all } \nu \in S \end{array} \right\}.$$

We will show that, $|\mathcal{A}_{\mathbf{q}}| < \delta\, T^{-n}|\mathbf{B}|$. We then will sum over all possible $\mathbf{q}$'s and the proof will be concluded.

We set

$$R = T^{\frac{1}{\kappa}}, \quad \text{and} \quad \mathbf{B}(\mathbf{x}) = \prod_{\nu \in S} B(x_\nu, \frac{1}{4R\|q\nabla f_\nu(x_\nu)\|_\nu}).$$

One obviously has $\mathcal{A}_{\mathbf{q}} \subseteq \cup_{\mathbf{x} \in \mathcal{A}_{\mathbf{q}}} \mathbf{B}(\mathbf{x})$. Let $\mathbf{x} \in \mathcal{A}_{\mathbf{q}}$, then there exists q_0 such that $|\tilde{\mathbf{q}} \cdot \widetilde{\mathbf{f}}(\mathbf{x})|_S^\kappa < \delta T^{-n-1}$, where $\tilde{\mathbf{q}} = (q_0, \ldots, q_n)$. Let $\mathbf{y} \in \mathbf{B}(\mathbf{x})$; then, for any ν a place in S, we have

$$q \cdot f_\nu(y_\nu) = q \cdot f_\nu(x_\nu) + (q\nabla f_\nu(x_\nu)) \cdot (x_\nu - y_\nu) + \sum_{i,j} \Phi_{ij}(q \cdot f_\nu)\ (x_\nu^{(i)} - y_\nu^{(i)})(x_\nu^{(j)} - y_\nu^{(j)}).$$

Comparing the maximum possible values taking into consideration that $\epsilon < \frac{1}{2\kappa}$, we get that $|\tilde{\mathbf{q}} \cdot \widetilde{\mathbf{f}}(\mathbf{y})|_S < \frac{1}{4R}$ hence we have $|\tilde{q} \cdot \widetilde{f}_\nu(y_\nu)|_\nu < \frac{1}{4R}$ for all $\nu \in S$. Now if there are q_0^1 & q_0^2 so that $|\widetilde{q^{(i)}} \cdot \widetilde{f}_\nu|_\nu < \frac{1}{4R}$, we get $\prod_{\nu \in S} |q_0^1 - q_0^2|_\nu < \frac{1}{4T}$, which by the product formula gives $q_0^1 = q_0^2$.

We now want to give an upper bound for $|\mathbf{B}(\mathbf{x}) \cap \mathcal{A}_{\mathbf{q}}|$, where $\mathbf{x} \in \mathcal{A}_{\mathbf{q}}$. This will be done using (i) and (ii) below

(i) For any $\mathbf{y} \in \mathbf{B}(\mathbf{x})$ and $\nu \in S$, one has $\|\nabla q.f_\nu(y_\nu) - \nabla q.f_\nu(x_\nu)\|_\nu < \|\nabla q.f_\nu(x_\nu)\|_\nu/4$. To see this, let $\mathbf{z} = (z_\nu)$ where $y_\nu = x_\nu + z_\nu$. In this setting, one has

$$\partial_i q \cdot f_\nu(y_\nu) = \partial_i q \cdot f_\nu(x_\nu) + \sum_j \Phi_j^{(1)}(\partial_i q \cdot f_\nu)(t_\nu^{(j)}, t_\nu^{(j-1)}) z_\nu^j$$

$$= \partial_i q \cdot f_\nu(x_\nu) + \sum_j (\Phi_{ji}(q \cdot f_\nu(t_\nu^{(j)}, t_\nu^{(j-1)}, t_\nu^{(j-1)})) + \Phi_{ji}(q \cdot f_\nu(t_\nu^{(j)}, t_\nu^{(j)}, t_\nu^{(j-1)}))) z_\nu^j,$$

where $t_\nu^{(i)}$'s are certain expressions in terms of coordinates of x_ν and y_ν. Hence

$$|\partial_i q \cdot f_\nu(y_\nu) - \partial_i q \cdot f_\nu(x_\nu)|_\nu < |z_\nu|_\nu \leq \frac{1}{4R\|\nabla q \cdot f_\nu(x_\nu)\|_\nu} \leq \frac{\|\nabla q \cdot f_\nu(x_\nu)\|_\nu}{4},$$

as we claimed.

(ii) We now bound $|(\mathcal{A}_\mathbf{q})_\nu \cap B(x_\nu, \frac{1}{4R\|q\nabla f_\nu(x_\nu)\|_\nu})|$ from above for all $\nu \in S$, where $(\mathcal{A}_\mathbf{q})_\nu$ is the projection of $\mathcal{A}_\mathbf{q}$ to the place ν. Without loss of generality we may assume $|\nabla q f_\nu(x_\nu)|_\nu = |q \cdot \partial_1 f_\nu(x_\nu)|_\nu$. Let $\mathbf{y} \in \mathbf{B}(\mathbf{x}) \cap \mathcal{A}_\mathbf{q}$, then we have

$$\tilde{q} \cdot \tilde{f}_\nu(y_\nu + \alpha e_1) - \tilde{q} \cdot \tilde{f}_\nu(y_\nu) = q \cdot \partial_1 f_\nu(y_\nu)\alpha + \Phi_{11} q \cdot f(y_\nu + \alpha e_1, y_\nu, y_\nu)\alpha^2.$$

As before a norm comparison, using the fact $\epsilon < \frac{1}{2\kappa}$, gives us

$$|\tilde{q} \cdot \tilde{f}_\nu(y_\nu + \alpha e_1) - \tilde{q} \cdot \tilde{f}_\nu(y_\nu)|_\nu = |q \cdot \partial_1 f_\nu(y_\nu)|_\nu |\alpha|_\nu \tag{1}$$

Now set $\widehat{y_\nu} = (\widehat{y_\nu^{(1)}, y_\nu^{(2)}}, \ldots, y_\nu^{(d_\nu)})$ for fixed $\{y_\nu^2, \ldots, y_\nu^{d_\nu}\}$. Using (1) we conclude that the measure of

$$\{y_\nu^1 \in \mathbb{Q}_\nu | \; y_\nu = (y_\nu^{(1)}, \widehat{y_\nu}) \in B(x_\nu, \frac{1}{4R\|q\nabla f_\nu(x_\nu)\|_\nu}) \cap (\mathcal{A}_\mathbf{q})_\nu\}$$

is at most $\frac{C'' \; (\delta T^{-n-1})^{\frac{1}{\kappa}}}{\|\nabla q \cdot f_\nu(x_\nu)\|_\nu}$, where C'' is a universal constant. This gives us

$$|(\mathcal{A}_\mathbf{q})_\nu \cap B(x_\nu, \frac{1}{4R\|q\nabla f_\nu(x_\nu)\|_\nu})| \leq C' \delta^{\frac{1}{\kappa}} R \, T^{\frac{-n-1}{\kappa}} |B(x_\nu, \frac{1}{4R\|q\nabla f_\nu(x_\nu)\|_\nu})|.$$

Recall that for non-Archimedean valuations two balls are either disjoint or one contains the other. We get $|(\mathcal{A}_\mathbf{q})_\nu| \leq C' \delta^{\frac{1}{\kappa}} R \, T^{\frac{-n-1}{\kappa}} |\mathbf{B}_\nu|$. Multiplying these inequalities for various ν's we have

$$|\mathcal{A}_\mathbf{q}| \leq C\delta T^{-n} |\mathbf{B}|.$$

We now sum up over all possible $\mathbf{q}$'s and get $|\mathcal{A}| \leq C\delta|\mathbf{B}|$, as we wished.

4 Good Functions

In this section we will state conditions which guarantee a "polynomial like" behavior of certain classes of maps on local fields. The definition of a "*good function*" which is given below generalizes conditions on the class of functions considered in [EMS97]. This definition was suggested in [KM98]. In what follows we just recall statements which are needed in the course of proof of theorem 5.1. The proofs can be found in [MS08].

Definition 4.1. (cf. [KM98, Section 3]) Let C and α be positive real numbers. A function $\mathbf{f}$ defined on an open set $\mathbf{V}$ of $X = \prod_{\nu \in S} \mathbb{Q}_\nu^{m_\nu}$ is called (C, α)-good, if for any open ball $\mathbf{B} \subset \mathbf{V}$ and any $\varepsilon > 0$ one has

$$|\{\mathbf{x} \in \mathbf{B}| \ \|\mathbf{f}(\mathbf{x})\| < \varepsilon \cdot \sup_{\mathbf{x} \in \mathbf{B}} \|\mathbf{f}(\mathbf{x})\|\}| \leq C\, \varepsilon^\alpha |\mathbf{B}|.$$

Remark 4.2. The following are consequences of the definition. Let $X, \mathbf{V}$ and $\mathbf{f}$ be as in definition 4.1. Then

(i) $\mathbf{f}$ is (C, α)-good on $\mathbf{V}$ if and only if $\|\mathbf{f}\|$ is (C, α)-good.

(ii) If $\mathbf{f}$ is (C, α)-good on $\mathbf{V}$, then so is $\lambda \mathbf{f}$ for any $\lambda \in \mathbb{Q}_S$.

(iii) Let I be a countable index set, if $\mathbf{f}_i$ is (C, α)-good on $\mathbf{V}$ for any $i \in I$, then so is $\sup_{i \in I} \|\mathbf{f}_i\|$.

(iv) If $\mathbf{f}$ is (C, α)-good on $\mathbf{V}$ and $c_1 \leq \|\mathbf{f}(\mathbf{x})\|_S / \|\mathbf{g}(\mathbf{x})\|_S \leq c_2$, for any $x \in \mathbf{V}$, then $\mathbf{g}$ is $(C(c_2/c_1)^\alpha, \alpha)$-good on $\mathbf{V}$.

As we mentioned above the definition of good functions was motivated by class of functions which have polynomial like behavior. The next lemma guarantees that indeed polynomials are good.

Lemma 4.3. *(cf. [KT07, Lemma 2.4]) Let ν be any place of $\mathbb{Q}$ and $p \in \mathbb{Q}_\nu[x_1, \ldots, x_d]$ be a polynomial of degree not greater than l. Then there exists $C = C_{d,l}$ independent of p, such that p is $(C, 1/dl)$-good on $\mathbb{Q}_\nu$.*

Next theorem "relates" the definition of a good function to conditions on its derivatives.

Theorem 4.4. *(cf. [MS08, Theorem 4.4]) Let $V_1, \ldots, V_d$ be nonempty open sets in $\mathbb{Q}_\nu$. Let $k \in \mathbb{N}$, $A_1, \ldots, A_d, A'_1, \ldots, A'_d$ be positive real numbers and $f \in C^k(V_1 \times \cdots \times V_d)$ be such that*

$$A_i \leq |\Phi_i^k f|_\nu \leq A'_i \ \text{ on } \bigtriangledown^{k+1} V_i \times \prod_{j \neq i} V_j, \ i = 1, \ldots, d.$$

Then f is (C, α)-good on $V_1 \times \cdots \times V_d$, where C and α depend only on k, d, A_i, and A'_i .

We need to show some families of functions are good with uniform constants. The following gives a condition to guarantee such assertion. The proof of this uses compactness arguments and Theorem 4.4 above. In our setting we actually will use the following corollary.

Theorem 4.5. *(cf. [MS08, Theorem 4.5]) Let U be an open neighborhood of $x_0 \in \mathbb{Q}_\nu^m$ and let $\mathcal{F} \subset C^l(U)$ be a family of functions $f : U \to \mathbb{Q}_\nu$ such that*

1. *$\{\nabla f : \ f \in \mathcal{F}\}$ is compact in $C^{l-1}(U)$*
2. *$\inf_{f \in \mathcal{F}} \sup_{|\beta| \leq l} |\partial_\beta f(x_0)| > 0$.*

Then there exist a neighborhood $V \subseteq U$ of x_0 and positive numbers $C = C(\mathcal{F})$ and $\alpha = \alpha(\mathcal{F})$ such that for any $f \in \mathcal{F}$

(i) f is (C, α)-good on V.

(ii) ∇f is (C, α)-good on V.

Corollary 4.6. *Let $f_1, f_2, \ldots, f_n$ be analytic functions from a neighborhood U of x_0 in $\mathbb{Q}_\nu^m$ to $\mathbb{Q}_\nu$, such that $1,\ f_1, f_2, \ldots, f_n$ are linearly independent on any neighborhood of x_0. Then*

(i) There exist a neighborhood V of x_0, C & $\alpha > 0$ such that any linear combination of $1, f_1, f_2, \ldots, f_n$ is (C, α)-good on V.

(ii) There exist a neighborhood V' of x_0, C' & $\alpha' > 0$ such that for any $c_1, c_2, \ldots, c_n \in \mathbb{Q}_\nu$, $\| \sum_{k=1}^n c_i \nabla f_i \|$ is (C', α')-good on V'.

We now recall the notion of *skew gradient* from [BKM01, Section 4]. For $i = 1, 2$ let $g_i : \mathbb{Q}_\nu^d \to \mathbb{Q}_\nu$ be two C^1 functions. Define then $\widetilde{\nabla}(g_1, g_2) := g_1 \nabla g_2 - g_2 \nabla g_1$. This, in some sense, measures how far two functions are from being linearly dependent. The following is the main technical result of section 4 in [MS08]. Let us remark that this theorem is responsible for the fact that the results of this paper are in the setting of analytic functions rather than C^k functions. In Archimedean case the proof of this fact uses polar coordinates which is not available in non-Archimedean setting.

Theorem 4.7. *(cf. [MS08, Theorem 4.7]) Let U be a neighborhood of $x_0 \in \mathbb{Q}_\nu^m$ and $f_1, f_2, \ldots, f_n$ be analytic functions from U to $\mathbb{Q}_\nu$, such that $1,\ f_1, f_2, \ldots, f_n$ are linearly independent on any open subset of U. Let $F = (f_1, \ldots, f_n)$ and*

$$\mathcal{F} = \{(D_1 \cdot F,\ D_2 \cdot F + a) |\ \|D_1\| = \|D_2\| = \|D_1 \wedge D_2\| = 1,\ D_1, D_2 \in \mathbb{Q}_\nu^n,\ a \in \mathbb{Q}_\nu\}.$$

Then there exists a neighborhood $V \subseteq U$ of x_0 such that

(i) For any neighborhood $B \subseteq V$ of x_0, there exists $\rho = \rho(\mathcal{F}, B)$ such that $\sup_{x \in B} \| \widetilde{\nabla} g(x) \| \geq \rho$ for any $g \in \mathcal{F}$.

(ii) There exist C and α, positive numbers, such that for any $g \in \mathcal{F}$, $\|\widetilde{\nabla} g\|$ is (C, α)-good on V.

5 Theorem 1.5 and lattices

In this section we prove Theorem 1.5. This is done with the aid of converting the problem into a question about quantitative recurrence properties of some "special flows" on the space of discrete $\mathbb{Z}_{\tilde{S}}$-modules. This dynamical translation was the breakthrough by Kleinbock and Margulis, see [KM98]. This point of view was then followed in [BKM01], [KT07] and [MS08].

Until now we essentially worked with a single non-Archimedean place. From this point on we need to work with all the places in $\tilde{S} = \{\infty\} \cup S$, simultaneously.

Let us fix some further notation to be used in the sequel. Let $d_\infty = 0$ and let $m_\nu + d_\nu + 1$ for all $\nu \in \tilde{S}$. We extend $\mathbf{f}$ in the statement of Theorem 1.5 to a map on $\prod_{\nu\in\tilde{S}} \mathbb{Q}_\nu^{d_\nu}$ by setting $f_\infty = 0$ and continue to denote this map by $\mathbf{f}$. Let $\{e_\nu^0, e_\nu^{*1}, \dots, e_\nu^{*d_\nu}, e_\nu^1, \dots, e_\nu^n\}$ be the standard basis for $\mathbb{Q}_\nu^{m_\nu}$. Let $\mathbf{e}_i = (e_\nu^i)_{\nu\in\tilde{S}}$ and define the $\mathbb{Z}_{\tilde{S}}$-module Λ to be the $\mathbb{Z}_{\tilde{S}}$-span of $\{\mathbf{e}_0, \dots, \mathbf{e}_n\}$. For any $\mathbf{x} \in \mathbf{U}$ define

$$\mathcal{U}_\mathbf{x} = \left(\begin{pmatrix} 1 & 0 & f_\nu(x_\nu) \\ 0 & I_{d_\nu} & \nabla f_\nu(x_\nu) \\ 0 & 0 & I_n \end{pmatrix} \right)_{\nu\in\tilde{S}}$$

Note that in the real place we have the identity matrix I_{n+1}. If 0_{d_ν} denotes the $d_\nu \times 1$ zero block then one has

$$\mathcal{U}_\mathbf{x} \left(\begin{pmatrix} p \\ 0_{d_\nu} \\ \vec{q} \end{pmatrix} \right)_{\nu\in\tilde{S}} = \left(\begin{pmatrix} p + f_\nu(x_\nu)\cdot\vec{q} \\ \nabla f_\nu(x_\nu)\vec{q} \\ \vec{q} \end{pmatrix} \right)_{\nu\in\tilde{S}}.$$

Let $\varepsilon > 0$ be given. Define the diagonal matrix

$$\mathbf{D} = (D_\nu)_{\nu\in\tilde{S}} = (\mathrm{diag}((a_\nu^{(0)})^{-1}, (a_\nu^*)^{-1}, \dots, (a_\nu^*)^{-1}, (a_\nu^{(1)})^{-1}, \dots, (a_\nu^{(n)})^{-1}))_{\nu\in\tilde{S}}$$

where $a_\nu^{(0)} = \begin{cases} \lceil\delta\rceil_\nu & \nu \in S \\ T_0/\varepsilon & \nu = \infty \end{cases}$, $a_\nu^* = \lceil K_\nu \rceil_\nu$, $a_\nu^{(i)} = \begin{cases} 1 & \nu \in S \\ T_i/\varepsilon & \nu = \infty \end{cases}$ $1 \le i \le n$, and for a positive real number a and $\nu \in S$ we let $\lceil a \rceil_\nu$ (resp. $\lfloor a \rfloor_\nu$) denote a power of p_ν with the smallest (resp. largest) ν-adic norm bigger (resp. smaller) than a. The constants δ, K_ν and T_i above are as in the statement of Theorem 1.5.

The following, which will be proved in section 6, proves Theorem 1.5.

Theorem 5.1. *Let $\mathbf{U}$ and $\mathbf{f}$ be as in theorem 1.5; then for any $\mathbf{x} = (x_\nu)_{\nu\in S}$, there exists a neighborhood $\mathbf{V} = \prod_{\nu\in S} V_\nu \subseteq \mathbf{U}$ of $\mathbf{x}$, and a positive number α with the following property: for any $\mathbf{B} \subseteq \mathbf{V}$ there exists $E > 0$ such that for any $\mathbf{D} = (\mathrm{diag}((a_\nu^{(0)})^{-1}, (a_\nu^*)^{-1}, \dots, (a_\nu^*)^{-1}, (a_\nu^{(1)})^{-1}, \dots, (a_\nu^{(n)})^{-1}))_{\nu\in\tilde{S}}$ with $1 \le |a_\infty^{(i)}|_\infty$, $0 < |a_\nu^{(0)}|_\nu \le 1 \le |a_\nu^{(1)}|_\nu \le \cdots \le |a_\nu^{(n)}|_\nu$ for all $\nu \in S$ which satisfy*

(i) $0 < \prod_{\nu\in S} |a_\nu^*|_\nu \le |a_\infty^{(0)} a_\infty^{(1)} \cdots a_\infty^{(n)}|_\infty^{-1} \prod_{\nu\in S} |a_\nu^{(0)} a_\nu^{(1)} \cdots a_\nu^{(n-1)}|_\nu^{-1}$.

(ii) $1 \le \min_i \left| \frac{1}{a_\infty^{(i)}} \right|_\infty \prod_{\nu\in S} |a_\nu^{(0)}|_\nu^{-1}$

and for any positive number ε, one has

$$|\{\mathbf{y} \in \mathbf{B} |\; c(\mathbf{D}\mathcal{U}_\mathbf{y}\lambda) < \varepsilon \text{ for some } \lambda \in \Lambda \setminus \{0\}\}| \le E\, \varepsilon^\alpha |B|.$$

Proof of Theorem 1.5 modulo Theorem 5.1. Choose ε as in Theorem 1.5 and define $a_\nu^{(i)}$'s, a_ν^* and $\mathbf{D}$ as above. Our assumptions in Theorem 1.5 guarantee that $\mathbf{D}$ satisfies the conditions above. Now if $\lambda = \left(\begin{pmatrix} p \\ 0_{d_\nu} \\ \vec{q} \end{pmatrix} \right)_{\nu\in\tilde{S}}$ is such that $(p, \vec{\mathbf{q}})$ satisfies the conditions in (1.5) then we have $c(\mathbf{D}\mathcal{U}_{\mathbf{y}}\lambda) < \varepsilon$. Recall that $c(\mathbf{x}) \leq \|\mathbf{x}\|_{\tilde{S}}^{\kappa}$, now $\mathbf{V}$ and α/κ as in Theorem 5.1 satisfy conditions of Theorem 1.5.

$\square$

6 Proof of Theorem 5.1

In the previous section we reduced the proof of Theorem 1.5 to Theorem 5.1. This section contains the proof of the latter. Theorem 5.1 is a far reaching quantitative generalization of recurrence properties of unipotent flows on homogenous spaces. We refer to [KM98] for further discussion and complementary remarks. Let us start with the following

Definition 6.1. (cf. [KT07, Section 6]) Let Ω be the set of all discrete $\mathbb{Z}_{\tilde{S}}$-submodules of $\prod_{\nu\in\tilde{S}}\mathbb{Q}_{\nu}^{m_{\nu}}$. A function θ from Ω to the positive real numbers is called a *norm-like map* if the following three properties hold:

i) For any Δ, Δ' with $\Delta' \subseteq \Delta$ and the same $\mathbb{Z}_{\tilde{S}}$-rank, one has $\theta(\Delta) \leq \theta(\Delta')$.

ii) For any Δ and $\gamma \notin \Delta_{\mathbb{Q}_{\tilde{S}}}$, one has $\theta(\Delta + \mathbb{Z}_{\tilde{S}}\gamma) \leq \theta(\Delta)\theta(\mathbb{Z}_{\tilde{S}}\gamma)$.

iii) For any Δ, the function $g \mapsto \theta(g\Delta)$ is a continuous function of $g \in \mathrm{GL}(\prod_{\nu\in\tilde{\mathrm{S}}}\mathbb{Q}_{\nu}^{\mathrm{m}_{\nu}})$.

Theorem 6.2. *(cf. [KT07, Theorem 8.3]) Let $\mathbf{B} = \mathbf{B}(\mathbf{x}_0, r_0) \subset \prod_{\nu\in S}\mathbb{Q}_{\nu}^{d_{\nu}}$ and $\widehat{\mathbf{B}} = \mathbf{B}(\mathbf{x}_0, 3^m r_0)$ for $m = \min_{\nu}(m_{\nu})$. Assume that $\mathbf{H} : \widehat{\mathbf{B}} \to \mathrm{GL}(\prod_{\nu\in\tilde{\mathrm{S}}}\mathbb{Q}_{\nu}^{\mathrm{m}_{\nu}})$ is a continuous map. Also let θ be a norm-like map defined on the set Ω of discrete $\mathbb{Z}_{\tilde{S}}$-submodules of $\prod_{\nu\in\tilde{S}}\mathbb{Q}_{\nu}^{m_{\nu}}$, and $\mathfrak{P}$ be a subposet of Ω. For any $\Gamma \in \mathfrak{P}$ denote by ψ_{Γ} the function $\mathbf{x} \mapsto \theta(\mathbf{H}(\mathbf{x})\Gamma)$ on $\widehat{\mathbf{B}}$. Now suppose for some $C, \alpha > 0$ and $\rho > 0$ one has*

(i) for every $\Gamma \in \mathfrak{P}$, the function ψ_{Γ} is (C, α)-good on $\widehat{\mathbf{B}}$;

(ii) for every $\Gamma \in \mathfrak{P}$, $\sup_{\mathbf{x}\in\mathbf{B}} \|\psi_{\Gamma}(\mathbf{x})\|_{\tilde{S}} \geq \rho$;

(iii) for every $\mathbf{x} \in \widehat{\mathbf{B}}$, $\#\{\Gamma \in \mathfrak{P} | \; \|\psi_{\Gamma}(\mathbf{x})\|_{\tilde{S}} \leq \rho\} < \infty$.

Then for any positive $\varepsilon \leq \rho$ one has

$$|\{\mathbf{x} \in \mathbf{B} | \; \theta(\mathbf{H}(\mathbf{x})\lambda) < \varepsilon \text{ for some } \lambda \in \Lambda \setminus \{0\}\}| \leq mC(N_{((d_{\nu}),S)}D^2)^m\left(\frac{\varepsilon}{\rho}\right)^{\alpha}|\mathbf{B}|,$$

where D may be taken to be $\prod_{\nu\in S}(3p_{\nu})^{d_{\nu}}$, and $N_{((d_{\nu}),S)}$ is the Besicovich constant for the space $\prod_{\nu\in S}\mathbb{Q}_{\nu}^{d_{\nu}}$.

The idea of the proof of Theorem 6.2 is very similar to Margulis' proof of recurrence properties of unipotent flows on homogenous spaces, but the proof is more technical. We will prove Theorem 5.1 using this theorem. However we need to set the stage for using this theorem.

The poset: let Λ be as in section 5 and let $\mathfrak{P}$ be the poset of primitive $\mathbb{Z}_{\tilde{S}}$-modules of Λ.

The norm-like map: For any $\nu \in \tilde{S}$ we let $\mathcal{I}^*_\nu$ be the ideal generated by $\{e^{*i}_\nu \wedge e^{*j}_\nu \text{ for } 1 \leq i, j \leq d_\nu\}$. Note that $\mathcal{I}^*_\infty = 0$. Let $\pi_\nu : \bigwedge \mathbb{Q}^{m_\nu}_\nu \to \bigwedge \mathbb{Q}^{m_\nu}_\nu / \mathcal{I}^*_\nu$ be the natural projection. For $\mathbf{x} \in \prod_{\nu\in\tilde{S}} \bigwedge \prod_{\nu\in\tilde{S}} \mathbb{Q}^{m_\nu}_\nu$ define $\theta(\mathbf{x}) = \prod_{\nu\in\tilde{S}} \theta_\nu(x_\nu)$ where $\theta_\nu(x_\nu) = \|\pi_\nu(x_\nu)\|_{\pi_\nu(\mathfrak{B}_\nu)}$ and $\mathfrak{B}_\nu$ is the standard basis of $\bigwedge \mathbb{Q}^{m_\nu}_\nu$. Finally for any discrete $\mathbb{Z}_{\tilde{S}}$-submodule Δ of $\prod_{\nu\in\tilde{S}} \mathbb{Q}^{m_\nu}_\nu$, let $\theta(\Delta) = \theta(\mathbf{x}^{(1)} \wedge \cdots \wedge \mathbf{x}^{(r)})$, where $\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r)}\}$ is a $\mathbb{Z}_{\tilde{S}}$-basis of Δ. Using the product formula, it is readily seen that $\theta(\Delta)$ is well-defined. This is our norm-like map.

The family $\mathcal{H}$: Let $\mathcal{H}$ be the family of functions

$$\mathbf{H} : \mathbf{U} = \prod_{\nu\in\tilde{S}} U_\nu \to \mathrm{GL}(\prod_{\nu\in\tilde{S}} \mathbb{Q}^{m_\nu}_\nu) \quad \text{where} \quad \mathbf{H}(\mathbf{x}) = \mathbf{D}\mathcal{U}_{\mathbf{x}},$$

where $\mathbf{D}$ and $\mathcal{U}_{\mathbf{x}}$ are as in Theorem 5.1.

Note that the restriction of θ to $\prod_{\nu\in\tilde{S}} \mathbb{Q}^{m_\nu}_\nu$ is the same as the function c. Hence Theorem 6.2 reduces the proof of Theorem 5.1 to finding a neighborhood $\mathbf{V}$ of $\mathbf{x}$ which satisfies the following

(I) There exist $C, \alpha > 0$, such that all the functions $\mathbf{y} \mapsto \theta(\mathbf{H}(\mathbf{y})\Delta)$, where $\mathbf{H} \in \mathcal{H}$ and $\Delta \in \mathfrak{P}$ are (C, α)-good on $\mathbf{V}$.

(II) For all $\mathbf{y} \in \mathbf{V}$ and $\mathbf{H} \in \mathcal{H}$, one has $\#\{\Delta \in \mathfrak{P} |\ \theta(\mathbf{H}(\mathbf{y})\Delta) \leq 1\} < \infty$.

(III) For every ball $\mathbf{B} \subseteq \mathbf{V}$, there exists $\rho > 0$ such that $\sup_{\mathbf{y}\in\mathbf{B}} \theta(\mathbf{H}(\mathbf{y})\Delta) \geq \rho$ for all $\mathbf{H} \in \mathcal{H}$ and $\Delta \in \mathfrak{P}$.

If $\mathbf{x} \in \prod_{\nu\in S} \mathbb{Q}^{d_\nu}_\nu$ define $\mathbf{V} = \prod_{\nu\in S} V_\nu$, where V_ν is small enough such that assertions of Corollary 4.6 and Theorem 4.7 hold. We now verify (I), (II), (III) for this choice of $\mathbf{V}$.

Proof of (I). Let Δ be a primitive submodule of Λ and let $k = \mathrm{rank}_{\mathbb{Z}_{\tilde{S}}} \Delta$. Denote by $(\mathbf{D}\Delta)_\nu$ the $\mathbb{Q}_\nu$-span of the projection of $\mathbf{D}\Delta$ to the place $\nu \in \tilde{S}$, note that $\dim_{\mathbb{Q}_\nu}(\mathbf{D}\Delta)_\nu = k$. Let W_ν (resp. W^*_ν) be the $\mathbb{Q}_\nu$-span of $\{e^1_\nu, \ldots, e^{n-1}_\nu\}_{\mathbb{Q}_\nu}$ (resp. $\{e^{*1}_\nu, \ldots, e^{*d_\nu}_\nu\}$). Let $x^{(1)}_\nu, \ldots, x^{(k-1)}_\nu \in (\mathbf{D}\Delta)_\nu \cap W_\nu \oplus \mathbb{Q}_\nu e^n_\nu$ be an orthonormal set, see section 2 for the definition in the non-arthimedean setting. Complete this to an orthonormal basis for $(\mathbf{D}\Delta)_\nu \oplus \mathbb{Q}_\nu e^0_\nu$ by adding e^0_ν and $x^{(0)}_\nu$ if needed. Let $\{\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(k)}\}$ be a $\mathbb{Z}_{\tilde{S}}$-basis for Δ. We have $\theta(\mathbf{D}\Delta) = \theta(\mathbf{D}\mathcal{Y})$, where $\mathcal{Y} = \mathbf{y}^{(1)} \wedge \cdots \wedge \mathbf{y}^{(k)}$. Let $a_\nu, b_\nu \in \mathbb{Q}_\nu$ be such that

$$(\mathbf{D}\mathcal{Y})_\nu = a_\nu e^0_\nu \wedge x^{(1)}_\nu \wedge \cdots \wedge x^{(k-1)}_\nu + b_\nu x^{(0)}_\nu \wedge \cdots \wedge x^{(k-1)}_\nu.$$

If $g(x) = (g_1(x), g_2(x))$ for g_1 and g_2 two functions from an open subset of $\mathbb{Q}^{d_\nu}_\nu$ to $\mathbb{Q}_\nu$ define $\widetilde{\nabla}^*(g)(x) = g_1(x)\nabla^* g_2(x) - g_2(x)\nabla^* g_1(x)$ where $\nabla^* \bar{g}(x_\nu) = \sum_{i=1}^{d_\nu} \partial_i \bar{g}(x_\nu) e^{*i}_\nu$.

Let us also define $\hat{\mathbf{f}}(\mathbf{x}) = (\hat{f}_\nu(x_\nu))_{\nu\in S}$, where

$$\hat{f}_\nu(x_\nu) = (1, 0_{d_\nu}, \frac{a^{(1)}_\nu}{a^{(0)}_\nu} f^{(1)}_\nu(x_\nu), \ldots, \frac{a^{(n)}_\nu}{a^{(0)}_\nu} f^{(n)}_\nu(x_\nu)).$$

Manipulation of the formulas gives

$$(\mathbf{D}\mathcal{U}_{\mathbf{x}}\mathbf{D}^{-1})_\nu w = w + (\hat{f}_\nu(x_\nu)\cdot w)e_\nu^0 + \frac{a_\nu^{(0)}}{a_\nu^*}\nabla^*(\hat{f}_\nu(x_\nu)w),$$

whenever w is in $W_\nu \oplus \mathbb{Q}_\nu e_\nu^n$. Therefore we have

$$\begin{aligned}\pi_\nu((H(\mathbf{x})\mathcal{Y})_\nu) &= (a_\nu + b_\nu \hat{f}_\nu(x_\nu)x_\nu^{(0)})e_\nu^0 \wedge x_\nu^{(1)} \wedge \cdots \wedge x_\nu^{(k-1)} + b_\nu x_\nu^{(0)} \wedge \cdots \wedge x_\nu^{(k-1)} \\ &+ b_\nu \sum_{i=1}^{k-1} \pm(\hat{f}_\nu(x_\nu)x_\nu^{(i)})e_\nu^0 \wedge \bigwedge_{s\neq i} x_\nu^{(s)} + b_\nu \frac{a_\nu^{(0)}}{a_\nu^*}\sum_{i=0}^{k-1} \pm\nabla^*(\hat{f}_\nu(x_\nu)x_\nu^{(i)}) \wedge \bigwedge_{s\neq i} x_\nu^{(s)} \\ &+ \frac{a_\nu^{(0)}}{a_\nu^*}\sum_{i=1}^{k-1} \pm\widetilde{\nabla}^*(\hat{f}_\nu(x_\nu)x_\nu^{(i)}, a_\nu + b_\nu\hat{f}_\nu(x_\nu)x_\nu^{(0)}) \wedge e_\nu^0 \wedge \bigwedge_{s\neq 0,i} x_\nu^{(s)} \qquad (2) \\ &+ b_\nu \frac{a_\nu^{(0)}}{a_\nu^*}\sum_{i,j=1, j>i}^{k-1} \pm\widetilde{\nabla}^*(\hat{f}_\nu(x_\nu)x_\nu^{(i)}, \hat{f}_\nu(x_\nu)x_\nu^{(j)}) \wedge e_\nu^0 \wedge \bigwedge_{s\neq i,j} x_\nu^{(s)}.\end{aligned}$$

The orthogonality assumption gives that the norm of the above vector would be the maximum of norms of each of its summands. Hence we need to show each summand is a good function. Note that there is nothing to prove in the case $\nu = \infty$. If $\nu \in S$ however our choice of $\mathbf{V}$ and conditions on $\mathbf{f}$ guarantee that we may apply Corollary 4.6 and Theorem 4.7 hence each summand is (C_ν, α_ν)-good as we wanted.

Proof of (II). First line in Equation (2), gives that

$$\theta(\mathbf{D}\mathcal{U}_{\mathbf{x}}\Delta) \geq \prod_{\nu\in\tilde{S}} \max\{|a_\nu + b_\nu\hat{f}_\nu(x_\nu)\cdot x_\nu^{(0)}|, |b_\nu|\}$$

Thus $\theta(\mathbf{D}\mathcal{U}_{\mathbf{x}}\Delta) \leq 1$ implies that $\prod_{\nu\in\tilde{S}}\max\{|a_\nu|,|b_\nu|\}$ has an upper bound. Hence Corollary 7.9 of [KT07] finishes the proof of (II).

Proof of (III). Let $\mathbf{B} \subseteq \mathbf{V}$ be a ball containing $\mathbf{x}$. Define

$$\rho_1 = \inf\{|f_\nu(x_\nu)\cdot C_\nu + c_\nu^0|_\nu \mid \mathbf{x}\in\mathbf{B}, \nu\in S, C_\nu \in \mathbb{Q}_\nu^n, \|C_\nu\| = 1, c_\nu^0 \in \mathbb{Q}_\nu\},$$

$$\rho_2 = \inf\{\sup_{\mathbf{x}\in\mathbf{B}} \|\nabla f_\nu(x_\nu)C_\nu\| \,|\nu\in S, C_\nu\in\mathbb{Q}_\nu^n, \|C_\nu\| = 1\},$$

Further let $M = \sup_{\mathbf{x}\in\mathbf{B}}\max\{\|\mathbf{f}(\mathbf{x})\|_S, \|\nabla\mathbf{f}(\mathbf{x})\|_S\}$ and ρ_3 be the constant obtained by theorem 4.7(a).

Assume first that $\operatorname{rank}_{\mathbb{Z}_{\tilde{S}}}\Delta = 1$. Hence Δ can be represented by a vector $\mathbf{w} = (w_\nu)_{\nu\in S}$, with $w_\nu^i \in \mathbb{Z}_{\tilde{S}}$ for all i's and any $\nu\in\tilde{S}$. Now

$$c(\mathbf{D}\mathcal{U}_{\mathbf{x}}\mathbf{w}) \geq \min_i \left|\frac{1}{a_\infty^{(i)}}\right|_\infty \prod_{\nu\in S}\left|\frac{w_\nu^{(0)} + \sum_{i=1}^n f_\nu^{(i)}(x_\nu)w_\nu^{(i)}}{a_\nu^{(0)}}\right|_\nu \geq \rho_1^\kappa.$$

The proof in this case is complete.

Hence we may assume $\mathrm{rank}_{\mathbb{Z}_{\tilde{S}}}\Delta = k > 1$. With the notations as in part (I) let $x_\nu^{(1)}, \ldots, x_\nu^{(k-2)}$ be an orthonormal set in $W_\nu \cap \Delta_\nu$. We extend this to an orthonormal set in $(W_\nu \oplus \mathbb{Q}_\nu e_\nu^n) \cap \Delta_\nu$ by adding $x_\nu^{(k-1)}$. Now if necessary choose a vector $x_\nu^{(0)}$ such that $\{e_\nu^0, x_\nu^{(0)}, x_\nu^{(1)}, \ldots, x_\nu^{(k-1)}\}$ is an orthonormal basis for $\Delta_\nu + \mathbb{Q}_\nu e_\nu^0$ Let $\mathcal{Y} = \mathbf{y}^{(1)} \wedge \cdots \wedge \mathbf{y}^{(k)}$ be as before. Since D_ν leaves $W_\nu, W_\nu^*, \mathbb{Q}_\nu e_\nu^0$, and $\mathbb{Q}_\nu e_\nu^n$ invariant, one has

$$\theta(\mathbf{D}\mathcal{U}_{\mathbf{x}}\Delta) = \theta(\mathbf{D}\mathcal{U}_{\mathbf{x}}\mathcal{Y}) = \prod_{\nu \in \tilde{S}} \theta_\nu(D_\nu \mathcal{U}_{\mathbf{x}}^\nu \mathcal{Y}_\nu) = \prod_{\nu \in \tilde{S}} \|D_\nu \pi_\nu(\mathcal{U}_{\mathbf{x}}^\nu \mathcal{Y}_\nu)\|_\nu.$$

Let $a_\nu, b_\nu \in \mathbb{Q}_\nu$ be so that

$$\mathcal{Y}_\nu = a_\nu e_\nu^0 \wedge x_\nu^{(1)} \wedge \cdots \wedge x_\nu^{(k-1)} + b_\nu x_\nu^{(0)} \wedge \cdots \wedge x_\nu^{(k-1)}.$$

Note that $\prod_{\nu \in \tilde{S}} \{|a_\nu|_\nu, |b_\nu|_\nu\} \geq 1$. Let $\check{\mathbf{f}}(\mathbf{x}) = (\check{f}_\nu(x_\nu))_{\nu \in \tilde{S}}$ where

$$\check{f}_\nu(x_\nu) = (1, 0_{d_\nu}, f_\nu^{(1)}(x_\nu), \ldots, f_\nu^{(n)}(x_\nu)).$$

We have

$$\begin{aligned}
\pi_\nu(\mathcal{U}_{\mathbf{x}}^\nu \mathcal{Y}_\nu) &= (a_\nu + b_\nu \check{f}_\nu(x_\nu) x_\nu^{(0)}) e_\nu^0 \wedge x_\nu^{(1)} \wedge \cdots \wedge x_\nu^{(k-1)} + b_\nu x_\nu^{(0)} \wedge \cdots \wedge x_\nu^{(k-1)} \\
&\quad + b_\nu \sum_{i=1}^{k-1} \pm (\check{f}_\nu(x_\nu) x_\nu^{(i)}) e_\nu^0 \wedge \bigwedge_{s \neq i} x_\nu^{(s)} + b_\nu \sum_{i=0}^{k-1} \pm \nabla^*(\check{f}_\nu(x_\nu) x_\nu^{(i)}) \wedge \bigwedge_{s \neq i} x_\nu^{(s)} \\
&\quad + e_\nu^0 \wedge \check{\mathcal{Y}}_\nu(x_\nu), \\
&\text{where } \check{\mathcal{Y}}_\nu(x_\nu) = \sum_{i=1}^{k-1} \pm \widetilde{\nabla}^*(\check{f}_\nu(x_\nu) x_\nu^{(i)}, a_\nu + b_\nu \check{f}_\nu(x_\nu) x_\nu^{(0)}) \wedge \bigwedge_{s \neq 0, i} x_\nu^{(s)} \\
&\quad + b_\nu \sum_{i,j=1, j>i}^{k-1} \pm \widetilde{\nabla}^*(\check{f}_\nu(x_\nu) x_\nu^{(i)}, \check{f}_\nu(x_\nu) x_\nu^{(j)}) \wedge \bigwedge_{s \neq i,j} x_\nu^{(s)}.
\end{aligned}$$

Claim: For all $\nu \in S$ one has

$$\sup \|e_\nu^n \wedge \check{\mathcal{Y}}_\nu(x_\nu)\|_\nu \geq \rho_0 \cdot \max\{|a_\nu|_\nu, |b_\nu|_\nu\}.$$

Proof of the claim: Let $\nu \in S$ we have

$$e_\nu^n \wedge \check{\mathcal{Y}}_\nu(x_\nu) = \pm z_\nu^{(*)}(x_\nu) \wedge e_\nu^n \wedge x_\nu^{(1)} \wedge x_\nu^{(2)} \cdots \wedge x_\nu^{(k-2)} + \begin{array}{l}\text{other terms where one} \\ \text{or two } x_\nu^{(i)} \text{ are missing,}\end{array}$$

where

$$\begin{aligned}
z_\nu^{(*)}(x_\nu) &= \widetilde{\nabla}^*(\check{f}_\nu(x_\nu) x_\nu^{k-1},\ a_\nu + b_\nu \check{f}_\nu(x_\nu) x_\nu^{(0)}) \\
&= b_\nu \widetilde{\nabla}^*(\check{f}_\nu(x_\nu) x_\nu^{k-1},\ \check{f}_\nu(x_\nu) x_\nu^{(0)}) - a_\nu \nabla^*(\check{f}_\nu(x_\nu) x_\nu^{(k-1)})
\end{aligned}$$

The first expression implies that $\sup_{x_\nu \in B_\nu} \|z_\nu^{(*)}(x_\nu)\|_\nu \geq \rho_3\ |b_\nu|_\nu$, and the second expression gives $\sup_{x_\nu \in B_\nu} \|z_\nu^{(*)}(x_\nu)\|_\nu \geq \rho_2 |a_\nu|_\nu - 2M^2|b_\nu|_\nu$. Thus there exists ρ_0 such that

$$\max\{\rho_2|a_\nu|_\nu - 2M^2|b_\nu|_\nu, \rho_3\ |b_\nu|_\nu\} \geq \rho_0 \cdot \max\{|a_\nu|_\nu, |b_\nu|_\nu\}.$$

This shows the claim.

Let $\nu \in S$ be any place; then $\|D_\nu(e_\nu^n \wedge \check{\mathcal{Y}}_\nu(x_\nu))\|_\nu \leq \|D_\nu \check{\mathcal{Y}}_\nu(x_\nu)\|_\nu / |a_\nu^{(n)}|_\nu$. Hence the smallest (in norm) eigenvalue of the action of D_ν on $W_\nu^* \wedge (\bigwedge^{k-1}(\mathbb{Q}_\nu e_\nu^0 \oplus W_\nu \oplus \mathbb{Q}_\nu e_\nu^n))$ is $\left(a_\nu^{(*)} a_\nu^{(n-k+2)} \cdots a_\nu^{(n)}\right)^{-1}$.

Let $\mathcal{R} = \frac{\max\{|a_\infty|,|b_\infty|\}}{|a_\infty^{(0)} a_\infty^{(1)} \cdots a_\infty^{(n)}|_\infty}$ we have

$$\sup_{x \in \mathbf{B}} \theta(\mathbf{D}\mathcal{U}_{\mathbf{x}}\mathcal{Y}) \geq \mathcal{R} \prod_{\nu \in S} \|D_\nu(e_\nu^0 \wedge \check{\mathcal{Y}}_\nu(x_\nu))\|_\nu \geq \mathcal{R} \prod_{\nu \in S} \frac{|a_\nu^{(n)}|_\nu \|D_\nu(e_\nu^n \wedge \check{\mathcal{Y}}_\nu(x_\nu))\|_\nu}{|a_\nu^{(0)}|_\nu}$$

$$\geq \mathcal{R} \prod_{\nu \in S} \frac{|a_\nu^{(n)}|_\nu \|e_\nu^n \wedge \check{\mathcal{Y}}_\nu(x_\nu)\|_\nu}{|a_\nu^{(0)} a_\nu^{(*)} a_\nu^{(n-k+3)} \cdots a_\nu^{(n)}|_\nu} \geq$$

$$\frac{\rho_0^\kappa \max\{|a|_\infty, |b|_\infty\}}{|a_\infty^{(0)} a_\infty^{(1)} \cdots a_\infty^{(n)}|_\infty} \prod_{\nu \in S} \frac{\max\{|a|_\nu, |b|_\nu\}}{|a_\nu^{(0)} a_\nu^{(*)} a_\nu^{(n-k+3)} \cdots a_\nu^{(n-1)}|_\nu} \geq \rho_0^\kappa.$$

This finishes the proof of part (III).

As mentioned before now Theorem 6.2 completes the proof of Theorem 5.1.

7 Regular systems

In this section, we will prove Theorem 1.6 and will state a general result about regular systems, Theorem 7.3. Through out this section, as we are working with only one place, we shall use U and f instead of $\mathbf{U}$ and $\mathbf{f}$ and they will be as Theorem 1.1, and as before let $\widetilde{f} = (f, 1)$. Let us first recall the definition of a *regular system of resonant sets*. This is a generalization of the concept of a regular system of points of Baker and Schmidt [BS70] for the real line.

Definition 7.1. (cf. [BBKM02, Definition 3.1]) Let U be an open subset of $\mathbb{Q}_\nu^d$, $\mathcal{R}$ be a family of subsets of $\mathbb{Q}_\nu^d$, $N : \mathcal{R} \to \mathbb{R}_+$ be a function and let s be a number satisfying $0 \leq s < d$. The triple $(\mathcal{R}, N, s)$ is called a *regular system* in U if there exists constants $K_1, K_2, K_3 > 0$ and a function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{x\to\infty} \lambda(x) = +\infty$ such that for any ball $B \subset U$ and for any $T > T_0 = T_0(\mathcal{R}, N, s, B)$ is a sufficiently large number, there exists

$$R_1, \ldots, R_t \in \mathcal{R} \quad \text{with} \quad \lambda(T) \leq N(R_i) \leq T \quad \text{for} \quad i = 1, \ldots, t$$

and disjoint balls

$$B_1, \ldots, B_t \quad \text{with} \quad 2B_i \subset B \quad \text{for} \quad i = 1, \ldots, t$$

such that

$$\operatorname{diam}(B_i) = T^{-1} \quad \text{for} \quad i = 1, \ldots, t$$
$$t \geq K_1 |B| T^d$$

and such that, for any $\gamma \in \mathbb{R}$ with $0 < \gamma < T^{-1}$, one has

$$K_2 \gamma^{d-s} T^{-s} \leq |B(R_i, \gamma) \cap B_i|$$

$$|B(R_i, \gamma) \cap 2B_i| \leq K_3 \gamma^{d-s} T^{-s}$$

where $B(R_i, \gamma)$ is the γ neighborhood of R_i. Moreover the elements of $\mathcal{R}$ will be called *resonant sets.*

The construction of the desired regular system, which in some sense is the main result of this section, will make essential use of the following.

Theorem 7.2. *Let U and f be as in statement of Theorem 1.2. In particular, f is non-degenerate on U. Then, for any $x_0 \in U$, there exist a sufficiently small ball $V \subset U$ centered at x_0 and a constant $C_0 > 0$ such that for any ball $B \subseteq V$ and any $\delta > 0$, for all sufficiently large Q, one has*

$$|\mathcal{A}_f(\delta; B; Q)| \leq C_0 \delta |B|,$$

where

$$\mathcal{A}_f(\delta; B; Q) = \bigcup_{\tilde{q} \in \mathbb{Z}^{n+1} : 0 < \|\tilde{q}\|_\infty \leq Q} \{x \in B |\ |\tilde{q} \cdot \widetilde{f}(x)| < \delta Q^{-n-1}\}$$

Proof. Let V be an open ball about x_0 such that the assertions of Theorems 1.4 and 1.5 hold. We will show V satisfies the conclusion of the theorem. For any $B \subseteq V$ and $0 < \varepsilon < 1/2$, let

$$\mathcal{A}^1(\delta; B; Q; \epsilon) = \left\{ x \in B |\ \exists\, \tilde{q},\ \|\tilde{q}\|_\infty < Q, \quad \begin{array}{l} |\tilde{q} \cdot \widetilde{f}(x)|_\nu \ < \delta Q^{-n-1} \\ \|q \nabla f(x)\|_\nu > |\tilde{q}|_\infty^{-\epsilon} \end{array} \right\}$$

and

$$\mathcal{A}^2(\delta; B; Q; \epsilon) = \left\{ x \in B |\ \exists \tilde{q},\ \|\tilde{q}\| < Q : \begin{array}{l} |\tilde{q} \cdot \widetilde{f}(x)| < \delta Q^{-n-1} \\ \|\nabla f_\nu(x) q\|_\nu \leq |\tilde{q}|_\infty^{-\epsilon} \end{array} \right\}.$$

One obviously has $\mathcal{A}(\delta; B; Q) \subseteq \mathcal{A}^1(\delta; B; Q; \epsilon) \cup \mathcal{A}^2(\delta; B; Q; \epsilon)$. Now one applies the bounds from Theorems 1.4 and 1.5 for $|\mathcal{A}^1(\delta; B; Q; \epsilon)|$ and $|\mathcal{A}^2(\delta; B; Q; \epsilon)|$ respectively. These give

$$|\mathcal{A}^1(\delta; B; Q; \epsilon)| \leq C_1 \delta |B| \quad \text{and} \quad |\mathcal{A}^1(\delta; B; Q; \epsilon)| \leq C_2 (\delta Q^{-\epsilon})^{\frac{\alpha}{n+1}} |B|$$

where $\alpha > 0$, C_1 is a positive universal constant, and $C_2 = C_2(B)$ is a positive number, which just depends on B. Hence, for sufficiently large Q (depending on B), the second expression can be made smaller than $C_1 \delta |B|$, which completes our proof.

□

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Thanks to the non-degeneracy assumption, replacing U with a smaller neighborhood, we may and will assume $f_1(x) = x_1$. Moreover, we can choose B_0 such that theorem 7.2 holds. Therefore the aforementioned theorem will guarantee that for any $B \subset B_0$, one has

$$|\mathcal{G}(B; \delta; Q)| \geq \frac{1}{2}|B| \tag{3}$$

for large enough Q, where

$$\mathcal{G}(B; \delta; Q) = \frac{3}{4}B \setminus \mathcal{A}_f(\delta; \frac{3}{4}B; Q)$$

Let $x \in \mathcal{G}(B; \delta; Q)$, applying a Dirichlet's principle argument one gets an absolute constant C such that for sufficiently large Q one can solve the following system of inequalities:

$$\begin{cases} |q \cdot f(x) + q_0|_\nu < C\delta^2 Q^{-n-1} \\ |q_i|_\infty < \delta^{-1}Q \qquad i = 0, 1, \ldots, n \\ |q_i|_\nu < \delta \qquad\qquad i = 2, \ldots, n \end{cases}$$

This, thanks to the fact that $x \in \mathcal{G}((B; \delta; Q))$, implies that $T = \delta^{-n-1}Q^{n+1}$ will satisfy $Q^{n+1} \leq N(R_{q,q_0}) \leq T$.

First claim: Let (q, q_0) satisfy the above system of inequalities. Define the function $F(x) = q \cdot f(x) + q_0$, then one has $|\partial_1 F(x)|_\nu > \frac{\delta}{2}$.

Assume the contrary so $|\partial_1 F(x)|_\nu \leq \frac{\delta}{2}$. This assumption gives $|q_1|_\nu < \delta$. Now since we have $|q \cdot f(x) + q_0|_\nu < C\delta^2 Q^{-n-1}$, if Q is sufficiently large, we will have $|q_0|_\nu < \delta$. This says that we can replace (q, q_0) by $(q', q'_0) = \frac{1}{p_\nu^l}(q, q_0)$ and have

$$\begin{cases} |q' f(x) + q'_0|_\nu < C\delta Q^{-n-1} \\ |q'_i|_\infty < Q \qquad i = 0, 1, \ldots, n \end{cases}$$

This however contradicts our assumption that $x \in \mathcal{G}((B; \delta; Q))$. Hence we have that $|\partial_1 F(x)|_\nu > \frac{\delta}{2}$. The first claim is proved.

Second claim: There exists $z \in R_{q,q_0}$ such that $|z - x|_\nu < 2C\delta Q^{-n-1}$, for large enough Q.

Using uniform continuity and the ultrametric inequality we get that there exists $r_1 > 0$ such that if $\|x - y\|_\nu < r_1$ then $|\partial_1 F(y)|_\nu > \frac{\delta}{2}$. As $x \in \frac{3}{4}B$ we have $B(x, \operatorname{diam} B) \subset B$. Define $r_0 = \min(r_1, \operatorname{diam} B)$, so we have $|\partial_1 F(y)|_\nu > \frac{\delta}{2}$ for all $y \in B(x, r_0)$.

Now if $x = (x_1, \ldots, x_d)$ and $|\theta|_\nu < r_0$ then $x_\theta = (x_1 + \theta, x_2, \ldots, x_d) \in B(x, r_0)$. Let $g(\theta) = F(x_\theta)$. Then

$$|g(0)|_\nu = |F(x)|_\nu < C\delta^2 Q^{-n-1} \text{ and } |g'(0)|_\nu = |\partial_1 F(x)|_\nu > \frac{\delta}{2}.$$

We now apply Newton's method, see [W98], and get: there exists θ_0 such that $g(\theta_0) = 0$ and $|\theta_0|_\nu < 2C\delta Q^{-n-1}$. So if $Q > \frac{2C}{\delta^{1/n+1}}$ then we have $x_{\theta_0} \in B(x, r_0)$. Hence there is $z \in R_{q,q_0}$ with $|z - x|_\nu < 2C\delta Q^{-n-1}$.

Third claim: There is a constant K_2 so that for any $0 < \gamma < T^{-1}$ we have

$$K_1\gamma T^{-(d-1)} \leq |B(R_{q,q_0}, \gamma) \cap B(z, T^{-1}/2)|$$

If $d = 1$ we are done by taking $K_1 = 1/2$ so we assume $d > 1$. Let $z = (z_1, \ldots, z_d)$ and $z' = (z_2, \ldots, z_d)$ where z is as in second claim above. Now for any $y' = (y_2, \ldots, y_d) \in \mathbb{Q}_\nu^{d-1}$ such that $|y' - z'|_\nu < C_1T^{-1}$ let $y = (y_1, y') = (y_1, y_2, \ldots, y_d)$ where $y_1 \in \mathbb{Q}_\nu$. If $|y_1 - z_1|_\nu \leq T^{-1}/4$ then $y \in B(z\,, T^{-1}/4)$.

We now want to show that for any y' with $|y' - z'|_\nu < C_1T^{-1}$ one can find $y_1(y') \in \mathbb{Q}_\nu$ such that $y = (y_1(y'), y') \in R_{q,q_0} \cap B(z, T^{-1}/4)$. First note that $B(z, T^{-1}) \subset B(x, r_0)$. So if $|y_1(y') - z_1|_\nu < T^{-1}/4$ then $(y_1(y'), y') \in B(x, r_0)$. This thanks to our previous observations gives $|\partial_1 F(y)|_\nu > \delta/2$. Now, as in the proof of Theorem 1.4, we have

$$F(y) = F(z) + \nabla F(z) \cdot (y - z) + \sum_{i,j} \Phi_{ij} F(y_i - z_i)(y_j - z_j).$$

Comparing the maximum of the norms using $F(z) = 0$ and $|y' - z'|_\nu < C_1T^{-1}$ we get that if $|y_1(y') - z_1|_\nu < T^{-1}/4$ then $|F(y)|_\nu < T^{-1}/4$. Again Newton's method helps to find $y_1(y')$ with $|y_1(y') - z_1|_\nu \leq T^{-1}/4$ such that $F(y_1(y'), y') = 0$. For any $0 < \gamma < T^{-1}$ define

$$\mathcal{A}(\gamma) = \{(y_1(y') + \theta, y') |\ \|y' - z'\|_\nu < C_1T^{-1},\ |\theta|_\nu \leq \gamma/2\}$$

The above gives $\mathcal{A}(\gamma) \subset B(R_{q,q_0}, \gamma) \cap B(z, T^{-1}/4)$. So an application of Fubini finishes the proof of the third claim.

The proof of the theorem now goes as in [BBKM02], we recall the steps here for the sake of completeness. Assume Q is large enough so that Theorem 7.2 holds. Choose a collection

$$(q_1, q_{0,1}, z_1), \ldots, (q_t, q_{0,t}, z_t) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} \times B \ \text{ with } \ z_i \in R_{q_i, q_{0,i}}$$

such that

$$Q^{n+1} = T\delta^{n+1} \leq N(R_{q_i, q_{0,i}}) \leq T = \delta^{-n-1}Q^{n+1} \ \ (1 \leq i \leq t)$$

and such that for any γ with $0 < \gamma < T^{-1}$ we have

$$K_2\gamma T^{-(d-1)} \leq |B(R_{q_i, q_{0,i}}, \gamma) \cap B(z_i, T^{-1}/2)| \ \ (1 \leq i \leq t)$$

$$|B(R_{q_i, q_{0,i}}, \gamma) \cap B(z_i, T^{-1})| \geq K_3\gamma T^{-(d-1)} \quad (1 \leq i \leq t)$$

Now by our above discussion for any point $x \in \mathcal{G}(B; \delta; Q)$ there is a triple

$$(q, q_0, z) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} \times B \ \text{ with } \ z \in R_{q,q_0}$$

which satisfies the above claims. Since t was chosen to be maximal there is an index $i \in \{1, \dots, t\}$ such that

$$B(z_i, T^{-1}/2) \cap B(z, T^{-1}/2) \neq \emptyset$$

As a result we have $\|z - z_i\| < T^{-1}/2$. This together with the second claim above gives $\|x - z_i\| < C_2 T^{-1}$. Thus

$$\mathcal{G}(B; \delta; Q) \subset \bigcup_{i=1}^{i=t} B\left(z_i, C_2 T^{-1}\right)$$

This inclusion plus (3) above give

$$|B|/2 \leq |\mathcal{G}(B; \delta; Q)| \leq t \cdot |B(0, C_2)| T^{-d}$$

Therefore $t \geq K_1 |B| T^d$, where $K_1 = |2B(0, C_2)|$.

Now $R_i = R_{q_i, q_{0,i}}$ and $B_i = B(z_i, T^{-1}/2)$ serve as the resonant sets and the desired balls in the definition 7.1. This finishes the proof of Theorem 1.6.

The following is a general result on regular systems which is Theorem 4.1 in [BBKM02]. The proof in there is only given for $\mathbb{R}^d$ however the same proof works for $\mathbb{Q}_\nu^d$ and we will not reproduce the proof here.

Theorem 7.3. *(cf. [BBKM02, Theorem 4.1])*
Let U be an open subset of $\mathbb{Q}_\nu^d$, and let $(\mathcal{R}, N, S)$ be a regular system in U. Let $\widetilde{\Psi} : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function such that the sum

$$\sum_{k=1}^{\infty} k^{d-s-1} \widetilde{\Psi}(k)^{d-s}$$

diverges. Then for almost all points $x \in U$ the inequality

$$\operatorname{dist}(x, R) < \widetilde{\Psi}(N(R))$$

has infinitely many solutions $R \in \mathcal{R}$.

8 Proof of the main theorems

We finally come to the proof of Theorems 1.1 and 1.2.

8.1 Proof of the convergence part

Take $\mathbf{x}_0 \in \mathbf{U}$. Choose a neighborhood $\mathbf{V} \subseteq \mathbf{U}$ of $\mathbf{x}_0$ and a positive number α, as in theorem 1.5, and pick a ball $\mathbf{B} = \prod_{\nu \in S} B_\nu \subseteq \mathbf{V}$ containing $\mathbf{x}_0$ such that the ball with the same center and triple the radius is contained in $\mathbf{U}$. We will show that $\mathbf{B} \cap \mathcal{W}_{\mathbf{f}, \Psi}$ has measure zero. For any $\tilde{\mathbf{q}} \in \mathbb{Z}^{n+1} \setminus \{0\}$, let

$$A_{\tilde{\mathbf{q}}} = \{(x_\nu)_{\nu \in S} \in B | \ |(f_\nu(x_\nu)) \cdot \tilde{\mathbf{q}}|_S < \Psi(\tilde{\mathbf{q}})\}.$$

We need to prove that the set of points $\mathbf{x}$ in $\mathbf{B}$ which belong to infinitely many $A_{\tilde{\mathbf{q}}}$ for $\tilde{\mathbf{q}} \in \mathbb{Z}^{n+1} \setminus \{0\}$ has measure zero. Now let

$$A_{\geq \tilde{\mathbf{q}}} = \{\mathbf{x} \in A_{\tilde{\mathbf{q}}} | \ \|\nabla f_\nu(x_\nu) q\|_\nu \geq |\tilde{\mathbf{q}}|_\infty^{-\epsilon}\} \ \ \&$$

$$A_{<\tilde{\mathbf{q}}} = A_{\tilde{\mathbf{q}}} \setminus A_{\geq \tilde{\mathbf{q}}}.$$

Furthermore for any $t \in \mathbb{N}$ let

$$\bar{A}_{\geq t} = \bigcup_{\tilde{\mathbf{q}} \in \mathbb{Z}^{n+1}, 2^t \leq \|\tilde{\mathbf{q}}\|_\infty < 2^{t+1}} A_{\geq \tilde{\mathbf{q}}} \quad \& \quad \bar{A}_{<t} = \bigcup_{\tilde{\mathbf{q}} \in \mathbb{Z}^{n+1}, 2^t \leq \|\tilde{\mathbf{q}}\|_\infty < 2^{t+1}} A_{<\tilde{\mathbf{q}}}$$

Recall that Ψ is non-increasing. Hence using theorem 1.4, with

$$2^t \leq \|\tilde{\mathbf{q}}\|_\infty \leq 2^{t+1} \text{ and } \delta = 2^{t(n+1)} \Psi(2^t, \dots, 2^t),$$

we see that $|\bar{A}_{\geq t}| \leq C 2^{t(n+1)} \Psi(2^t, \dots, 2^t)$. Now a use of Borel-Cantelli lemma, gives that almost every $\mathbf{x} \in \mathrm{B}$ is in at most finitely many sets $A_{\geq \tilde{\mathbf{q}}}$.

We will be done if we show that the sum of the measures of $\bar{A}_{<t}$'s is convergent. The conditions posed on Ψ imply easily that $\Psi(\tilde{\mathbf{q}}) \leq \|\tilde{\mathbf{q}}\|_\infty^{-(n+1)}$ for large enough $\|\tilde{\mathbf{q}}\|_\infty$. So if $2^t \leq \|\tilde{\mathbf{q}}\|_\infty < 2^{t+1}$ then $\Psi(\tilde{\mathbf{q}}) \leq 2^{-t(n+1)}$ for large enough t. Now for such t we may write $\bar{A}_t = \cup_{\nu \in S} \bar{A}_{t,\nu}$ where each $\bar{A}_{t,\nu}$ is contained in the set defined in 1.5, with $\delta = 2^{-t(n+1)}$, $T_i = 2^{t+1}$, $K_\nu = 2^{-\epsilon t}$ and $K_\omega = 1$ for $\omega \in S \setminus \{\nu\}$. It is not hard to verify the inequalities in the hypothesis of theorem 1.5. Moreover, one has

$$\varepsilon^{n+1} = \max\{\delta^{n+1}, \delta^\kappa T_0 \cdots T_n \prod_{\nu \in S} K_\nu\} = 2^{-\epsilon t} \delta^\kappa T_0 \cdots T_n = C' 2^{-\epsilon t},$$

for some universal constant C. So by theorem 1.5 and the choice of $\mathbf{V}$ and $\mathbf{B}$ measure of $\bar{A}_t$ is at most

$$C 2^{\frac{-\alpha \epsilon t}{\kappa(n+1)}} |\mathbf{B}|.$$

Therefore the sum of measures of $\bar{A}_t$'s is finite, thus another use of Borel-Cantelli lemma completes the proof of Theorem 1.1.

8.2 Proof of the divergence part.

Replacing U by a smaller neighborhood, if needed, we assume that Theorem 1.6 holds for U. Define now the sequence

$$\widetilde{\Psi}(x) = x^{-n/(n+1)} \psi(x^{1/(n+1)})$$

As ψ was non-increasing we get that $\widetilde{\Psi}$ is non-increasing as well, and we have

$$\sum_{k=1}^{\infty} k^{d-s-1} \widetilde{\Psi}(k)^{d-s} = \sum_{k=1}^{\infty} \widetilde{\Psi}(k) = \sum_{\ell=1}^{\infty} \sum_{(\ell-1)^{n+1} < k \leq \ell^{n+1}} \widetilde{\Psi}(k) \geq$$

$$\sum_{\ell=1}^{\infty} \sum_{(\ell-1)^{n+1} < k \leq \ell^{n+1}} \ell^{-n} \psi(\ell) \geq \sum_{\ell=1}^{\infty} \psi(\ell) = \infty$$

Now Theorem 7.3 says, for almost every $x \in U$ there are infinitely many elements $(q, q_0) \in \mathbb{Z}^n \times \mathbb{Z}$ satisfying

$$\operatorname{dist}(x, R_{q,q_0}) < \widetilde{\Psi}(\|\tilde{q}\|_\infty^{n+1}) \tag{4}$$

this says, there is a point $z \in R_{q,q_0}$ such that

$$\|x - z\|_\nu < \widetilde{\Psi}(\|\tilde{q}\|_\infty^{n+1}).$$

We have

$$F(x) = F(z) + \nabla F(x) \cdot (x - z) + \sum_i \Phi_{ij}(F)(t_i, t_j) \cdot (x_i - z_i)(x_j - z_j).$$

Recall that we have $\|\tilde{q}\|_\nu \leq 1$, $\|\nabla f\|_\nu \leq 1$, $|\Phi_{i,j}(f)|_\nu \leq 1$ and $F(z) = 0$. Hence we get

$$|\tilde{q} \cdot f(x)|_\nu = |F(x)|_\nu \leq \|x - z\|_\nu < \widetilde{\Psi}(\|\tilde{q}\|_\infty^{n+1}) = \|\tilde{q}\|_\infty^{-n} \psi(\|\tilde{q}\|_\infty) = \Psi(\tilde{q}) \tag{5}$$

Note that for almost every $x \in U$ there are infinitely many $(q, q_0) \in \mathbb{Z}^n \times \mathbb{Z}$, which satisfy (4). So we get that for almost every $x \in U$ there are infinitely many $(q, q_0) \in \mathbb{Z}^n \times \mathbb{Z}$ satisfying (5). This completes the proof of Theorem 1.2.

9 A few remarks and open problems

1. In this article, we worked with product of non-degenerate p-adic analytic manifolds. However most of the argument is valid for the product of non-degenerate C^k manifolds. The only part in which we use analyticity extensively is in the proof of Theorem 4.7.

2. The divergence result in here was obtained for the case S is a singleton only. However the convergence part of this paper which gives the simultaneous approximation should be optimal and the divergence should hold in the more general setting of the simultaneous approximation as well.

3. We considered measured supported on non-degenrate manifolds in here. There are other natural measures that one can consider. Indeed D. Y. Kleinbock, E. Lindenstrauss, B. Weiss in [KLW04] proved extremality of certain non-planar fractal measures. It is interesting to prove a Khintchine type theorem for fractal measures.

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