ISOLATIONS OF GEODESIC PLANES IN THE FRAME BUNDLE OF A HYPERBOLIC 3-MANIFOLD

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Abstract. We present a quantitative isolation property of the lifts of properly immersed geodesic planes in the frame bundle of a geometrically finite hyperbolic 3-manifold. Our estimates are polynomials in the tight areas and Bowen-Margulis-Sullivan densities of geodesic planes, with degree given by the modified critical exponents.

Contents

1. Introduction

Let $\mathbb{H}^3$ denote the hyperbolic 3-space, and let $G := \mathrm{PSL}_2(\mathbb{C})$, which can be identified with the group $\mathrm{Isom}^+(\mathbb{H}^3)$ of all orientation preserving isometries of $\mathbb{H}^3$. Any complete orientable hyperbolic 3-manifold can be presented as a quotient $M = \Gamma\backslash\mathbb{H}^3$ where Γ is a torsion-free discrete subgroup of G. An oriented geodesic plane in M is the image of a totally geodesic immersion of the hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$ equipped with an orientation under the quotient map $\mathbb{H}^3 \to \Gamma\backslash\mathbb{H}^3$. In this paper, all geodesic planes are assumed to be oriented. Set $X := \Gamma\backslash G$. Via the identification of X with the oriented frame bundle $\mathrm{F}M$, a geodesic plane in M arises as the image of a unique

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$\mathrm{PSL}_2(\mathbb{R})$-orbit under the base point projection map

$$\pi : X \simeq \mathrm{F}M \to M.$$

Moreover a *properly immersed* geodesic plane in M corresponds to a *closed* $\mathrm{PSL}_2(\mathbb{R})$-orbit in X.

Setting $H := \mathrm{PSL}_2(\mathbb{R})$, the main goal of this paper is to obtain a quantitative isolation result for closed H-orbits in X when Γ is a *geometrically finite* group. Fix a left invariant Riemannian metric on G, which projects to the hyperbolic metric on $\mathbb{H}^3$. This induces the distance d on X so that the canonical projection $G \to X$ is a local isometry. We use this Riemannian structure on G to define the volume of a closed H-orbit in X. For a closed subset $S \subset X$ and $\varepsilon > 0$, $B(S, \varepsilon)$ denotes the ε-neighborhood of S.

The case when M is compact. We first state the result for compact hyperbolic 3-manifolds. In this case, Ratner [24] and Shah [28] independently showed that every H-orbit is either compact or dense in X. Moreover, there are only countably many compact H-orbits in X. Mozes and Shah [22] proved that an infinite sequence of compact H-orbits becomes equidistributed in X. Our questions concern the following quantitative isolation property: for given compact H-orbits Y and Z in X,

(1) How close can Y approach Z?
(2) Given $\varepsilon > 0$, what portion of Y enters into the ε-neighborhood of Z?

It turns out that volumes of compact orbits are the only complexity which measures their quantitative isolation property. The following theorem was proved by Margulis in an unpublished note:

Theorem 1.1 (Margulis)**.** *Let Γ be a cocompact lattice in G. For every $1/3 \le s < 1$, the following hold for any compact H-orbits $Y \neq Z$ in X:*

(1)
$$d(Y, Z) \gg \alpha_s^{-4/s} \cdot \mathrm{Vol}(Y)^{-1/s}\,\mathrm{Vol}(Z)^{-1/s}$$
where $\alpha_s = (\frac{1}{1-s})^{1/(1-s)}$.

(2) *For all $0 < \varepsilon < 1$,*
$$m_Y(Y \cap B(Z, \varepsilon)) \ll \alpha_s^4 \cdot \varepsilon^s \cdot \mathrm{Vol}(Z)$$
where m_Y denotes the H-invariant probability measure on Y.

In both statements, the implied constants depend only on the injectivity radius of $\Gamma\backslash G$ (see (11.9) and (11.10) for more details).

Remark 1.2. (1) By recent works ([17], [2]), there may be infinitely many compact H-orbits only when Γ is an arithmetic lattice.

(2) Theorem 1.1 for *some* exponent s is proved in [10, Lemma 10.3]. The proof in [10] is based on the effective ergodic theorem which relies on the arithmeticity of Γ via uniform spectral gap on compact H-orbits; the exponent s obtained in their approach however is much smaller than 1.

(3) Margulis' proof does not rely on the arithmeticity of Γ and is based on the construction of a certain function on Y which measures the distance $d(y, Z)$ for $y \in Y$ (cf. (1.13)). A similar function appeared first in the work of Eskin, Mozes and Margulis in the study of a quantitative version of the Oppenheim conjecture [12], and later in several other works (e.g., [11], [4], and [13]).

General geometrically finite case. We now consider a general hyperbolic 3-manifold $M = \Gamma\backslash\mathbb{H}^3$. Denote by $\Lambda \subset \partial\mathbb{H}^3$ the limit set of Γ and by $\operatorname{core} M$ the convex core of M, i.e.,

$$\operatorname{core} M = \Gamma\backslash \operatorname{hull}\Lambda \subset M$$

where $\operatorname{hull}\Lambda \subset \mathbb{H}^3$ denotes the convex hull of Λ. In the rest of the introduction, we assume that M is geometrically finite, that is, the unit neighborhood of $\operatorname{core} M$ has finite volume.

Let $Y \subset X$ be a closed H-orbit and $S_Y = \Delta_Y\backslash\mathbb{H}^2$ be the associated hyperbolic surface, where $\Delta_Y < H$ is the stabilizer in H of a point in Y. We assume that Y is non-elementary, that is, Δ_Y is not virtually cyclic; otherwise, we cannot expect an isolation phenomenon for Y, as there is a continuous family of parallel elementary closed H-orbits in general when M is of infinite volume. It is known that S_Y is always geometrically finite [23, Thm. 4.7].

Let $0 < \delta(Y) \leq 1$ denote the critical exponent of S_Y, i.e., the abscissa of the convergence of the series $\sum_{\gamma\in\Delta_Y} e^{-sd(o,\gamma(o))}$ for some $o \in \mathbb{H}^2$. We define the following *modified critical exponent* of Y:

$$\delta_Y := \begin{cases} \delta(Y) & \text{if } S_Y \text{ has no cusp} \\ 2\delta(Y) - 1 & \text{otherwise;} \end{cases} \tag{1.3}$$

note that $0 < \delta_Y \leq \delta(Y) \leq 1$, and $\delta_Y = 1$ if and only if S_Y has finite area.

In generalizing Theorem 1.1(1), we first observe that the distance $d(Y, Z)$ between two closed H-orbits Y, Z may be zero, e.g., if they both have cusps going into the same cuspidal end of X. To remedy this issue, we use the thick-thin decomposition of $\operatorname{core} M$. For $p \in M$, we denote by $\operatorname{inj} p$ the injectivity radius at p. For all $\varepsilon > 0$, the ε-thick part

$$(\operatorname{core} M)_\varepsilon := \{p \in \operatorname{core} M : \operatorname{inj} p \geq \varepsilon\} \tag{1.4}$$

is compact, and for all sufficiently small $\varepsilon > 0$, the ε-thin part given by $\operatorname{core} M - (\operatorname{core} M)_\varepsilon$ is contained in finitely many disjoint cuspidal ends, i.e., images of horoballs in $\Gamma\backslash\mathbb{H}^3$. Let $X_0 \subset X$ denote the renormalized frame bundle $\mathrm{RF}M$ (see (2.1)). Using the fact that the projection of X_0 is contained in $\operatorname{core} M$ under π, we define the ε-thick part of X_0 as follows:

$$X_\varepsilon := \{x \in X_0 : \pi(x) \in (\operatorname{core} M)_\varepsilon\}.$$

The following theorem extends Theorem 1.1 to all geometrically finite hyperbolic manifolds:

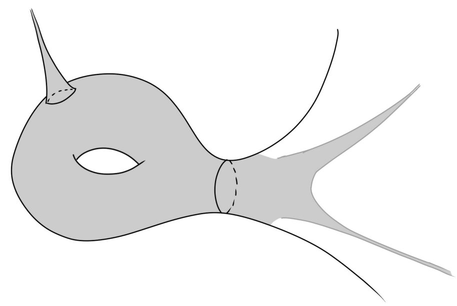

FIGURE 1. $S \cap \mathcal{N}(\operatorname{core} M)$

Theorem 1.5. *Let M be a geometrically finite hyperbolic 3-manifold. Let $Y \neq Z$ be non-elementary closed H-orbits in X, and denote by m_Y the probability Bowen-Margulis-Sullivan measure on Y. For every $\frac{\delta_Y}{3} \leq s < \delta_Y$ the following hold.*

(1) *For all $0 < \varepsilon \ll 1$, we have*

$$d(Y \cap X_\varepsilon, Z) \gg \alpha_{Y,s}^{-\star/s} \cdot \left(\frac{v_{Y,\varepsilon}}{\operatorname{area}_t Z} \right)^{1/s} \tag{1.6}$$

where

- $v_{Y,\varepsilon} = \min_{y \in Y \cap X_\varepsilon} m_Y(B_Y(y, \varepsilon))$ *where $B_Y(y, \varepsilon)$ is the ε-ball around y in the induced metric on Y.*
- $\operatorname{area}_t Z$ *denotes the tight area of S_Z relative to M.*
- $\alpha_{Y,s} := \left(\frac{\mathsf{s}_Y}{\delta_Y - s} \right)^{1/(\delta_Y - s)}$ *where s_Y is the shadow constant of Y.*

(2) *For all $0 < \varepsilon \ll 1$,*

$$m_Y(Y \cap B(Z, \varepsilon)) \ll \alpha_{Y,s}^\star \cdot \varepsilon^s \cdot \operatorname{area}_t Z.$$

In both statements, the implied constants and $\star$ depend only on Γ.

Remark.

(1) We give a proof of a more general version of Theorem 1.5(1) where Z is allowed to be equal to Y (see Corollary 10.5 for a precise statement).
(2) When X has finite volume, we have $\delta_Y = 1$ and m_Y is H-invariant so that $v_{Y,\varepsilon} \asymp \varepsilon^3 \operatorname{Vol}(Y)^{-1}$. Moreover, the tight area $\operatorname{area}_t Z$ and the shadow constant s_Y are simply the usual area of S_Z and a fixed constant (in fact, 2) respectively. Therefore Theorem 1.5 recovers Theorem 1.1. Moreover, the *exponent* $\star$ depends only on G as well; this follows since the proofs of Theorem 9.18 and theorems in Section

10, of which Theorem 1.5 is a special case, show that $\star$ depends only on s_Y, p_Y and δ_Y, which are all absolute constants in the finite volume case.

We now give definitions of the tight area $\operatorname{area}_t Z$ and the shadow constant s_Y for a general geometrically finite case; these are new geometric invariants introduced in this paper.

Definition 1.7 (Tight area of S)**.** For a properly immersed geodesic plane S of M, the *tight-area* of S relative to M is given by

$$\operatorname{area}_t(S) := \operatorname{area}(S \cap \mathcal{N}(\operatorname{core} M))$$

where $\mathcal{N}(\operatorname{core} M) = \{p \in M : d(p,q) \leq \operatorname{inj}(q) \text{ for some } q \in \operatorname{core} M\}$ is the tight neighborhood of $\operatorname{core} M$.

We show that $\operatorname{area}_t(S)$ is finite in Theorem 3.3, by proving that $S \cap \mathcal{N}(\operatorname{core} M)$ is contained in the union of a bounded neighborhood of $\operatorname{core}(S)$ and finitely many cusp-like regions (see Fig.1). We remark that the area of the intersection $S \cap B(\operatorname{core} M, 1)$ is not finite in general.

Definition 1.8 (Shadow constant of Y)**.** For a closed H-orbit Y in X, let $\Lambda_Y \subset \partial \mathbb{H}^2$ denote the limit set of Δ_Y, $\{\nu_p : p \in \mathbb{H}^2\}$ the Patterson-Sullivan density for Δ_Y, and $B_p(\xi, \varepsilon)$ the ε-neighborhood of $\xi \in \partial \mathbb{H}^2$ with respect to the Gromov metric at p. The shadow constant of Y is defined as follows:

$$\mathsf{s}_Y := \sup_{\xi \in \Lambda_Y, p \in [\xi, \Lambda_Y], 0 < \varepsilon \leq 1/2} \frac{\nu_p(B_p(\xi, \varepsilon))^{1/\delta_Y}}{\varepsilon \cdot \nu_p(B_p(\xi, 1/2))^{1/\delta_Y}}, \tag{1.9}$$

where $[\xi, \Lambda_Y]$ is the union of all geodesics connecting ξ to a point in Λ_Y.

We show that $\mathsf{s}_Y < \infty$ in Theorem 4.8.

Remark 1.10. If Y is convex cocompact, then for all $0 < \varepsilon < 1$, $v_{Y,\varepsilon} \asymp \varepsilon^{1+2\delta_Y}$ with the implied constant depending on Y. When Y has a cusp, Sullivan's shadow lemma (cf. Proposition 4.11) implies that $\lim_{\varepsilon \to 0} \frac{\log v_{Y,\varepsilon}}{\log \varepsilon}$ does not exist.

A hyperbolic 3-manifold M is called *convex cocompact acylindrical* if $\operatorname{core} M$ is a compact manifold with no essential discs or cylinders which are not boundary parallel. For such a manifold, there exists a uniform positive lower bound for $\delta(Y) = \delta_Y$ for all non-elementary closed H-orbits Y [20]; therefore the dependence of δ_Y can be removed in Theorem 1.5 if one is content with taking some s which works uniformly for all such orbits.

Examples of X with infinitely many closed H-orbits are provided by the following theorem which can be deduced from ([20], [21], [3]):

Theorem 1.11. *Let M_0 be an arithmetic hyperbolic 3-manifold with a properly immersed geodesic plane. Any geometrically finite acylindrical hyperbolic 3-manifold M which covers M_0 contains infinitely many non-elementary properly immersed geodesic planes.*

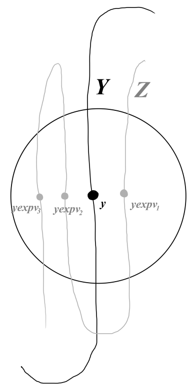

FIGURE 2. $I_Z(y)$

It is easy to construct examples of M satisfying the hypothesis of this theorem. For instance, if M_0 is an arithmetic hyperbolic 3-manifold with a properly embedded compact geodesic plane P, M_0 is covered by a geometrically finite acylindrical manifold M whose convex core has boundary isometric to P.

Finally, we mention the following application of Theorem 1.5 in view of recent interests in related counting problems [8].

Corollary 1.12. *Let* $\mathrm{Vol}(M) < \infty$*, and let* $\mathcal{N}(T)$ *denote the number of properly immersed totally geodesic planes* P *in* M *of area at most* T*. Then for any* $1/2 < s < 1$*, we have*

$$\mathcal{N}(T) \ll_s T^{(6/s)-1} \quad \text{for all } T > 1;$$

see Corollary 10.7 for a detailed information on the dependence of the implied constant.

We remark that when $\mathrm{Vol}(M) < \infty$, the heuristics suggest $s = \dim G/H = 3$ in Theorem 1.5 and hence $\mathcal{N}(T) \ll T$ in Corollary 1.12. Indeed, when $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$, the asymptotic $\mathcal{N}(T) \sim c \cdot T$, as suggested in [26], has been obtained by Jung [14] based on subtle number theoretic arguments. We refer to Remark 10.11 for an analogous statement to Corollary 1.12 for a general geometrically finite manifold.

Discussion on proofs. We discuss some of the main ingredients of the proof of Theorem 1.5. First consider the case when $X = \Gamma\backslash G$ is compact (the account below deviates slightly from Margulis' original argument). Let ε_X be the minimum injectivity radius of points in X. The Lie algebra of G decomposes as $\mathfrak{sl}_2(\mathbb{R}) \oplus i\mathfrak{sl}_2(\mathbb{R})$. Hence, for each $y \in Y$, the set

$$I_Z(y) := \{v \in i\mathfrak{sl}_2(\mathbb{R}) : 0 < \|v\| < \varepsilon_X, \;\; y\exp(v) \in Z\}$$

keeps track of all points of $Z \cap B(y, \varepsilon_X)$ in the direction transversal to H (see Fig. 2).

Therefore, the following function $f_s : Y \to [2, \infty)$ $(0 < s < 1)$ encodes the information on the distance $d(y, Z)$:

$$f_s(y) = \begin{cases} \sum_{v \in I_Z(y)} \|v\|^{-s} & \text{if } I_Z(y) \neq \emptyset \\ \varepsilon_X^{-s} & \text{otherwise} \end{cases}. \tag{1.13}$$

A function of this type is referred to as a *Margulis function* in the literature.

The proof of Theorem 1.1 is based on the following fact: the average of f_s is controlled by the volume of Z, i.e.,

$$m_Y(f_s) \ll_s \operatorname{Vol}(Z). \tag{1.14}$$

We prove the estimate in (1.14) using the following super-harmonicity type inequality: for any $1/3 \leq s < 1$, there exist $t = t_s > 0$ and $b = b_s > 1$ such that for all $y \in Y$,

$$\mathsf{A}_t f_s(y) \leq \frac{1}{2} f_s(y) + b \operatorname{Vol}(Z) \tag{1.15}$$

where $(\mathsf{A}_t f_s)(y) = \int_0^1 f_s(y u_r a_t) dr$, $u_r = \left(\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right)$, and $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$.

The proof of (1.15) is based on the inequality (11.1), which is essentially a lemma in linear algebra. We refer to the Appendix (section 11), where a more or less complete proof of Theorem 1.1 is given.

For a general geometrically finite hyperbolic manifold, many changes are required, and several technical difficulties arise. In general, there is no positive lower bound for the injectivity radius on X, and the shadow constant of Y appears in the linear algebra lemma (Lemma 5.6). These facts force us to incorporate the height of y as well as the shadow constant of Y in the definition of the Margulis function (see Def. 9.1). The correct substitutes for the volume measures on Y and Z turn out to be the Bowen-Margulis-Sullivan probability measure m_Y and the tight area of Z respectively.

It is more common in the existing literature on the subject to define the operator A_t using averages over large spheres in $\mathbb{H}^2$. Our operator A_t however is defined using averages over expanding horocyclic pieces; this choice is more amenable to the change of variables and iteration arguments for Patterson-Sullivan measures. Indeed, for a locally bounded Borel function f on $Y \cap X_0$ and for any $y \in Y \cap X_0$,

$$(\mathsf{A}_t f)(y) = \frac{1}{\mu_y([-1,1])} \int_{-1}^{1} f(y u_r a_t) d\mu_y(r)$$

where μ_y is the Patterson-Sullivan measure on yU (see (4.2))

When X is compact and hence m_Y is H-invariant, (1.14) follows by simply integrating (1.15) with respect to m_Y. In general, we resort to Lemma 7.3 the proof of which is based on an iterated version of (1.15) for A_{nt_0}, $n \in \mathbb{N}$, for some $t_0 > 0$ as well as on the fact that the Bowen-Margulis-Sullivan measure m_Y is a_{t_0}-ergodic.

In fact, the main technical result of this paper can be summarized as follows:

Proposition 1.16. *Let Γ be a geometrically finite subgroup of G. Let $Y \neq Z$ be non-elementary closed H-orbits in $X = \Gamma\backslash G$, and set $Y_0 := Y \cap X_0$. For any $\frac{\delta_Y}{3} \leq s < \delta_Y$, there exist $t_s > 0$ and a locally bounded Borel function $F_s : Y_0 \to (0, \infty)$ with the following properties:*

(1) *For all $y \in Y_0$,*
$$d(y, Z)^{-s} \leq \mathsf{s}_Y^\star F_s(y).$$

(2) *For all $y \in Y_0$ and $n \geq 1$,*
$$(\mathsf{A}_{nt_s} F_s)(y) \leq \frac{1}{2^n} F_s(y) + \alpha^\star_{Y,s} \operatorname{area}_t(S_Z).$$

(3) *There exists $1 < \sigma \ll \mathsf{s}_Y^\star$ such that for all $y \in Y_0$ and for all $h \in H$ with $\|h\| \geq 2$ and $yh \in Y_0$,*
$$\sigma^{-1} F_s(y) \leq F_s(yh) \leq \sigma F_s(y).$$

Finally we mention that the reason that we can take the exponent s arbitrarily close to δ_Y lies in the two ingredients of our proof: firstly, the linear algebra lemma (Lemma 5.6) is obtained for all $\delta_Y/3 \leq s < \delta_Y$ and secondly, for any $y \in Y \cap X_0$, we can find $|r| < 1$ so that $yu_r \in X_0$ and the height of yu_r can be lowered to be $O(1)$ by the geodesic flow of time comparable to the logarithmic height of y; see Lemma 8.4 for the precise statement.

Organization. We end this introduction with an outline of the paper. In §2, we fix some notation and conventions to be used throughout the paper. In §3, we show the finiteness of the tight area of a properly immersed geodesic plane. In §4, we show the finiteness of the shadow constant of a closed H-orbit. In §5, we prove a lemma from linear algebra; this lemma is a key ingredient to prove a local version of our main inequality. §6 is devoted to the study of the height function in X_0. In §7, the definition of the Markov operator and a basic property of this operator are discussed. In §8, we prove the return lemma, and use it to obtain a uniform control on the number of sheets of Z in a neighborhood of y. In §9, we construct the desired Margulis function and prove the main inequalities. In §10, we give a proof of Theorem 1.5. In the Appendix (§11), we provide a proof of Theorem 1.1.

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2. Notation and preliminaries

In this section, we review some definitions and introduce notation which will be used throughout the paper.

We set $G = \mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$, and $H = \mathrm{PSL}_2(\mathbb{R})$. We fix $\mathbb{H}^2 \subset \mathbb{H}^3$ with an orientation so that $\{g \in G : g(\mathbb{H}^2) = \mathbb{H}^2\} = H$. Let A denote the following one-parameter subgroup of G:

$$A = \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Set $K_0 = \mathrm{PSU}(2)$ and M_0 the centralizer of A in K_0. We fix a point $o \in \mathbb{H}^2 \subset \mathbb{H}^3$ and a unit tangent vector $v_o \in \mathrm{T}_o(\mathbb{H}^3)$ so that their stabilizer subgroups are K_0 and M_0 respectively. The isometric action of G on $\mathbb{H}^3$ induces identifications $G/K_0 = \mathbb{H}^3$, $G/M_0 = \mathrm{T}^1\,\mathbb{H}^3$, and $G = \mathrm{F}\,\mathbb{H}^3$ where $\mathrm{T}^1\,\mathbb{H}^3$ and $\mathrm{F}\mathbb{H}^3$ denote, respectively, the unit tangent bundle and the oriented frame bundle over $\mathbb{H}^3$. Note also that $H \cap K_0 = \mathrm{PSO}(2)$ and that $H(o) = \mathbb{H}^2$.

The right translation action of A on G induces the geodesic/frame flow on $\mathrm{T}^1\,\mathbb{H}^3$ and $\mathrm{F}\mathbb{H}^3$, respectively. Let $v_o^{\pm} \in \partial\mathbb{H}^3$ denote the forward and backward end points of the geodesic given by v_o. For $g \in G$, we define

$$g^{\pm} := g(v_o^{\pm}) \in \partial\mathbb{H}^3.$$

Let $\Gamma < G$ be a discrete torsion-free subgroup. We set

$$M := \Gamma\backslash\mathbb{H}^3 \quad \text{and} \quad X := \Gamma\backslash G \simeq \mathrm{F}M.$$

We denote by $\pi : X \to M$ the base point projection map. Denote by $\Lambda = \Lambda(\Gamma)$ the limit set of Γ. The convex core of M is given by $\mathrm{core}\, M = \Gamma\backslash \mathrm{hull}(\Lambda)$. Let X_0 denote the renormalized frame bundle $\mathrm{RF}M$, i.e.,

$$X_0 = \{[g] \in X : g^{\pm} \in \Lambda\}, \tag{2.1}$$

that is, X_0 is the union of all the A-orbits whose projections to M stay inside $\mathrm{core}\, M$. We remark that X_0 does not surject onto $\mathrm{core}\, M$ in general.

In the whole paper, we assume that Γ is geometrically finite, that is, the unit neighborhood of $\mathrm{core}\, M$ has finite volume. This is equivalent to the condition that Λ is the union of the radial limit points and bounded parabolic limit points: $\Lambda = \Lambda_{rad} \bigcup \Lambda_{bp}$ (cf. [5], [18]). A point $\xi \in \Lambda$ is called *radial* if the projection of a geodesic ray toward to ξ accumulates on $M = \Gamma\backslash\mathbb{H}^3$, *parabolic* if it is fixed by a parabolic element of Γ, and *bounded parabolic* if it is parabolic and $\mathrm{Stab}_\Gamma(\xi)$ acts co-compactly on $\Lambda - \{\xi\}$. In particular, for Γ geometrically finite, the set of parabolic limit points Λ_p is equal to Λ_{bp}. For $\xi \in \Lambda_p$, the rank of the free abelian subgroup $\mathrm{Stab}_\Gamma(\xi)$ is referred to as the rank of ξ.

A geometrically finite group Γ is called *convex cocompact* if $\mathrm{core}\, M$ is compact, or equivalently, if $\Lambda = \Lambda_{rad}$.

We denote by N the expanding horospherical subgroup of G for the action of A:

$$N = \left\{ u_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} : s \in \mathbb{C} \right\}.$$

For $\xi \in \Lambda_p$, a horoball $\tilde{\mathfrak{h}}_\xi \subset G$ based at ξ is of the form

$$\tilde{\mathfrak{h}}_\xi(T) = gNA_{(-\infty,-T]}K_0 \text{ for some } T \geq 1 \tag{2.2}$$

where $g \in G$ is such that $g^- = \xi$ and $A_{(-\infty,-T)} = \{a_t : -\infty < t \leq -T\}$. Its image $\tilde{\mathfrak{h}}_\xi(o)$ in $\mathbb{H}^3$ is called a horoball in $\mathbb{H}^3$ based at ξ. By a horoball $\mathfrak{h}_\xi$ in X and in M, we mean their respective images of horoballs $\tilde{\mathfrak{h}}_\xi$ and $\tilde{\mathfrak{h}}_\xi(o)$ in X and M under the corresponding projection maps.

Thick-thin decomposition of X_0. We fix a Riemannian metric d on G which induces the hyperbolic metric on $\mathbb{H}^3$. By abuse of notation, we use d to denote the distance function on X induced by d, as well as on M. For a subset $S \subset \spadesuit$ and $\varepsilon > 0$, $B_\spadesuit(S, \varepsilon)$ denotes the set $\{x \in \spadesuit : d(x, S) \leq \varepsilon\}$. When $\spadesuit$ is a subgroup of G and $S = \{e\}$, we simply write $B_\spadesuit(\varepsilon)$ instead of $B_\spadesuit(S, \varepsilon)$. When there is no room for confusion for the ambient space $\spadesuit$, we omit the subscript $\spadesuit$.

For $p \in M$, we denote by $\operatorname{inj} p$ the injectivity radius at $p \in M$, that is: the supremum $r > 0$ such that the projection map $\mathbb{H}^3 \to M = \Gamma\backslash\mathbb{H}^3$ is injective on the ball $B_{\mathbb{H}^3}(\tilde{p}, r)$ where $\tilde{p} \in \mathbb{H}^3$ is such that $p = [\tilde{p}] = \tilde{p}\Gamma$. For $S \subset M$ and $\varepsilon > 0$, we call the subsets $\{p \in S : \operatorname{inj}(p) \geq \varepsilon\}$ and $\{p \in S : \operatorname{inj}(p) < \varepsilon\}$ the ε-thick part and the ε-thin part of S respectively.

As M is geometrically finite, $\operatorname{core} M$ is contained in a union of its ε-thick part $(\operatorname{core} M)_\varepsilon$ and finitely many disjoint horoballs for all small $\varepsilon > 0$ (cf. [18]). If $p = gu_s a_{-t} o$ is contained in a horoball $\mathfrak{h}_\xi = gNA_{(-\infty,-T]}(o)$, then $\operatorname{inj}(p) \asymp e^{-t}$ for all $t \gg T$, this is a standard fact see, e.g., [15, Prop. 5.1].

Let $\varepsilon_M > 0$ be the supremum of ε with respect to which such a decomposition of $\operatorname{core} M$ holds. We call the ε_M-thick part of $\operatorname{core} M$ the compact core of M, and denote by M_{cpt}.

For $x = [g] \in X$, we denote by $\operatorname{inj}(x)$ the injectivity radius of $\pi(x) \in M$. For $\varepsilon > 0$, we set

$$X_\varepsilon := \{x \in X_0 : \operatorname{inj}(x) \geq \varepsilon\}.$$

We set $\varepsilon_X = \varepsilon_M/2$; note that $X_0 - X_{\varepsilon_X}$ is either empty or is contained in a union of horoballs in X.

Convention. By an absolute constant, we mean a constant which depends at most on G and Γ. We will use the notation $A \asymp B$ when the ratio between the two lies in $[C^{-1}, C]$ for some absolute constant $C \geq 1$. We write $A \ll B^\star$ (resp. $A \asymp B^\star$, $A \ll \star B$) to mean that $A \leq CB^L$ (resp. $C^{-1}B^L \leq A \leq CB^L$, $A \leq C \cdot B$) for some absolute constants $C > 0$ and $L > 0$.

3. Tight area of a properly immersed geodesic plane

In this section, we show that the tight area of a properly immersed geodesic plane of M is finite.

For a closed subset $Q \subset M$, we define *the tight neighborhood of* Q by

$$\mathcal{N}(Q) := \{p \in M : d(p, q) \leq \operatorname{inj}(q) \text{ for some } q \in Q\}.$$

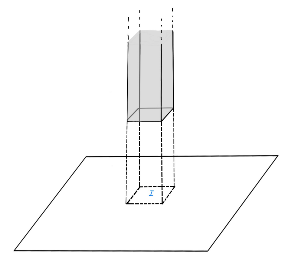

FIGURE 3. Chimney

We are mainly interested in the tight neighborhood of $\operatorname{core} M$. If M is convex cocompact, $\mathcal{N}(\operatorname{core} M)$ is compact. In order to describe the shape of $\mathcal{N}(\operatorname{core} M)$ in the presence of cusps, fix a set $\xi_1, \cdots, \xi_\ell$ of Γ-representatives of Λ_p, cf. [18]. Then $\operatorname{core} M$ is contained in the union of M_{cpt} and a disjoint union $\bigcup \mathfrak{h}_{\xi_i}$ of horoballs based at ξ_is.

Consider the upper half-space model $\mathbb{H}^3 = \{(x_1, x_2, y) : y > 0\} = \mathbb{R}^2 \times \mathbb{R}_{>0}$, and let $\infty \in \Lambda_p$. Let $p : \mathbb{H}^3 \to M$ denote the canonical projection map. As ∞ is a bounded parabolic fixed point, there exists a bounded rectangle, say, $I \subset \mathbb{R}^2$ and $r > 0$ (depending on ∞) such that

(1) $p(I \times \{y > r\}) \supset \mathcal{N}(\mathfrak{h}_\infty \cap \operatorname{core} M)$ and
(2) $p(I \times \{r\}) \subset B(M_{\mathrm{cpt}}, R)$

where R depends only on M. We call this set $\mathfrak{C}_\infty := I \times \{y \geq r\}$ a chimney for ∞ (cf. Figure 3).

Note that increasing R if necessary, we have

$$\mathcal{N}(\operatorname{core} M) \subset B(M_{\mathrm{cpt}}, R) \cup \left(\bigcup_{1 \leq i \leq \ell} p(\mathfrak{C}_{\xi_i}) \right) \tag{3.1}$$

where $\mathfrak{C}_{\xi_i}$ is a chimney for ξ_i.

Definition 3.2. For a properly immersed geodesic plane S of M, we define the *tight-area* of S relative to M as follows:

$$\operatorname{area}_t(S) := \operatorname{area}(S \cap \mathcal{N}(\operatorname{core} M)).$$

Theorem 3.3. *For a properly immersed non-elementary geodesic plane S of M, we have*

$$1 \ll \operatorname{area}_t(S) < \infty$$

where the implied multiplicative constant depends only on M.

Proof. Since no horoball can contain a complete geodesic, it follows that S intersects the compact core M_{cpt}. Therefore,

$$\operatorname{area}_t S \geq 4\pi \sinh^2(\varepsilon_X/2),$$

as $S \cap M_{\rm cpt}$ contains a hyperbolic disk of radius ε_X (see Section 2). This implies the lower bound.

We now turn to the proof of the upper bound. We use the notation in (3.1). Fix a geodesic plane $P \subset \mathbb{H}^3$ which covers S and let $\Delta = \mathrm{Stab}_\Gamma(P)$. Fix a Dirichlet domain D in P for the action of Δ. As $\Delta \backslash P$ is geometrically finite, the Dirichlet domain is a finite sided polygon; hence, $D \cap \mathrm{hull}(\Delta)$ has finite area, and the set $D - \mathrm{hull}(\Delta)$ is a disjoint union of finitely many flares, where a flare is a region bounded by three geodesics as shown in Figure 4. Fixing a flare $F \subset D - \mathrm{hull}(\Delta)$, it suffices to show that $\{x \in F : p(x) \in \mathcal{N}(\mathrm{core}\, M)\}$ has finite area. As S is properly immersed, the set $\{x \in F : d(p(x), M_{\rm cpt}) \le R\}$ is bounded. Therefore, fixing a chimney $\mathfrak{C}_{\xi_i}$ as above, it suffices to show that the set $\{x \in F : p(x) \in \mathfrak{C}_{\xi_i}\} = F \cap \Gamma \mathfrak{C}_{\xi_i}$ has finite area.

Without loss of generality, we may assume $\xi_i = \infty$. We will denote by ∂F the intersection of the closure of F and ∂P, and let $F_\varepsilon \subset \overline{F}$ denote the ε-neighborhood of ∂F in the Euclidean metric in the unit disc model of $\overline{P}$ (cf. Figure 4).

Fix $\varepsilon_0 > 0$ so that

$$F_{\varepsilon_0} \cap \{x \in D : d(p(x), M_{\rm cpt}) < R\} = \emptyset; \tag{3.4}$$

such ε_0 exists, as S is a proper immersion. Writing $\mathfrak{C}_\infty = I \times \{y \ge r\}$ as above, let $H_\infty := \mathbb{R}^2 \times \{y > r\}$, and set $\Gamma_\infty := \mathrm{Stab}_\Gamma(\infty)$.

We claim that

$$\#\{\gamma H_\infty : F_{\varepsilon_0/2} \cap \gamma \mathfrak{C}_\infty \ne \emptyset\} < \infty. \tag{3.5}$$

Suppose not. Since ΓH_∞ is closed in the space of all horoballs in $\mathbb{H}^3$, there exists a sequence of distinct $\gamma_i(\infty) \in \Gamma(\infty)$ such that $F_{\varepsilon_0/2} \cap \gamma_i \mathfrak{C}_\infty \ne \emptyset$ and the size of the horoballs $\gamma_i H_\infty$ goes to 0 in the Euclidean metric in the ball model of $\mathbb{H}^3$. Note that if ∞ has rank 2, then $\Gamma_\infty(I \times \{r\}) = \mathbb{R}^2 \times \{r\}$ and that if ∞ has rank 1, then $\Gamma_\infty(I \times \{r\})$ contains a region between two parallel horocycles in $\mathbb{R}^2 \times \{r\}$. Since $P \cap \gamma_i \mathfrak{C}_\infty \ne \emptyset$, it follows that $P \cap \gamma_i(\Gamma_\infty(I \times \{r\})) \ne \emptyset$. Moreover, if i is large enough so that the Euclidean size of $\gamma_i H_\infty$ is smaller than $\varepsilon_0/2$, the condition $F_{\varepsilon_0/2} \cap \gamma_i \mathfrak{C}_\infty \ne \emptyset$ implies that $F_{\varepsilon_0} \cap \gamma_i(\Gamma_\infty(I \times \{r\})) \ne \emptyset$. This yields a contradiction to (3.4) since $p(I \times \{r\})$ is contained in the R-neighborhood of $M_{\rm cpt}$, proving the claim.

By Claim 3.5, it is now enough to show that, fixing a horoball γH_∞, the intersection $F_{\varepsilon_0} \cap \gamma \Gamma_\infty \mathfrak{C}_\infty$ has finite area. Suppose that $F_{\varepsilon_0} \cap \gamma \Gamma_\infty \mathfrak{C}_\infty$ is unbounded in P; otherwise the claim is clear. Without loss of generality, we may assume $\gamma = e$, by replacing P by $\gamma^{-1} P$ if necessary. If $\infty \notin \partial P$, then $F_{\varepsilon_0} \cap \Gamma_\infty \mathfrak{C}_\infty$, being contained in $P \cap H_\infty$, is a bounded subset of P; contradiction. Therefore, $\infty \in \partial P$. Then, as $F_{\varepsilon_0} \cap \Gamma_\infty \mathfrak{C}_\infty \subset F_{\varepsilon_0} \cap H_\infty$ is unbounded, we have $\infty \in \partial F$. Since F is a flare, it follows that ∞ is not a limit point for Δ. This implies that the rank of ∞ in Λ_p is 1 [23, Lem. 6.2]. Therefore $\Gamma_\infty \mathfrak{C}_\infty$ is contained in a subset of the form $T \times \{y \ge r\}$ where T is a strip between two parallel lines L_1, L_2 in $\mathbb{R}^2$. Since ∞ is not a limit

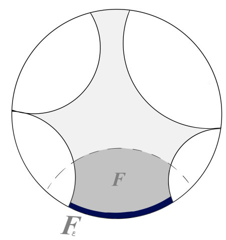

FIGURE 4. Flare F and F_ε

point for Δ, the vertical plane P is not parallel to the L_i. Therefore, the intersection $F_{\varepsilon_0} \cap \Gamma_\infty \mathfrak{C}_\infty$, being a subset of $P \cap (T \times \{y \geq r\})$, is contained in a cusp-like region, isometric to $\{(x, y) \in \mathbb{H}^2 : y \geq r\}$ and x is also bounded from above and below (recall that P is not parallel to the L_i). This finishes the proof. □

The proof of the above theorem demonstrates that the portion of S, especially of the flares of S, staying in the *tight* neighborhood of core M can go to infinity only in cusp-like shapes, by visiting the chimneys of horoballs of core M (Fig. 1). This is not true any more if we replace the tight neighborhood of core M by the unit neighborhood of core M. More precisely if Λ contains a parabolic limit point of rank one which is not stabilized by any element of $\pi_1(S)$, then some region of S with infinite area can stay inside the unit neighborhood of core M. This situation may be compared to the presence of divergent geodesics in finite area setting.

4. Shadow constants

In this section, fixing a closed non-elementary H-orbit Y in X, we recall the definition of Patterson-Sullivan measures μ_y on horocycles in Y, and relate its density with the shadow constant s_Y, which we show is a finite number.

Set $\Delta_Y := \mathrm{Stab}_H(y_0)$ to be the stabilizer of a point $y_0 \in Y$; note that despite the notation, Δ_Y is uniquely determined up to a conjugation by an element of H. As Γ is geometrically finite and $Y = Hy_0$ is a closed orbit, the subgroup Δ_Y is a geometrically finite subgroup of H, [23, Thm. 4.7]. We denote by $\Lambda_Y \subset \partial\mathbb{H}^2$ the limit set of Δ_Y. Let $0 < \delta(Y) \leq 1$ denote the critical exponent of Δ_Y, or equivalently, the Hausdorff dimension of Λ_Y.

We denote by $\{\nu_p = \nu_{Y,p} : p \in \mathbb{H}^2\}$ the Patterson-Sullivan density for Δ_Y, normalized so that $|\nu_o| = 1$. This means that the collection $\{\nu_p\}$ consists of

Borel measures on Λ_Y satisfying that for all $\gamma \in \Delta_Y$, $p, q \in \mathbb{H}^2$, $\xi \in \Lambda_Y$,

$$\frac{d\gamma_* \nu_p}{d\nu_p}(\xi) = e^{-\delta(Y)\beta_\xi(\gamma^{-1}(p),p)} \quad \text{and} \quad \frac{d\nu_q}{d\nu_p}(\xi) = e^{-\delta(Y)\beta_\xi(q,p)}$$

where $\beta_\xi(\cdot,\cdot)$ denotes the Busemann function. In the sequel we will refer to the first identity above as Γ-conformality of $\{\nu_p\}$.

As Δ_Y is geometrically finite, there exists a unique Patterson-Sullivan density up to a constant multiple.

PS-measures on U-orbits. Set

$$U := \left\{ u_r = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} : r \in \mathbb{R} \right\} = N \cap H$$

which is the expanding horocylic subgroup of H. Using the parametrization $r \mapsto u_r$, we may identify U with $\mathbb{R}$. Note that for all $r, t \in \mathbb{R}$,

$$a_{-t} u_r a_t = u_{e^t r}.$$

For any $h \in H$, the restriction of the visual map $g \mapsto g^+$ is a diffeomorphism between hU and $\partial\mathbb{H}^2 - \{h^-\}$. Using this diffeomorphism, we can define a measure μ_{hU} on hU:

$$d\mu_{hU}(hu_r) = e^{\delta(Y)\beta_{(hu_r)^+}(p, hu_r(p))} d\nu_p(hu_r)^+; \tag{4.1}$$

this is independent of the choice of $p \in \mathbb{H}^2$. We simply write $d\mu_h(r)$ for $d\mu_{hU}(hu_r)$. Note that these measures depend on the U-orbits but not on the individual points. By the Δ_Y-invariance and the conformal property of the PS-density, we have

$$d\mu_h(\mathcal{O}) = d\mu_{\gamma h}(\mathcal{O}) \tag{4.2}$$

for any $\gamma \in \Delta_Y$ and for any bounded Borel set $\mathcal{O} \subset \mathbb{R}$; therefore $\mu_y(\mathcal{O})$ is well-defined for $y \in \Delta_Y \backslash H$.

For any $y \in \Delta_Y \backslash H$ and any $t \in \mathbb{R}$, we have:

$$\mu_y([-e^t, e^t]) = e^{\delta(Y)t} \mu_{ya_{-t}}([-1,1]). \tag{4.3}$$

Set

$$Y_0 := \{[h] \in \Delta_Y \backslash H : h^\pm \in \Lambda_Y\} \tag{4.4}$$

where $h^\pm = \lim_{t \to \pm\infty} ha_t(o)$.

Shadow constant. As in the introduction, we define the modified critical exponent of Y:

$$\delta_Y = \begin{cases} \delta(Y) & \text{if } Y \text{ is convex cocompact} \\ 2\delta(Y) - 1 & \text{otherwise.} \end{cases} \tag{4.5}$$

If Y has a cusp, then $\delta(Y) > 1/2$, and hence $0 < \delta_Y \le \delta(Y) \le 1$.

Define

$$\mathsf{p}_Y = \sup_{y \in Y_0, 0 < r \le 2} \frac{\mu_y([-r,r])^{1/\delta_Y}}{r \cdot \mu_y([-1,1])^{1/\delta_Y}}; \tag{4.6}$$

the range $0 < r \le 2$ is motivated by our applications later, see e.g., (7.13).

Recall the shadow constant $\mathsf{s}_Y = \sup_{0<\varepsilon\le 1/2} \mathsf{s}_Y(\varepsilon)$ in (1.8) where

$$\mathsf{s}_Y(\varepsilon) := \sup_{\xi\in\Lambda_Y, p\in[\xi,\Lambda_Y]} \frac{\nu_p(B_p(\xi,\varepsilon))^{1/\delta_Y}}{\varepsilon\cdot\nu_p(B_p(\xi,1/2))^{1/\delta_Y}}. \tag{4.7}$$

where $[\xi,\Lambda_Y]$ is the union of all geodesics connecting ξ to a point in Λ_Y, and $B_p(\xi,\cdot)$ is as in (4.10).

The rest of this section is devoted to the proof of the following theorem using a uniform version of Sullivan's shadow lemma.

Theorem 4.8. *We have*

$$\mathsf{s}_Y \asymp \mathsf{p}_Y < \infty.$$

In principle, this definition of s_Y involves making a choice of $\Delta_Y = \mathrm{Stab}_H(y_0)$, i.e., the choice of $y_0 \in Y$, as Λ_Y is the limit set of Δ_Y. However we observe the following:

Lemma 4.9. *The constant s_Y is independent of the choice of $y_0 \in Y$.*

Proof. Let $y = y_0 h^{-1} \in Y$ for $h \in H$. Define s_Y' similar to s_Y using $\Delta_Y' = \mathrm{Stab}_H(y) = h\Delta_Y h^{-1}$ and put $\nu_p' := h_*\nu_{h^{-1}p}$ for each $p \in \mathbb{H}^2$. If $\xi \in \Lambda_Y$, then

$$\begin{aligned}\frac{d\Big((h\gamma h^{-1})_*\nu_p'\Big)}{d\nu_p'}(h\xi) &= \frac{d\Big((h\gamma)_*\nu_{h^{-1}p}\Big)}{dh_*\nu_{h^{-1}p}}(h\xi) = \frac{d\gamma_*\nu_{h^{-1}p}}{d\nu_{h^{-1}p}}(\xi)\\ &= e^{-\delta(Y)\beta_\xi(\gamma^{-1}(h^{-1}p),h^{-1}p)} = e^{-\delta(Y)\beta_{h\xi}(h\gamma^{-1}h^{-1}(p),p)}.\end{aligned}$$

Since the limit set of Δ_Y' is given by $h\Lambda_Y$, this implies that the family $\{\nu_p' : p \in \mathbb{H}^2\}$ is the Patterson-Sullivan density for Δ_Y'. Now for any $0 < \varepsilon \le 1$ and $\xi \in \Lambda_Y$, we have

$$\nu_{hp}'(B_{hp}(h\xi,\varepsilon)) = h_*\nu_p(B_{hp}(h\xi,\varepsilon)) = \nu_p(h^{-1}B_{hp}(h\xi,\varepsilon)) = \nu_p(B_p(\xi,\varepsilon)).$$

It follows that $\mathsf{s}_Y = \mathsf{s}_Y'$. $\square$

Shadow lemma. Consider the associated hyperbolic plane and its convex core:

$$S_Y := \Delta_Y\backslash\mathbb{H}^2 \quad \text{and} \quad \mathrm{core}(S_Y) := \Delta_Y\backslash\,\mathrm{hull}(\Lambda_Y).$$

We denote by C_Y the compact core of S_Y, defined as the minimal connected surface whose complement in $\mathrm{core}(S_Y)$ is a union of disjoint cusps. If S_Y is convex cocompact, then $C_Y = S_Y$. Let

$$d_Y := \max\{1, \mathrm{diam}(C_Y)\}.$$

We can write $\mathrm{core}(S_Y)$ as the disjoint union of the compact core $C_0 := C_Y$ and finitely many cusps, say, $C_1,\ldots,C_m$. Fix a Dirichlet domain $\mathcal{F}_Y \subset \mathbb{H}^2$ for Δ_Y containing the base point o. For each C_i, $0 \le i \le m$, choose the lift $\tilde{C}_i \subset \mathcal{F}_Y \cap \mathrm{hull}(\Lambda_Y)$ so that $\Delta_Y\backslash\Delta_Y\tilde{C}_i = C_i$. In particular, $\partial\tilde{C}_0$ intersects $\tilde{C}_i$ in an interval for $i \ge 1$. Let $\xi_i \in \Lambda_Y$ be the base point of the horodisc

$\tilde{C}_i$, i.e., $\xi_i = \partial\tilde{C}_i \cap \partial\mathbb{H}^2$. Let $F_{\xi_i} \subset \partial\mathbb{H}^2 - \{\xi_i\}$ be a minimal closed interval so that $\Lambda_Y - \{\xi_i\} \subset \mathrm{Stab}_{\Delta_Y}(\xi_i)F_{\xi_i}$.

For $p \in \mathbb{H}^2$, let d_p denote the Gromov distance on $\partial\mathbb{H}^2$: for $\xi \neq \eta \in \partial\mathbb{H}^2$,

$$d_p(\xi,\eta) = e^{-(\beta_\xi(p,q)+\beta_\eta(p,q))/2}$$

where q is any point on the geodesic connecting ξ and η. The diameter of $(\partial\mathbb{H}^2, d_p)$ is equal to 1.

For any $h \in H$, we have $d_p(\xi,\eta) = d_{h(p)}(h(\xi),h(\eta))$. For $\xi \in \partial\mathbb{H}^2$, and $r > 0$, set

$$B_p(\xi,r) = \{\eta \in \partial\mathbb{H}^2 : d_p(\eta,\xi) \le r\} \tag{4.10}$$

as was defined in the introduction. Also, denote by $V(p,\xi,r)$ the set of all $\eta \in \partial\mathbb{H}^2$ such that the distance between p and the orthogonal projection of η onto the geodesic $[p,\xi)$ is at least r. Note that

$$V(p,\xi,t) = B_p(\xi, \tfrac{e^{-t}}{\sqrt{1+e^{-2t}}}),$$

see ([27, Lemma 2.5] and the discussion following that lemma). Therefore,

$$V(p,\xi,r+1) \subset B_p(\xi,e^{-r}) \subset V(p,\xi,r-1) \quad \text{for all } r \ge 1.$$

The following is a uniform version of Sullivan's shadow lemma [29]. The proof of this proposition is similar to the proof of [27, Thm. 3.2]; since the dependence on the multiplicative constant is important to us, we give a sketch of the proof while making the dependence of constants explicit.

Proposition 4.11. *There exists a constant $c \asymp e^{\star d_Y}$ such that for all $\xi \in \Lambda_Y$, $p \in \tilde{C}_0$, and $t > 0$,*

$$\begin{aligned} c^{-1} \cdot \nu_p(F_{\xi_t})\beta_Y e^{-\delta(Y)t+(1-\delta(Y))d(\xi_t,\Delta_Y(p))} &\le \nu_p(V(p,\xi,t)) \\ &\le c \cdot \nu_p(F_{\xi_t}) e^{-\delta(Y)t+(1-\delta(Y))d(\xi_t,\Delta_Y(p))} \end{aligned}$$

where

- *$\{\xi_t\}$ is the unit speed geodesic ray $[p,\xi)$ so that $d(p,\xi_t) = t$;*
- *$F_{\xi_t} = \partial\mathbb{H}^2$ if $\xi_t \in \Delta_Y\tilde{C}_0$, and $F_{\xi_t} = F_{\xi_i}$ if $\xi_t \in \Delta_Y\tilde{C}_i$ for $1 \le i \le m$;*
- *$\beta_Y := \inf_{\eta\in\Lambda_Y, q\in\tilde{C}_0} \nu_q(B_q(\eta, e^{-d_Y}))$.*

Proof. Let p, $\xi \in \Lambda_Y$ and ξ_t be as in the statement. By the $\delta(Y)$-conformality of the PS density, we have

$$\nu_p(V(p,\xi,t)) = e^{-\delta(Y)t}\nu_{\xi_t}(V(p,\xi,t)).$$

Therefore it suffices to show

$$\nu_{\xi_t}(V(p,\xi,t)) \asymp \nu_p(F_{\xi_t}) \cdot e^{(1-\delta(Y))d(\xi_t,\Delta_Y(p))}$$

while making the dependence of the implied constant explicit.

Claim A. If $\xi_t \in \Delta_Y\tilde{C}_0$, then

$$e^{-\delta(Y)d_Y} \cdot \inf_{\eta\in\Lambda_Y} \nu_p(B(\eta,e^{-d_Y})) \ll \nu_{\xi_t}(V(p,\xi,t)) \ll e^{\delta(Y)d_Y}|\nu_p| \tag{4.12}$$

where the implied constants are absolute.

First note that this implies the claim in the proposition if $\xi_t \in \Delta_Y \tilde{C}_0$. Indeed $d(\xi_t, \Delta_Y(p)) \le d_Y$ and $F_{\xi_t} = \partial\mathbb{H}^2$ in this case. Moreover, by (4.12), we have

$$e^{-\star d_Y}\beta_Y e^{-\delta(Y)t} \le \nu_p(V(p,\xi,t)) = e^{-\delta(Y)t}\nu_{\xi_t}(V(p,\xi,t)) \le e^{\star d_Y}e^{-\delta(Y)t}$$

where we also used $|\nu_p| = e^{\star d_Y}$ (recall that $p \in \tilde{C}_0$). Thus the claim in the proposition follows in this case.

We now turn to the proof of Claim A. As $\xi_t \in \Delta_Y \tilde{C}_0$, there exists $\gamma \in \Delta_Y$ such that $d(\xi_t, \gamma p) \le d_Y$. Hence

$$\begin{aligned} e^{-\delta(Y)d_Y}\nu_{\xi_t}(V(p,\xi,t)) &\le \nu_{\gamma p}(V(p,\xi,t)) = \nu_p(V(\gamma^{-1}p, \gamma^{-1}\xi, t)) \\ &\le e^{\delta(Y)d_Y}\nu_{\xi_t}(V(p,\xi,t)). \end{aligned}$$

The upper bound in (4.12) follows from the first inequality, while the lower bound follows from the second inequality; indeed

$$V(\gamma^{-1}p, \gamma^{-1}\xi, t) = V(\gamma^{-1}\xi_t, \gamma^{-1}\xi, 0)$$

and the latter contains $B_p(\gamma^{-1}\xi, e^{-d_Y})$, since $d(p, \gamma^{-1}\xi_t) \le d_Y$ and $d_Y \ge 1$.

Claim B. Let ξ be a parabolic limit point in Λ_Y. Assume that for some $i \ge 1$, $\xi_t \in \tilde{C}_i$ for all large t.

We claim:

$$\nu_{\xi_t}(V(p,\xi,t)) \asymp \nu_p(F_\xi) \cdot e^{(1-\delta(Y))(d(\xi_t,\Delta_Y(p))+d_Y)} \tag{4.13}$$

and

$$\nu_{\xi_t}(\partial\mathbb{H}^2 - V(p,\xi,t)) \asymp \nu_p(F_\xi) \cdot e^{(1-\delta(Y))(d(\xi_t,\Delta_Y(p))+d_Y)} \tag{4.14}$$

where here and in what follows implied constants are of the form $e^{\pm\star d_Y}$ unless otherwise is stated explicitly.

Let $s_i \ge 0$ be such that $\xi_{s_i} \in \partial\tilde{C}_i$. Then for all $t \ge s_i$,

$$|d(\xi_t, \Delta_Y(p)) - (t - s_i)| \le d_Y.$$

Hence for (4.13), it suffices to show

$$\nu_{\xi_t}(V(p,\xi,t)) \asymp e^{(1-\delta(Y))(t-s_i)}\nu_p(F_\xi). \tag{4.15}$$

Note that if we set $\Delta_{Y,\xi} = \mathrm{Stab}_{\Delta_Y}(\xi)$,

$$\nu_{\xi_t}(V(p,\xi,t)) \asymp \sum_{\gamma\in\Delta_{Y,\xi},\gamma F_\xi \cap V(p,\tilde{x},t)\neq\emptyset} \nu_{\xi_t}(\gamma F_\xi).$$

Let F_ξ^* denote the image of F_ξ on the horocycle based at ξ passing through p via the inverse of the visual map. Since $p \in \tilde{C}_0$, there exists $\gamma \in \Delta_{Y,\xi}$ so that γF_ξ^* is contained in the closure of $\tilde{C}_0$. Hence,

$$\mathrm{diam} F_\xi^* \le d_Y = \max\{1, \mathrm{diam}(\tilde{C}_0)\}.$$

We now apply [27, Lemma 2.9] with $K = F_\xi^*$ and let K_3 be as in loc. cit. By the definition of K_3 given in the proof of [27, Lemma 2.9], we have $K_3 \ll \mathrm{diam} F_\xi^*$ where the implied constant is absolute. In view of [27, Lemma

2.9], thus, if $\gamma \in \Delta_{Y,\xi}$ is so that $\gamma F_\xi \cap V(p,\xi,t) \neq \emptyset$, then $d(p,\gamma p) \geq 2t - k d_Y$, where k is absolute. In consequence,

$$\nu_{\xi_t}(V(p,\xi,t)) \asymp \sum_{\gamma \in \Delta_{Y,\xi}, d(p,\gamma p) \geq 2t} \nu_{\xi_t}(\gamma F_\xi)$$

where the implied constant is absolute.

Now we use the fact that if $d(p,\gamma p) \geq 2t$, then for all $\eta \in F_\xi$,

$$|\beta_\eta(\gamma^{-1}\xi_t, \xi_t) - d(p,\gamma p) + 2t| \ll \operatorname{diam} F_\xi^* \leq d_Y$$

(cf. proof of [27, Lemma 2.9]). Since

$$\nu_{\xi_t}(\gamma F_\xi) = \int_{\gamma F_\xi} d\nu_{\xi_t} = \int_{F_\xi} e^{-\delta(Y)\beta_{\gamma\eta}(\xi_t, \gamma\xi_t))} d\nu_{\xi_t}(\eta),$$

and $\nu_{\xi_t}(F_\xi) = e^{-\delta(Y)t}\nu_p(F_\xi)$, we deduce, with multiplicative constant $\asymp e^{\delta(Y)d_Y}$,

$$\begin{aligned} \sum_{\gamma \in \Delta_{Y,\xi}, d(p,\gamma p) \geq 2t} \nu_{\xi_t}(\gamma F_\xi) &\asymp \sum_{\gamma \in \Delta_{Y,\xi}, d(p,\gamma p) \geq 2t} e^{2\delta(Y)t - \delta(Y)d(p,\gamma p)} \nu_{\xi_t}(F_\xi) \\ &\asymp \nu_p(F_\xi) e^{\delta(Y)t} \sum_{\gamma \in \Delta_{Y,\xi}, d(p,\gamma p) \geq 2t} e^{-\delta(Y)d(p,\gamma p)} \\ &\asymp \nu_p(F_\xi) e^{(1-\delta(Y))t} \end{aligned}$$

using $a_n := \#\{\gamma \in \Delta_{Y,\xi} : n < d(p,\gamma p) \leq n+1\} \asymp e^{n/2}$ in the last estimate. This proves (4.13).

The estimate (4.14) follows similarly now using

$$\nu_{\xi_t}(\partial \mathbb{H}^2 - V(p,\xi,t)) \asymp \sum_{\gamma \in \Delta_{Y,\xi}, d(p,\gamma p) \leq 2t} \nu_{\xi_t}(\gamma F)$$

and $\sum_{n=0}^{[2t]} a_n e^{-\delta(Y)n} \asymp e^{(1-2\delta(Y))t}$.

Note that when ξ is a parabolic limit point, (4.13) holds with multiplicative constant $\asymp e^{\star d_Y}$ (see the proof of [27, Prop. 3.4]).

As for the remaining case, i.e., ξ is a radial limit point but $\xi_t \in \Delta_Y \tilde{C}_i$ for some i, one can prove that (4.13) holds with multiplicative constant $\asymp e^{\star d_Y}$ (see the proof of [27, Lemma 3.6]). $\square$

Proposition 4.16. *Fix $p = p_Y \in \tilde{C}_0$. There exists $R_Y \asymp e^{\star d_Y}$ such that for all $y \in Y_0$, we have*

$$R_Y^{-1}\beta_Y e^{(1-\delta(Y))d(C_Y,\pi(y))}|\nu_p| \leq \mu_y([-1,1]) \leq R_Y e^{(1-\delta(Y))d(C_Y,\pi(y))}|\nu_p|$$

where π denotes the base point projection $\Delta_Y \backslash H = \mathrm{T}^1(S_Y) \to S_Y$.

Proof. The following argument is a slight modification of the proof of [19, Prop. 5.1]. Since the map $y \mapsto \mu_y[-1,1]$ is continuous on Y_0 and $\{[h] \in Y_0 : h^-$ is a radial limit point of $\Lambda_Y\}$ is dense in Y_0, it suffices to prove the claim for $y = [h]$, assuming that h^- is a radial limit point for Δ_Y.

Recall that $\mu_y([-1,1]) = e^{\delta(Y)t}\mu_{ya_{-t}}([-e^{-t}, e^{-t}])$ for all $t \in \mathbb{R}$. Let $t \geq 0$ be the minimal number so that $\pi(ya_{-t}) \in C_Y$; this exists as h^- is a radial limit point. Then

$$d(\pi(y), C_Y) \leq d(\pi(y), \pi(ya_{-t})) \leq d_Y + d(\pi(y), C_Y). \tag{4.17}$$

Set $\xi_t = ha_{-t}(o)$. Then

$$\mu_{ya_{-t}}[-e^{-t}, e^{-t}] \asymp \nu_{\xi_t}(V(\xi_t, h^+, t))$$

(cf. [27, Lemma 4.4]).

Since $ya_{-t} \in C_Y$, $F_{\xi_t} = \partial\mathbb{H}^2$. So $\nu_{\xi_t}(F_{\xi_t}) = |\nu_{\xi_t}| \asymp |\nu_p|$ up to a multiplicative constant $e^{\star d_Y}$. Therefore, for some implied constant $\asymp e^{\star d_Y}$, we have

$$\beta_Y e^{-\delta(Y)t+(1-\delta(Y))d(\pi(y),\pi(ya_{-t}))}|\nu_p| \ll \nu_{\xi_t}(V(\xi_t, h^+, t)) \ll e^{-\delta(Y)t+(1-\delta(Y))d(\pi(y),\pi(ya_{-t}))}|\nu_p|.$$

This estimate and (4.17), therefore, imply that

$$\beta_Y e^{(1-\delta(Y))d(\pi(y),C_Y)}|\nu_p| \ll \mu_y([-1,1]) \ll e^{(1-\delta(Y))d(\pi(y),C_Y)}|\nu_p|$$

with the implied constant $\asymp e^{\star d_Y}$, proving the claim. $\square$

We use the following result, essentially obtained by Schapira-Maucourant ([29], [19]):

Corollary 4.18. *Fix $\rho > 0$. Then for all $0 < \varepsilon \leq \rho$,*

$$R_Y^{-2} \cdot \beta_Y \leq \sup_{y \in Y_0} \frac{\mu_y([-\varepsilon, \varepsilon])}{\varepsilon^{\delta_Y}\mu_y([-1,1])} \leq \max\{1, \rho^2\} \cdot R_Y^2 \cdot \beta_Y^{-1} < \infty,$$

where R_Y is as in Proposition 4.16.

Proof. By (4.3), we have $\mu_y([-\varepsilon, \varepsilon]) = \varepsilon^{\delta(Y)}\mu_{ya_{-\log\varepsilon}}([-1,1])$. Hence the case when Y is convex cocompact follows from Proposition 4.16.

Now suppose that Y has a cusp. Let $y \in Y_0$. Using the triangle inequality, we get that $d(\pi(ya_{-\log\varepsilon}), C_Y) - d(\pi(y), C_Y) \leq |\log\varepsilon|$. Therefore, by Proposition 4.16, we have

$$\begin{aligned}\tfrac{\mu_{ya_{-\log\varepsilon}}([-1,1])}{\mu_y([-1,1])} &\leq R_Y^2\beta_Y^{-1} \cdot e^{(1-\delta(Y))(d(\pi(ya_{-\log\varepsilon}),C_Y)-d(\pi(y),C_Y))} \\ &\leq \begin{cases} R_Y^2 \cdot \beta_Y^{-1} \cdot \varepsilon^{\delta(Y)-1} & \text{if } 0<\varepsilon<1 \\ R_Y^2 \cdot \beta_Y^{-1} \cdot \varepsilon^{1-\delta(Y)} & \text{if } \varepsilon \geq 1 \end{cases}.\end{aligned}$$

As a consequence, we have

$$\tfrac{\mu_y([-\varepsilon,\varepsilon])}{\varepsilon^{2\delta(Y)-1}\mu_y([-1,1])} \leq \begin{cases} R_Y^2 \cdot \beta_Y^{-1} & \text{if } 0<\varepsilon<1 \\ R_Y^2 \cdot \beta_Y^{-1} \cdot \rho^2 & \text{if } \rho \geq 1 \text{ and } 1 \leq \varepsilon \leq \rho \end{cases}.$$

Recall from (4.5) that $\delta_Y = \delta(Y)$ when Y is cocompact and $\delta_Y = 2\delta(Y) - 1$ otherwise. The above thus establishes the upper bound.

By choosing $y \in Y_0$ such that $d(\pi(ya_{-\log \varepsilon}), C_Y) - d(\pi(y), C_Y) = |\log \varepsilon|$, we get the lower bound. $\square$

Theorem 4.8 follows from the following:

Proposition 4.19. *We have*

(1) *for any* $0 < \varepsilon \le 1/2$, $0 < \mathsf{s}_Y(\varepsilon) < \infty$.
(2) $\mathsf{s}_Y \asymp \mathsf{p}_Y \ll e^{\star d_Y/\delta_Y} \beta_Y^{-1/\delta_Y}$.

Proof. Let $y \in Y_0$ and $h \in H$ be so that $y = [h]$. Fix $0 < r \le 2$. Recall

$$\mu_y([-r, r]) = \int_{-r}^{r} e^{-\delta(Y)\beta_{hu_s^+}(h(o), hu_s(o))} d\nu_{h(o)}(hu_s^+).$$

Since $|\beta_{hu_r^+}(h(o), hu_r(o))| \le d(o, u_r(o))$, we have

$$e^{-\delta(Y)\beta_{hu_r^+}(h(o), hu_r(o))} \asymp 1$$

with the implied constant independent of all $0 < r \le 2$.

Since $d_o(u_r^+, e^+) = d_{h(o)}((hu_r)^+, h^+)$ where e is the identity (recall that $v_o^+ = e^+$), we have

$$\nu_{h(o)}(B_{h(o)}(h^+, \tfrac{c^{-1}r}{\sqrt{1+2r^2}})) \ll \mu_y([-r, r]) \ll \nu_{h(o)}(B_{h(o)}(h^+, \tfrac{cr}{\sqrt{1+2r^2}}))$$

for some $c > 1$ independent of r and h.

This implies that

$$\mu_y([-\varepsilon/c', \varepsilon/c']) \ll \nu_{h(o)}(B_{h(o)}(h^+, \varepsilon)) \ll \mu_y([-c'\varepsilon, c'\varepsilon])$$

as well as

$$\frac{\mu_y([-\varepsilon/c', \varepsilon/c'])}{\varepsilon^{\delta_Y}\mu_y([-c'/2, c'/2])} \ll \frac{\nu_{h(o)}(B_{h(o)}(h^+, \varepsilon))}{\varepsilon^{\delta_Y}\nu_{h(o)}(B_{h(o)}(h^+, 1/2))} \ll \frac{\mu_y([-c'\varepsilon, c'\varepsilon])}{\varepsilon^{\delta_Y}\mu_y([-1/(2c'), 1/(2c')])}$$

where $c' > 1$ is independent of $0 < \varepsilon < 1/2$ and $h \in H$.

First note that by Corollary 4.18, we have

$$\mu_y([-1/(2c'), 1/(2c')]) \asymp_{c'} \mu_y[-1, 1] \asymp_{c'} \mu_y([-c'/2, c'/2]).$$

Similarly, using Corollary 4.18, for any $0 < \varepsilon \le 1/2$, we have

$$\mu_y([-\varepsilon/c', \varepsilon/c']) \asymp_{c'} \mu_y[-4\varepsilon, 4\varepsilon] \asymp_{c'} \mu_y([-c'\varepsilon, c'\varepsilon]);$$

the choice of the constant 4 here is motivated by the definitions of p_Y and s_Y in (4.6) and (4.7), respectively.

Altogether we conclude that

$$\frac{\nu_{h(o)}(B_{h(o)}(h^+, \varepsilon))}{\varepsilon^{\delta_Y}\nu_{h(o)}(B_{h(o)}(h^+, 1/2))} \asymp \frac{\mu_y([-4\varepsilon, 4\varepsilon])}{(4\varepsilon)^{\delta_Y}\mu_y([-1, 1])}.$$

Taking supremum over $0 < \varepsilon \le 1/2$ and $h \in H$ with $h^\pm \in \Lambda_Y$, we conclude that $\mathsf{s}_Y \asymp \mathsf{p}_Y$.

The last claim follows from Corollary 4.18. $\square$

5. Linear algebra lemma

The goal of this section is to prove the linear algebra lemma (Lemma 5.6) and its slight variant (Lemma 5.13).

In this section, it is more convenient to identify G as $\mathrm{SO}(\mathsf{Q})^\circ$ for the quadratic form

$$\mathsf{Q}(x_1, x_2, x_3, x_4) = 2x_1x_4 - x_2^2 - x_3^2.$$

As Q has signature $(1,3)$, $\mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{SO}(\mathsf{Q})^\circ$ as real Lie groups. We consider the standard representation of G on the space $\mathbb{R}^4$ of row vectors and denote the Euclidean norm on $\mathbb{R}^4$ by $\|\cdot\|$. We have

$$H = \mathrm{Stab}_G(e_3) \simeq \mathrm{SO}(1,2)^\circ,$$
$$A = \{a_t = \mathrm{diag}(e^t, 1, 1, e^{-t}) : t \in \mathbb{R}\} < H \quad \text{and}$$
$$U = \left\{ u_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{r^2}{2} & r & 0 & 1 \end{pmatrix} : r \in \mathbb{R} \right\} < H.$$

Set

$$V := \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_4.$$

Then the restriction of the standard representation of G to H induces a representation of H on V, which is isomorphic to the adjoint representation of H on its Lie algebra $\mathfrak{sl}_2(\mathbb{R})$; in particular, it is irreducible.

Note that for each $t > 0$, $\mathbb{R}e_2 = \{v \in V : va_t = v\}$, $\mathbb{R}e_1$ is the subspace of all vectors with eigenvalues > 1, and $\mathbb{R}e_4$ is the subspace of all vectors with eigenvalues < 1.

Let $p : V \to \mathbb{R}e_1 \oplus \mathbb{R}e_2$ and $p^+ : V \to \mathbb{R}e_1$ denote the natural projections. Writing $v = v_1e_1 + v_2e_2 + v_4e_4$, a direct computation yields that for any $r \in \mathbb{R}$,

$$p(vu_r) = (v_1 + v_2r + \tfrac{v_4r^2}{2})e_1 + (v_2 + v_4r)e_2 \text{ and} \tag{5.1}$$
$$p^+(vu_r) = (v_1 + v_2r + \tfrac{v_4r^2}{2})e_1.$$

For a unit vector $v \in V$ and $\varepsilon > 0$, define

$$D(v,\varepsilon) = \{r \in [-1,1] : \|p(vu_r)\| \le \varepsilon\};$$
$$D^+(v,\varepsilon) = \{r \in [-1,1] : \|p^+(vu_r)\| \le \varepsilon\}.$$

Lemma 5.2. *For all $0 < \varepsilon < 1/2$ and a unit vector $v \in V$, we have*

$$\ell(D(v,\varepsilon)) \ll \varepsilon \quad \text{and} \quad \ell(D^+(v,\varepsilon)) \ll \varepsilon^{1/2}$$

where ℓ denotes the Lebesgue measure on $\mathbb{R}$.

Proof. Since we are allowed to choose the implied constant in the statement, it suffices to prove the lemma for $0 < \varepsilon \le 0.01$.

Writing $v = v_1e_1 + v_2e_2 + v_4e_4$, we have

$$\ell(D(v,\varepsilon)) \le \ell\{r \in [-1,1] : |v_1 + v_2r + \tfrac{v_4r^2}{2}| \le \varepsilon \text{ and } |v_2 + v_4r| \le \varepsilon\}.$$

If $|v_4| \geq 0.01$, then

$$\ell(D(v,\varepsilon)) \leq \ell\{r \in [-1,1] : |v_2 + v_4 r| \leq \varepsilon\} \leq 200\varepsilon.$$

If $|v_4| < 0.01$ but $0.1 \leq |v_2| \leq 1$, then for $r \in [-1,1]$, we have $|v_2 + v_4 r| \geq 0.09$, and hence for all $\varepsilon \leq 0.01$,

$$\ell(D(v,\varepsilon)) \leq \ell\{r \in [-1,1] : |v_2 + v_4 r| \leq \varepsilon\} = 0.$$

Now consider the case when $|v_4| \leq 0.01$ and $|v_2| \leq 0.1$. Then, since $\|v\| = 1$, we get that $|v_1| \geq 0.7$. Hence for all $r \in [-1,1]$, $|v_1 + v_2 r + v_4 r^2/2| > 0.5$. In consequence, for all $\varepsilon < 1/2$,

$$\ell(D(v,\varepsilon)) \leq \ell\{r \in [-1,1] : |v_1 + v_2 r + v_4 r^2/2| \leq \varepsilon\} = 0,$$

proving the estimate on $D(v,\varepsilon)$. To estimate $D^+(v,\varepsilon)$, observe that $p^+(vu_r) = (v_1 + v_2 r + \frac{v_4 r^2}{2})e_1$ is a polynomial map of degree at most 2. Moreover, since $\|v\| = 1$, we have

$$\max\{|v_1|, |v_2|, |v_4|\} \gg 1.$$

Therefore, $\sup_{r\in[-1,1]} \|p^+(vu_r)\| \gg 1$. The claim about $D^+(v,\varepsilon)$ now follows using Lagrange's interpolation, see [7] for a more general statement. $\square$

For the rest of this section, we fix a closed non-elementary H-orbit Y.

Lemma 5.3. *There exists an absolute constant $\hat{b}_0 > 0$ for which the following holds: for any $y \in Y_0$ and $0 < \varepsilon < 1$, we have*

$$\sup_{v\in V, \|v\|=1} \mu_y(D(v,\varepsilon)) \leq \hat{b}_0 \mathsf{p}_Y^{\delta_Y} \varepsilon^{\delta_Y} \mu_y([-1,1]), \tag{5.4}$$

and

$$\sup_{v\in V, \|v\|=1} \mu_y(D^+(v,\varepsilon)) \leq \hat{b}_0 \mathsf{p}_Y^{\delta_Y} \varepsilon^{\delta_Y/2} \mu_y([-1,1]) \tag{5.5}$$

where p_Y is given as in (4.6).

Proof. By (5.1), each set $D(v,\varepsilon)$ and $D^+(v,\varepsilon)$ consists of at most 2 intervals. By Lemma 5.2, $D(v,\varepsilon)$ (resp. $D^+(v,\varepsilon)$) may be covered by $\ll 1$ many intervals of length ε (resp. $\varepsilon^{1/2}$). Therefore (5.4) (resp. (5.5)) follows from the definition of p_Y. $\square$

We use Lemma 5.3 to prove the following lemma which will be crucial in the sequel.

Lemma 5.6 (Linear algebra lemma)**.** *For any $\frac{\delta_Y}{3} \leq s < \delta_Y$, $1 \leq \rho \leq 2$, and $t > 0$, we have*

$$\sup_{y\in Y_0, v\in V, \|v\|=1} \frac{1}{\mu_y([-\rho,\rho])} \int_{-\rho}^{\rho} \frac{1}{\|vu_r a_t\|^s} d\mu_y(r) \leq b_0 \frac{\mathsf{p}_Y^{\delta_Y} e^{-(\delta_Y - s)t/4}}{(\delta_Y - s)} \tag{5.7}$$

where $b_0 \geq 2$ is an absolute constant.

Proof. We first claim that it suffices to prove the claim for $\rho = 1$. Indeed, let $t_\rho = t - \log \rho$ and let $y_\rho = y a_{-\log \rho}$, and for every $v \in V$, let $v_\rho = v a_{-\log\rho}$. Recall also that $\mu_y[-r,r] = \rho^{\delta(Y)}\mu_{ya_{-\log\rho}}[-r/\rho, r/\rho]$ and that Y_0 is A-invariant. Thus,

$$\begin{aligned}\frac{1}{\mu_y([-\rho,\rho])}\int_{-\rho}^{\rho}\frac{1}{\|vu_ra_t\|^s}d\mu_y(r) &= \frac{1}{\mu_y([-\rho,\rho])}\int_{-\rho}^{\rho}\frac{1}{\|va_{-\log\rho}u_{\rho^{-1}r}a_{t_\rho}\|^s}d\mu_y(r)\\ &= \rho^{\delta(Y)}\|v_\rho\|^{-s}\frac{1}{\mu_{y_\rho}([-1,1])}\int_{-1}^{1}\frac{1}{\|v'_\rho u_r a_{t_\rho}\|^s}d\mu_{y_\rho}(r)\end{aligned}$$

where $v'_\rho = v_\rho/\|v_\rho\|$.

Since $\|v_\rho\|^{-s} \asymp 1$ (with absolute implied constants for $1 \le \rho \le 2$) and Y_0 is A-invariant, it thus suffices to prove the lemma for $\rho = 1$.

Fix $0 < s < \delta_Y$ and $t > 0$. We observe that for all $r \in \mathbb{R}$,

$$\|vu_ra_t\| \ge \|p(vu_r)\| \quad \text{and} \quad \|vu_ra_t\| \ge e^t\|p^+(vu_r)\|. \tag{5.8}$$

For simplicity, set $\beta_y := \frac{1}{\mu_y([-1,1])}$. The inequality (5.4) and the first estimate in (5.8) imply that for any $0 < \varepsilon \le 1$ and any unit vector $v \in V$, we have

$$\begin{aligned}\beta_y\int_{r\in D(v,\varepsilon)-D(v,\varepsilon/2)}\|vu_ra_t\|^{-s}d\mu_y(r) &\le \hat{b}_0{\mathsf{p}_Y}^{\delta_Y}\varepsilon^{\delta_Y}\cdot(\varepsilon/2)^{-s}\\ &\le 2\hat{b}_0\mathsf{p}_Y^{\delta_Y}\varepsilon^{\delta_Y-s}.\end{aligned}$$

We write $D(v,\varepsilon) = \bigcup_{k=0}^{\infty} D(v,\varepsilon/2^k) - D(v,\varepsilon/2^{k+1})$. Now applying the above estimate for each $\varepsilon/2^k$ and summing up the geometric series, we get that for any $0 < \varepsilon < 1$,

$$\beta_y\int_{r\in D(v,\varepsilon)}\|vu_ra_t\|^{-s}d\mu_y(r) \le \frac{2\hat{b}_0\mathsf{p}_Y^{\delta_Y}\varepsilon^{\delta_Y-s}}{1-2^{s-\delta_Y}}. \tag{5.9}$$

Moreover, using (5.5) and the first estimate in (5.8) again, for any $\kappa > 0$, we have

$$\beta_y\int_{r\in D^+(v,\kappa)-D(v,\varepsilon)}\|vu_ra_t\|^{-s}d\mu_y(r) \le 2\hat{b}_0\mathsf{p}_Y^{\delta_Y}\kappa^{\delta_Y/2}\varepsilon^{-s}. \tag{5.10}$$

Finally, the definition of $D^+(v,\kappa)$ and the second estimate in (5.8) imply

$$\beta_y\int_{r\in[-1,1]-D^+(v,\kappa)}\|vu_ra_t\|^{-s}d\mu_y(r) \le \kappa^{-s}e^{-st}. \tag{5.11}$$

Combining (5.9), (5.10), and (5.11) and using the inequality $\frac{1}{1-2^{-(\delta_Y-s)}} \le \frac{2}{\delta_Y-s}$, we deduce that for any $0 < \varepsilon, \kappa < 1$,

$$\beta_y\int_{-1}^{1}\|vu_ra_t\|^{-s}d\mu_y(r) \le \frac{2\hat{b}_0\mathsf{p}_Y^{\delta_Y}}{\delta_Y - s}\left(\varepsilon^{\delta_Y-s} + \kappa^{\delta_Y/2}\varepsilon^{-s} + \kappa^{-s}e^{-st}\right).$$

Let $\varepsilon = e^{-t/4}$ and $\kappa = \varepsilon^2$. As $\delta_Y/3 \le s < \delta_Y$, we have $e^{-s/2} \le e^{(s-\delta_Y)/4}$. This yields:

$$\beta_y \int_{-1}^{1} \|vu_r a_t\|^{-s} d\mu_y(r) \le \frac{6\hat{b}_0 \mathsf{p}_Y^{\delta_Y}}{\delta_Y - s} \cdot e^{-(\delta_Y - s)t/4},$$

as we claimed. □

We will extend the upper bound in Lemma 5.6 to all unit vectors $v \in e_1 G$, based on the fact that the vectors in $e_1 G$ are projectively away from the H-invariant point corresponding to $\mathbb{R}e_3$.

Lemma 5.12. *There exists an absolute constant $b_1 > 1$ such that for any vector $v \in e_1 G \subset \mathbb{R}^4$,*

$$\|v\| \le b_1 \|v_1\|$$

where v_1 is the projection of $v \in \mathbb{R}^4$ to $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_4$.

Proof. Since $\mathsf{Q}(e_1) = 0$ and $G = \mathrm{SO}(\mathsf{Q})^\circ$, we have $\mathsf{Q}(e_1 g) = 0$ for every $g \in G$. Since $\mathsf{Q}(e_3) = -1$, the set $\{\|v\|^{-1} v : v \in e_1 G\}$ is a compact subset of the unit sphere in $\mathbb{R}^4$ not containing $\pm e_3$. Therefore there exists an absolute constant $0 < \eta < 1$ such that if we write $v = v_1 + r e_3 \in e_1 G$, then $|r| \le \eta \|v\|$. Therefore $\|v_1\|^2 = \|v\|^2 - r^2 \ge (1-\eta^2)\|v\|^2$. Hence it suffices to set $b_1 = (1-\eta^2)^{-1/2}$. □

Lemma 5.13 (Linear algebra lemma II)**.** *For any $\frac{\delta_Y}{3} \le s < \delta_Y$, $1 \le \rho \le 2$, and $t > 0$, we have*

$$\sup_{y \in Y_0, v \in e_1 G, \|v\|=1} \frac{1}{\mu_y([-\rho,\rho])} \int_{-\rho}^{\rho} \frac{1}{\|vu_r a_t\|^s} d\mu_y(r) \le b_0 b_1 \frac{\mathsf{p}_Y^{\delta_Y} e^{-(\delta_Y - s)t/4}}{(\delta_Y - s)}$$

where $b_0 \ge 2$ and $b_1 > 1$ are absolute constants as in Lemmas 5.6 and 5.12 respectively.

Proof. Let $v \in e_1 G$ be a unit vector, and write $v = v_0 + v_1$ where $v_0 \in \mathbb{R}e_3$ and $v_1 \in V$. Since e_3 is H-invariant, we have $vh = v_0 + v_1 h \in \mathbb{R}e_3 \oplus V$ for all $h \in H$. Therefore,

$$\frac{1}{\mu_y([-\rho,\rho])} \int_{-\rho}^{\rho} \frac{1}{\|vu_r a_t\|^s} d\mu_y(r) \le \frac{1}{\mu_y([-\rho,\rho])} \int_{-\rho}^{\rho} \frac{1}{\|v_1 u_r a_t\|^s} d\mu_y(r)$$

$$\begin{aligned} &\le \frac{b_0 \mathsf{p}_Y^{\delta_Y} e^{-(\delta_Y - s)t/4}}{(\delta_Y - s)} \|v_1\|^{-s} \quad \text{by Lemma 5.6} \\ &\le \frac{b_0 b_1 \mathsf{p}_Y^{\delta_Y} e^{-(\delta_Y - s)t/4}}{(\delta_Y - s)} \|v\|^{-s} \quad \text{by Lemma 5.12.} \end{aligned}$$

□

6. Height function ω

In this section we define the height function $\omega : X_0 \to (0,\infty)$ and show that $\omega(x)$ is comparable to the reciprocal of the injectivity radius at x.

For this purpose, we continue to realize G as $\mathrm{SO}(\mathsf{Q})^\circ$ acting on $\mathbb{R}^4$ by the standard representation, as in Section 5. Observe that $\mathsf{Q}(e_1) = 0$ and the stabilizer of e_1 in G is equal to M_0N.

Fixing a set of Γ-representatives $\xi_1, \cdots, \xi_\ell$ in Λ_{bp}, choose elements $g_i \in G$ so that $g_i^- = \xi_i$ and $\|e_1 g_i^{-1}\| = 1$; this is possible since $\{g \in G : g^- = \xi_i\}$ is a conjugate of AM_0N.

Set

$$v_i := e_1 g_i^{-1} \in e_1 G. \tag{6.1}$$

Note that

$$\mathrm{Stab}_G(\xi_i) = g_i AM_0N g_i^{-1} \text{ and } \mathrm{Stab}_G(v_i) = g_i M_0 N g_i^{-1}.$$

By Witt's theorem, we have that for each i,

$$\{v \in \mathbb{R}^4 - \{0\} : \mathsf{Q}(v) = 0\} = v_i G \simeq g_i M_0 N g_i^{-1} \backslash G.$$

Lemma 6.2. *For each $1 \le i \le \ell$, the orbit $v_i\Gamma$ is a closed (and hence discrete) subset of $\mathbb{R}^4$.*

Proof. The condition $\xi_i \in \Lambda_{bp}$ implies that $\Gamma\backslash\Gamma g_i M_0 N$ is a closed subset of X. Equivalently, $\Gamma g_i M_0 N$ as well as $\Gamma g_i M_0 N g_i^{-1}$ is closed in G. Therefore, its inverse $g_i M_0 N g_i^{-1}\Gamma$ is a closed subset of G. In consequence, $v_i\Gamma \subset \mathbb{R}^4$ is a closed subset of $v_i G = \{v \in \mathbb{R}^4 - \{0\} : \mathsf{Q}(v) = 0\}$.

It remains to show that $v_i\Gamma$ does not accumulate on 0. Suppose on the contrary that there exists an infinite sequence $v_i\gamma_\ell$ converging to 0 for some $\gamma_\ell \in \Gamma$. Using the Iwasawa decomposition $G = g_i NAK_0$, we may write $\gamma_\ell = g_i n_\ell a_{t_\ell} k_\ell$ with $n_\ell \in N, t_\ell \in \mathbb{R}$ and $k_\ell \in K_0$. Since

$$v_i \gamma_\ell = e^{t_\ell}(e_1 k_\ell),$$

the assumption that $v_i\gamma_\ell \to 0$ implies that $t_\ell \to -\infty$.

On the other hand, as $\xi_i \in \Lambda_{bp}$, $\mathrm{Stab}_\Gamma(\xi_i) = \Gamma \cap g_i AM_0 N g_i^{-1}$ contains a parabolic element, say, $\gamma' \ne e$. Note that $n_0 := g_i^{-1}\gamma' g_i$ is then an element of N and hence a unipotent element, as any parabolic element of AM_0N belongs to N in the group $G \simeq \mathrm{PSL}_2(\mathbb{C})$. Now observe that, as N is abelian,

$$\gamma_\ell^{-1}\gamma'\gamma_\ell = k_\ell^{-1} a_{-t_\ell}(n_\ell^{-1} g_i^{-1}\gamma' g_i n_\ell) a_{t_\ell} k_\ell = k_\ell^{-1}(a_{-t_\ell} n_0 a_{t_\ell}) k_\ell.$$

Since $t_\ell \to -\infty$, the sequence $a_{-t_\ell} n_0 a_{t_\ell}$ converges to e. Since $\{k_\ell^{-1}\}$ is a bounded sequence, it follows that, up to passing to a subsequence, $\gamma_\ell^{-1}\gamma'\gamma_\ell$ is an infinite sequence converging to e, contradicting the discreteness of Γ. $\square$

Definition 6.3 (Height function)**.** Define the height function $\omega : X_0 \to [2,\infty)$ by

$$\omega(x) := \max_{1 \le i \le \ell} \omega_i(x)$$

where

$$\omega_i(x) = \max_{\gamma\in\Gamma}\left\{2, \|v_i\gamma g\|^{-1}\right\} \quad \text{for any } g \in G \text{ with } x = [g];$$

this is well-defined by Lemma 6.2.

If Γ has no parabolic elements, we define $\omega(x) = 2$ for all $x \in X_0$.

By the definition of ε_X, X_0 is contained in the union of X_{ε_X} and $\cup_{j=1}^{\ell}\mathfrak{h}_j$ where $\mathfrak{h}_j$ is a horoball based at ξ_j.

$$\text{Fix } T_j > 0 \text{ so that } \mathfrak{h}_j = [g_j]NA_{(-\infty,-T_j]}K_0.$$

Set $\tilde{\mathfrak{h}}_j := g_jNA_{(-\infty,-T_j]}K_0$.

The following is an immediate consequence of the thick-thin decomposition of M:

Lemma 6.4. *If $\tilde{\mathfrak{h}}_j \cap \gamma\tilde{\mathfrak{h}}_i \neq \emptyset$ for some $1 \le i, j \le \ell$ and $\gamma \in \Gamma$, then $i = j$, $\gamma \in \mathrm{Stab}_G(\xi_i) = \mathrm{Stab}\,\tilde{\mathfrak{h}}_i$, and hence $\tilde{\mathfrak{h}}_j = \gamma\tilde{\mathfrak{h}}_i$.*

Lemma 6.5. *For all $1 \le i, j \le \ell$ and $\gamma \in \Gamma$ such that $\tilde{\mathfrak{h}}_j \neq \gamma\tilde{\mathfrak{h}}_i$,*

$$\inf_{q\in\tilde{\mathfrak{h}}_i} \|v_j\gamma h\| \ge \eta_0 \tag{6.6}$$

where $\eta_0 := \min_{1\le m\le \ell} e^{-T_m}$.

Proof. Let $q \in \tilde{\mathfrak{h}}_i$ and $\gamma \in \Gamma$. Using $G = g_jNAK_0$, write $\gamma q = g_jua_sk \in g_jNAK_0$. Then $\|v_j\gamma q\| = e^s$. Hence if $\|v_j\gamma q\| < \eta_0$, then $s \le -T_j$. So $\gamma q \in \tilde{\mathfrak{h}}_j$. Therefore $\tilde{\mathfrak{h}}_j \cap \gamma\tilde{\mathfrak{h}}_i \neq \emptyset$. By Lemma 6.4, $\tilde{\mathfrak{h}}_j = \gamma\tilde{\mathfrak{h}}_i$. □

Proposition 6.7. *There is an absolute constant $\alpha \ge 2$ such that for all $x \in X_0$,*

$$\tfrac{1}{2\alpha} \cdot \mathrm{inj}(x) \le \omega(x)^{-1} \le \tfrac{\alpha}{2} \cdot \mathrm{inj}(x). \tag{6.8}$$

Proof. Fixing $1 \le j \le \ell$, it suffices to show the claim for all $x \in X_0 \cap \mathfrak{h}_j$.

Let $g \in g_iua_{-t}k \in \tilde{\mathfrak{h}}_i$ be so that $x = [g]$, where $ua_{-t}k \in NA_{(-\infty,-T_j]}K_0$.

Note that

$$\omega_i(x)^{-1} \le \|v_ig\| = \|e_1g_i^{-1}(g_iua_{-t}k)\| = \|e_1ua_{-t}k\| = e^{-t}.$$

In view of the definition of ω and ω_i, this together with Lemma 6.5 implies that

$$\omega(x) = \omega_i(x) = e^t.$$

Since $\mathrm{inj}(x) \asymp e^{-t}$, this finishes proof. □

7. Markov operators

In this section we define a Markov operator A_t and prove Proposition 7.5 which relates the average $m_Y(F)$ of a locally bounded, log-continuous, Borel function F on Y_0 with a super-harmonic type inequality for A_tF. This proposition will serve as a main tool in our approach to prove Theorem 1.5.

Fix a closed non-elementary H-orbit Y in X.

Bowen-Margulis-Sullivan measure m_Y. We denote by m_Y the Bowen-Margulis-Sullivan probability measure on $\Delta_Y\backslash H = \mathrm{T}^1(S_Y)$, which is the unique probability measure of maximal entropy (that is $\delta(Y)$) for the geodesic flow. We will also use the same notation m_Y to denote the push-forward of the measure to Y via the map $\mathrm{Stab}_H(y_0)\backslash H \to Y$ given by $[h] \to y_0 h$. Considered as a measure on Y, m_Y is well-defined, independent of the choice of $y_0 \in Y$.

Recall the definition of Y_0 in (4.4); note that $Y_0 = \operatorname{supp} m_Y$. In the following, all of our Borel functions are assumed to be defined everywhere in their domains. By a locally bounded function, we mean a function which is bounded on every compact subset.

Definition 7.1 (Markov Operator)**.** *Let $t \in \mathbb{R}$ and $\rho > 0$. For a locally bounded Borel function $\psi : Y_0 \to \mathbb{R}$, we define*

$$(\mathsf{A}_{t,\rho}\psi)(y) := \frac{1}{\mu_y([-\rho,\rho])} \int_{-\rho}^{\rho} \psi(yu_r a_t) d\mu_y(r). \tag{7.2}$$

We set $\mathsf{A}_t := \mathsf{A}_{t,1}$.

Note that $\mathsf{A}_{t,\rho}\psi$ is a locally bounded Borel function on Y_0. Although $\lim_{n\to\infty} \mathsf{A}_{nt}(\psi) = m_Y(\psi)$ for any $\psi \in C_c(Y_0)$ and any $t > 0$ [23], the *Margulis function F* we will be constructing is not a continuous function on Y_0, and hence we cannot use such an equidistribution statement to control $m_Y(F)$. We will use the following lemma instead:

Lemma 7.3. *Let $F : Y_0 \to [2, \infty)$ be a locally bounded Borel function. Assume that there exist some $t > 0$ and $D > 0$ such that*

$$\limsup_{n\to\infty} \mathsf{A}_{nt} F(y) \le D \quad \textit{for all } y \in Y_0. \tag{7.4}$$

Then

$$m_Y(F) \le 8D.$$

Proof. For every $k \ge 2$, let $F_k : Y_0 \to [2, \infty)$ be given by

$$F_k(y) := \min\{F(y), k\}.$$

As F_k is bounded, it belongs to $L^1(Y_0, m_Y)$. Since the action of A is mixing for m_Y by the work of Babillot [1], we have m_Y is a_t-ergodic for each $t \ne 0$. Hence, by the Birkhoff ergodic theorem, for m_Y-a.e. $y \in Y_0$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F_k(ya_{nt}) = \int F_k \, dm_Y.$$

Therefore, using Egorov's theorem, for every $\varepsilon > 0$, there exist $N_\varepsilon > 1$ and a measurable subset $Y'_\varepsilon \subset Y_0$ with $m_Y(Y'_\varepsilon) > 1 - \varepsilon^2$ such that for every $y \in Y'_\varepsilon$ and all $N > N_\varepsilon$, we have

$$\frac{1}{N} \sum_{n=1}^{N} F_k(ya_{nt}) > \frac{1}{2} \int F_k \, dm_Y.$$

Now by the maximal ergodic theorem [16, App. A.1], if ε is small enough, there exists a measurable subset $Y_\varepsilon \subset Y'_\varepsilon$ with $m(Y_\varepsilon) > 1-\varepsilon$ so that for all $y \in Y_\varepsilon$, we have

$$\mu_y\{r \in [-1,1] : yu_r \in Y'_\varepsilon\} > \tfrac{1}{2}\mu_y([-1,1]).$$

Altogether, if $y \in Y_\varepsilon$ and $N > N_\varepsilon$, we have

$$\tfrac{1}{N}\sum_{n=1}^{N} \mathsf{A}_{nt}F_k(y) = \tfrac{1}{\mu_y([-1,1])}\int_{-1}^{1} \tfrac{1}{N}\sum_{n=1}^{N} F_k(yu_r a_{nt})d\mu_y(r) > \tfrac{1}{4}\int F_k\, dm_Y.$$

Fix $y \in Y_\varepsilon$. By the hypothesis (7.4), there exists $n_0 = n_0(y)$ such that for all $n \ge n_0$, we have

$$\mathsf{A}_{nt}F_k(y) \le \mathsf{A}_{nt}F(y) \le 2D.$$

Therefore, we deduce that for all sufficiently large $N \gg 1$,

$$\tfrac{1}{4}\int F_k\, dm_Y \le \tfrac{1}{N}\left(\sum_{n=1}^{n_0} \mathsf{A}_{nt}F_k(y) + \sum_{n=n_0+1}^{N} \mathsf{A}_{nt}F_k(y)\right) \le \tfrac{kn_0}{N} + \tfrac{2D(N-n_0)}{N}.$$

By sending $N \to \infty$, we get that for all $k > 2$,

$$\int F_k\, dm_Y \le 8D.$$

Since $\{F_k : k = 3,4,..\}$ is an increasing sequence of positive functions converging to F point-wise, the monotone convergence theorem implies

$$\int F\, dm_Y = \lim_{k\to\infty}\int F_k\, dm_Y \le 8D$$

as we claimed. □

We remark that in [12], the Markov operator A_t was defined using the integral over the translates $\mathrm{SO}(2)a_t$, whereas we use the integral over the translates $U_{[-\rho,\rho]}a_t$ of a horocyclic piece. The proof of the following proposition, which is an analogue of [12, §5.3], is the main reason for our digression from their definition, as the handling of the PS-measure on U is more manageable than that of the PS-measure on $\mathrm{SO}(2)$ in performing change of variables.

Proposition 7.5. *Let $F : Y_0 \to [2,\infty)$ be a locally bounded Borel function satisfying the following properties:*

(a) *There exists $\sigma \ge 2$ such that for all $h \in B_H(2)$ and $y \in Y_0$,*

$$\sigma^{-1}F(y) \le F(yh) \le \sigma F(y).$$

(b) *There exist $t \ge 2$ and $D_0 > 0$ such that for all $y \in Y_0$ and $1 \le \rho \le 2$,*

$$\mathsf{A}_{t,\rho}F(y) \le \frac{1}{8\sigma \mathsf{p}_Y^{\delta_Y}} \cdot F(y) + D_0$$

where p_Y is as in (4.6).

Then

$$m_Y(F) \le 64 D_0 \mathsf{p}_Y^{\delta_Y}.$$

In view of Lemma 7.3, Proposition 7.5 is an immediate consequence of the following:

Proposition 7.6. *Let F be as in Proposition 7.5. Then for all $y \in Y_0$ and $n \ge 1$, we have*

$$\mathsf{A}_{nt} F(y) \le \frac{1}{2^n} F(y) + 8 D_0 \mathsf{p}_Y^{\delta_Y}. \tag{7.7}$$

Proof. The main step of the proof is the following estimate.
Claim: For any $1 \le \rho \le \frac{3}{2}$, $y \in Y_0$ and $n \in \mathbb{N}$, we have

$$\mathsf{A}_{(n+1)t,\rho} F(y) \le \tfrac{1}{2} \mathsf{A}_{nt,\rho+e^{-nt}} F(y) + \hat{D} \tag{7.8}$$

where $\hat{D} := 4 D_0 \mathsf{p}_Y^{\delta_Y}$; recall that $e^{-nt} \le 1/2$.

Let us first assume this claim and prove the proposition. We observe

- $\sum_{j\ge 1} e^{-jt} \le 1/2$ (as $t \ge 2$),
- $(8\sigma \mathsf{p}_Y^{\delta_Y})^{-1} \le 1/2$, and
- $D_0 \le \hat{D}$.

Using the assumption (b) of Proposition 7.5 with $\rho_n = 1 + \sum_{j=1}^{n-1} e^{-jt}$ ($n \ge 2$), we deduce that for any $n \ge 2$,

$$\begin{aligned}
\mathsf{A}_{nt} F(y) &\le \tfrac{1}{2^{n-1}} \mathsf{A}_{t,\rho_n} F(y) + \hat{D}(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^{n-2}}) \\
&\le \tfrac{1}{2^{n-1}} \Big((8\sigma \mathsf{p}_Y^{\delta_Y})^{-1} F(y) + D_0 \Big) + \hat{D}(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^{n-2}}) \\
&\le \tfrac{1}{2^n} F(y) + 2\hat{D} \qquad (7.9)
\end{aligned}$$

which establishes the proposition.

We now prove the claim (7.8). For $y \in Y_0$ and $\rho > 0$, set

$$b_y(\rho) := \mu_y([-\rho, \rho]) \text{ and } b_y = b_y(1).$$

To ease the notation, we prove (7.8) with $\rho = 1$; the proof in general is similar. By assumption (a) and (b) of Proposition 7.5, we have

$$\mathsf{A}_t F(y) \le c_0 F(y) + D_0 \le \Big(\frac{c_0 \sigma}{b_y} \int_{-1}^{1} F(y u_r) d\mu_y(r) \Big) + D_0 \tag{7.10}$$

where $c_0 = (8\sigma \mathsf{p}_Y^{\delta_Y})^{-1}$.

Set $\rho_n := e^{-nt}$. Let $\{[r_j - \rho_n, r_j + \rho_n] : j \in J\}$ be a covering of

$$[-1, 1] \cap \operatorname{supp}(\mu_y)$$

with $r_j \in [-1, 1] \cap \operatorname{supp}(\mu_y)$ and with multiplicity bounded by 2. For each $j \in J$, let $z_j := y u_{r_j}$. Then

$$\sum_j b_{z_j}(\rho_n) = \sum_j \mu_y([r_j - \rho_n, r_j + \rho_n]) \le 2 b_y(2). \tag{7.11}$$

Moreover, we get

$$
\begin{aligned}
\mathsf{A}_{(n+1)t}F(y) &= \frac{1}{b_y}\int_{-1}^{1} F(yu_r a_{(n+1)t})d\mu_y(r)\\
&\le \frac{1}{b_y}\sum_j \int_{-\rho_n}^{\rho_n} F(z_j u_r a_{(n+1)t})d\mu_{z_j}(r)\\
&= \frac{1}{b_y}\sum_j \int_{-\rho_n}^{\rho_n} F(z_j a_{nt} u_{re^{nt}} a_t)d\mu_{z_j}(r). \qquad (7.12)
\end{aligned}
$$

We now make the change of variables $s = re^{nt}$. In view of (7.12), we have

$$\mathsf{A}_{(n+1)t}F(y) \le \frac{1}{b_y}\sum_j \frac{b_{z_j}(\rho_n)}{b_{z_j a_{nt}}}\int_{-1}^{1} F(z_j a_{nt} u_s a_t)d\mu_{z_j a_{nt}}(s).$$

Applying (7.10) with the base point $z_j a_{nt}$, we get from the above that

$$\mathsf{A}_{(n+1)t}F(y) \le \frac{1}{b_y}\sum_j \frac{b_{z_j}(\rho_n)c_0\sigma}{b_{z_j a_{nt}}}\int_{-1}^{1} F(z_j a_{nt} u_s)d\mu_{z_j a_{nt}}(s) + \frac{1}{b_y}\sum_j b_{z_j}(\rho_n)D_0. \qquad (7.13)$$

By (7.11), we have $\frac{1}{b_y}\sum_j b_{z_j}(\rho_n)D_0 \le \hat{D}$.

Therefore, reversing the change of variable, i.e., now letting $r = e^{-nt}s$, we get from (7.13) the following:

$$
\begin{aligned}
\mathsf{A}_{(n+1)t}F(y) &\le \frac{1}{b_y}\sum_j c_0\sigma \int_{-\rho_n}^{\rho_n} F(z_j u_r a_{nt})d\mu_{z_j}(r) + \hat{D}\\
&\le \frac{2c_0\sigma}{b_y}\int_{-(1+\rho_n)}^{1+\rho_n} F(yu_r a_{nt})d\mu_y(r) + \hat{D}\\
&= \frac{2c_0\sigma b_y(1+\rho_n)}{b_y}\mathsf{A}_{nt,1+\rho_n}F(y) + \hat{D}.
\end{aligned}
$$

Since

$$\sup_{y\in Y_0}\frac{2c_0\sigma b_y(2)}{b_y} = (4\mathsf{p}_Y^{\delta_Y})^{-1}\sup_{y\in Y_0}\frac{b_y(2)}{b_y} \le \frac{1}{2},$$

we get

$$\mathsf{A}_{(n+1)t}F(y) \le \frac{1}{2}\mathsf{A}_{nt,1+\rho_n}F(y) + \hat{D}.$$

The proof is complete. □

8. Return lemma and number of nearby sheets

We fix closed non-elementary H-orbits Y and Z in X. Since Z is closed, a fixed ball around $y \in Y_0$ intersects only finitely many sheets of Z (Fig. 2). The aim of this section is to show that the number of sheets of Z

in $B(y, \mathrm{inj}(y))$ is controlled by the tight area of S_Z with a multiplicative constant depending on p_Y and δ_Y.

The main ingredient is a return lemma which says that for any $y \in Y_0$, there exists some point in $\{yu_r \in Y_0 : r \in [-1, 1]\}$ whose minimum return time to a fixed compact subset under the geodesic flow is comparable to $\log(\omega(y))$ (see Lemma 8.4).

Return lemma. We use the notation of section 6.

Recall that $\mathrm{Lie}(G) = i\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$. We define a norm $\|\cdot\|$ on $\mathrm{Lie}(G)$ using an inner product with respect to which $\mathfrak{sl}_2(\mathbb{R})$ and $i\mathfrak{sl}_2(\mathbb{R})$ are orthogonal to each other. Given a vector $w \in \mathrm{Lie}(G)$, we write

$$w = i\mathrm{Im}(w) + \mathrm{Re}(w) \in i\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R}).$$

Since the exponential map $\mathrm{Lie}(G) \to G$ defines a local diffeomorphism, there exists an absolute constant $c_1 \geq 2$ satisfying the following two properties:

(1) for all $x \in X$, and all $w = i\mathrm{Im}(w) + \mathrm{Re}(w) \in \mathrm{Lie}(G)$ with $\|w\| \leq \max(1, \varepsilon_X)$,

$$c_1^{-1}\|w\| \leq d(x, x\exp(i\mathrm{Im}(w))\exp(\mathrm{Re}(w))) \leq c_1\|w\|; \tag{8.1}$$

(2) If $d(x, x') \leq \varepsilon_X/c_1$, then $x' = x\exp(i\mathrm{Im}(w))\exp(\mathrm{Re}(w))$ for some $w \in \mathrm{Lie}(G)$.

We choose an absolute constant $d_X \geq 24$ so that

$$X_{\varepsilon_X} \subset \{x \in X_0 : \omega(x) \leq d_X\}.$$

Let $D_1 := D_1(Y)$ be given by

$$D_1 = c_1\alpha\left(\tfrac{6b_1}{\kappa\eta_0} + d_X\right) \tag{8.2}$$

where κ is defined by $\hat{b}_0\mathsf{p}_Y^{\delta_Y}\kappa^{\delta_Y/2} = 1/2$, $0 < \eta_0 < 1$ is as in (6.6), $\alpha \geq 1$ is as in (6.8), and c_1 is as in (8.1). We note that by increasing $\hat{b}_0$ if necessary, we may and will assume that $\kappa \in (0, 1)$. Moreover we put $\eta_0 = \frac{1}{2}$ when Y is convex cocompact.

Define

$$\mathcal{K}_Y = \{y \in Y_0 : \omega(y) \leq D_1/(c_1\alpha)\}. \tag{8.3}$$

Note that $X_{\varepsilon_X} \cap Y_0 \subset \mathcal{K}_Y$.

The choices of the above parameters are motivated by our applications in the following lemmas. Indeed the choice of κ is used in (8.6). The multiplicative parameter $c_1\alpha$, which features in the definitions of D_1 and $\mathcal{K}_Y$, is tailored so that we may utilize Lemma 8.10 in the proof of Lemma 8.13.

Lemma 8.4 (Return lemma)**.** *For every $y \in Y_0$, there exists some $|r| \leq 1$ so that $yu_ra_{-t} \in \mathcal{K}_Y$ where $t = \log(\eta_0\omega(y)/6)$.*

Proof. Let $y \in Y_0 - \mathcal{K}_Y$. By the definition of ω, there exist $1 \leq i \leq \ell$ and $g \in \tilde{\mathfrak{h}}_i$ so that $y = [g]$ and

$$\omega(y) = \omega_i(y),$$

see §6 for the notation. Set $v := v_i g$. Then

$$\|v\|^{-1} = \omega_i(y) = \omega(y).$$

Let us write $v = w + se_3$ where $w \in V$ and $s \in \mathbb{R}$. Recall from Lemma 5.12 that there exists $b_1 > 1$ so that

$$\|w\| \geq b_1^{-1}\|v\|. \tag{8.5}$$

Let $\kappa > 0$ be as used in (8.2). Then (5.5) implies that

$$\mu_y(D^+(\tfrac{w}{\|w\|}, \kappa)) \leq \tfrac{1}{2}\mu_y([-1,1]). \tag{8.6}$$

Therefore, there exists $r \in \operatorname{supp}(\mu_y) \cap \Big([-1,1] - D^+(\tfrac{w}{\|w\|}, \kappa)\Big)$. This means that $yu_r \in Y_0$, moreover, we have, using (8.5),

$$\|p^+(vu_r)\| = \|p^+(wu_r)\| > \kappa\|w\| \geq \kappa b_1^{-1}\|v\|.$$

Set $t := \log(\eta_0\omega(y)/6)$. Then

$$\begin{aligned}\kappa b_1^{-1}\|v\| \cdot \tfrac{\eta_0\omega(y)}{6} &= \kappa b_1^{-1}\|v\|e^t \leq \|p^+(vu_r)a_t\| \\ &\leq \|vu_r a_t\| \leq \|vu_r\|e^t \leq 2\|v\| \cdot \tfrac{\eta_0\omega(y)}{6},\end{aligned}$$

where we use $\|vu_r\| \leq 2\|v\|$ in the last inequality.

Hence, using the fact that $\omega(y) = \|v\|^{-1}$,

$$\tfrac{\kappa b_1^{-1}\eta_0}{6} \leq \|vu_r a_t\| = \|v_i g u_r a_t\| \leq \tfrac{\eta_0}{3}.$$

This in particular implies that $gu_r a_t \in \tilde{\mathfrak{h}}_i$. By Lemma 6.5, whenever $\gamma \in \Gamma$ and $1 \leq j \leq \ell$ satisfy that $\tilde{\mathfrak{h}}_j \neq \gamma\tilde{\mathfrak{h}}_i$, we have

$$\|v_j\gamma g u_r a_t\| \geq \eta_0;$$

note that $i = j$ is allowed.

This and the above upper bound thus imply

$$\omega(yu_r a_t) = \|v_i g u_r a_t\|^{-1}.$$

Therefore,

$$\omega(yu_r a_t) \leq \frac{6b_1}{\kappa\eta_0} \leq D_1/(c_1\alpha)$$

proving the claim. □

Number of nearby sheets. Recalling that $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{R}) \oplus i\mathfrak{sl}_2(\mathbb{R})$, we set $V = i\mathfrak{sl}_2(\mathbb{R})$ and consider the action of H on V via the adjoint representation; so $v \cdot h = h^{-1}vh$ for $v \in V$ and $h \in H$. We use the relation $g(\exp v)h = gh\exp(v \cdot h)$ which is valid for all $g \in G, v \in V, h \in H$.

If $D \geq \alpha/2$ for α as in Proposition 6.7, then $D^{-1}\omega(y)^{-1} \leq \frac{1}{2}\operatorname{inj}(y)$.

Definition 8.7. *For $y \in Y_0$ and $D \geq \alpha/2$, we define*

$$I_Z(y, D) = \{v \in V - \{0\} : \|v\| < D^{-1}\omega(y)^{-1},\ y\exp(v) \in Z\}. \tag{8.8}$$

Since V is the orthogonal complement to $\mathrm{Lie}(H)$, the set $I_Z(y,D)$ can be understood as the number of sheets of Z in the ball around y of radius $D^{-1}\omega(y)^{-1}$.

It turns out that $\#I_Z(y,D)$ can be controlled in terms of the tight area of S_Z, uniformly over all $y\in Y_0$ for an appropriate $D>1$.

Notation 8.9. *We set*

$$\tau_Z := \mathrm{area}_t(S_Z).$$

Theorem 3.3 shows that $1 \ll \tau_Z < \infty$ where the implied constant depends only on M.

We begin with the following lemma:

Lemma 8.10. *With $c_1\ge 2$ and $\alpha\ge 2$ given respectively in* (8.1) *and* (6.7)*, we have that for all $y\in Y_0$,*

$$\#I_Z(y,c_1\alpha) \ll \omega(y)^3\tau_Z. \tag{8.11}$$

Proof. Let $c_1\ge 1$ and α be the absolute constants given in (8.1) and (6.7) respectively. It follows that for any $y\in Y_0$ and $v\in I_Z(y,\alpha)$,

$$d(y, y\exp(v)) \le c_1\|v\| \le c_1(c_1\alpha)^{-1}\cdot\omega(y)^{-1} < \tfrac{1}{2}\cdot \mathrm{inj}(y). \tag{8.12}$$

It follows that for each $v\in I_Z(y,c_1\alpha)$, $\mathrm{inj}(y\exp v)\ge \mathrm{inj}(y)/2$. Hence the balls $B_Z(y\exp v, \mathrm{inj}(y)/2)$, $v\in I_Z(y,c_1\alpha)$ are disjoint from each other, and hence

$$\#I_Z(y,\alpha)\cdot \mathrm{Vol}(B_H(e,\mathrm{inj}(y)/2)) = \mathrm{Vol}\Big\{\bigcup B_Z(y\exp v, \mathrm{inj}(y)/2) : v\in I_Z(y,\alpha)\Big\}.$$

On the other hand, if we set $\rho_y := \min\{1, \mathrm{inj}(y)/2\}$, then

$$\pi\Big(\{\bigcup B_Z(y\exp v,\rho_y) : v\in I_Z(y,c_1\alpha)\}\Big) \subset S_Z\cap \mathcal{N}(\mathrm{core}(M)).$$

Therefore

$$\#I_Z(y,c_1\alpha) \le \mathrm{Vol}(B_H(e,\rho_y))^{-1}\cdot \tau_Z \ll \rho_y^{-3}\tau_Z \ll \omega(y)^3\tau_Z;$$

we have used that $2\pi(\cosh r-1)\ge r^3$ for all $r>0$ and Proposition 6.7 respectively in the last two estimates. □

Let D_1 be as in (8.2). By the choice of κ, we have $D_1 \ll \mathsf{p}_Y^2$ (see the discussion following (8.2)).

Lemma 8.13 (Number of sheets)**.** *For $D_1 = D_1(Y)\ll \mathsf{p}_Y^2$ as in* (8.2)*, we have*

$$\sup_{y\in Y_0} \#I_Z(y,D_1) \le c_0\cdot \mathsf{p}_Y^6\cdot \tau_Z$$

where $c_0\ge 2$ is an absolute constant.

Proof. Let $\mathcal{K}_Y$ be as in (8.3):

$$\mathcal{K}_Y = \{y\in Y_0 : \omega(y) \le (c_1\alpha)^{-1}D_1\}.$$

If $y\in\mathcal{K}_Y$, then, by Lemma 8.10,

$$\#I_Z(y,D_1) \le \#I_Z(y,c_1\alpha) \ll D_1^3\tau_Z \ll \mathsf{p}_Y^6\tau_Z.$$

Now suppose that $y \in Y_0 - \mathcal{K}_Y$. By Lemma 8.4, there exist $|r| < 1$ and $t = \log(\eta_0 \cdot \omega(y)/6)$, where $0 < \eta_0 \le 1$ is as in (6.6), such that

$$yu_r a_t \in \mathcal{K}_Y.$$

We claim that if $v \in I_Z(y, D_1)$, then $v(u_r a_t) \in I_Z(yu_r a_t, c_1\alpha)$. Firstly, note that, plugging $t = \log(\eta_0 \cdot \omega(y)/6)$ and using $0 < \eta \le 1$,

$$\|v(u_r a_t)\| \le 3e^t\|v\| = \tfrac{3\eta_0\, \omega(y)\, \|v\|}{6} < \omega(y) \cdot \|v\|.$$

Hence for $v \in I_Z(y, D_1)$, as $\omega(y)\|v\| < D_1^{-1}$,

$$\|v(u_r a_t)\| < \omega(y) \cdot \|v\| \le D_1^{-1} \le (c_1\alpha)^{-1}\omega(yu_r a_t)^{-1}.$$

where we used the fact that $(c_1\alpha)^{-1}D_1 > \omega(yu_r a_t)$.

Since $y(\exp v)u_r a_t = (yu_r a_t)\exp(v(u_r a_t)) \in Z$, this implies that $v(u_r a_t) \in I_Z(yu_r a_t, c_1\alpha)$. Therefore the map $v \mapsto v(u_r a_t)$ is an injective map from $I_Z(y, D_1)$ into $I_Z(yu_r a_t, c_1\alpha)$. Consequently,

$$\#I_Z(y, D_1) \le \#I_Z(yu_r a_t, c_1\alpha) \ll \mathsf{p}_Y^6 \cdot \tau_Z.$$

This finishes the proof. $\square$

9. Margulis function: construction and estimate

Throughout this section, we fix closed non-elementary H-orbits Y, Z in X and

$$\frac{\delta_Y}{3} \le s < \delta_Y.$$

In this section, we define a family of Margulis functions $F_{s,\lambda} = F_{s,\lambda,Y,Z}$, $\lambda > 1$ and show that the hypothesis of Proposition 7.5 is satisfied for a certain choice of λ, which we will denote by λ_s. As a consequence, we will get an estimate on $m_Y(F_{s,\lambda_s})$ in Theorem 9.18.

We set

$$I_Z(y) := \{v \in V - \{0\} : \|v\| < D_1^{-1}\omega(y)^{-1},\ y\exp(v) \in Z\}$$

for $D_1 > 1$ as given in Lemma 8.13.

Definition 9.1 (Margulis function)**.** (1) Define $f_s := f_{s,Y,Z} : Y_0 \to (0, \infty)$ by

$$f_s(y) := \begin{cases} \sum_{v \in I_Z(y)} \|v\|^{-s} & \text{if } I_Z(y) \ne \emptyset \\ \omega(y)^s & \text{otherwise.} \end{cases}$$

(2) For $\lambda \ge 1$, define $F_{s,\lambda} = F_{s,\lambda,Y,Z} : Y_0 \to (0, \infty)$ as follows:

$$F_{s,\lambda}(y) = f_s(y) + \lambda\, \omega(y)^s. \tag{9.2}$$

Note that for all $y \in Y_0$

$$\omega(y)^s \le f_s(y) < \infty. \tag{9.3}$$

Since Y and Z are closed orbits, both f_s and $F_{s,\lambda}$ are locally bounded. Moreover, they are also Borel functions. Indeed, ω^s is continuous on Y_0, and f_s is continuous on the open subset $\{y \in Y_0 : I_Z(y) \neq \emptyset\}$ as well as on its complement.

In this section, we specify choices of parameters t_s and λ_s so that the average $\mathsf{A}_{t_s} F_{s,\lambda_s}$ satisfies the hypothesis of Proposition 7.5 with controlled size of the additive term (Lemma 9.14).

Notation 9.4 (Parameters)**.** (1) *For $0 < c < 1$, define $t(c,s) > 0$ by*

$$\frac{b_0 b_1 \mathsf{p}_Y^{\delta_Y} e^{-(\delta_Y - s)t(c,s)/4}}{(\delta_Y - s)} = c$$

where b_0 and b_1 are given in Lemma 5.13.

(2) *For $0 < c < 1$ and $t > 0$, define $\lambda(t,c,s) > 0$ by*

$$\lambda(t,c,s) := \left(2c_0 D_1 p_Y^6 \tau_Z\right) \frac{e^{2ts}}{c}$$

where c_0 is given by (8.13)*.*

As it is evident from the above, the definition of $t(c,s)$ is motivated by the linear algebra lemma 5.13. Indeed, for any vector $v \in e_1 G$ and $t \geq t(c,s)$, we have we have

$$\sup_{1 \leq \rho \leq 2} \frac{1}{\mu_y[-\rho,\rho]} \int_{-\rho}^{\rho} \frac{1}{\|vu_r a_t\|^s} d\mu_y(r) \leq c\|v\|^{-s}. \tag{9.5}$$

The choice of $\lambda(t,c,s)$ is to control the additive difference between $f_s(yu_r a_t)$ and $\sum_{v \in I_Z(y)} \|vu_r a_t\|^{-s}$ uniformly over all $r \in [-1,1]$ such that $yu_r \in Y_0$, so that we will get:

$$\mathsf{A}_t f_s(y) \leq c \cdot f_s(y) + \tfrac{\lambda(t,c,s)c}{2} \omega(y)^s$$

(see Lemma 9.11, (9.15) and (9.16)).

Markov operator for the height function. In this subsection, we use notation from section 6.

It will be convenient to introduce the following notation:

Notation 9.6. Let $Q \subset G$ be a compact subset.

(1) Let $d_Q \geq 1$ be the infimum of all $d \geq 1$ such that for all $g \in Q$ and $v \in \mathbb{R}^4$,

$$d^{-1}\|v\| \leq \|vg\| \leq d\|v\|. \tag{9.7}$$

Note that $d_Q \asymp \max_{g \in Q} \|g\|$, up to an absolute multiplicative constant.

(2) We also define $c_Q \geq 1$ to be the infimum of all $c \geq 1$ such that for any $x \in X_0$, $g \in Q$ with $xg \in X_0$, and for all $1 \leq i \leq \ell$

$$c^{-1}\omega_i(x) \leq \omega_i(xg) \leq c\,\omega_i(x). \tag{9.8}$$

We note that $c_Q \asymp \max_{g \in Q} \|g\|$ up to an absolute multiplicative constant.

Lemma 9.9. *For any $0 < c \le 1/2$ and $t \ge t(c,s)$, there exists $D_2 \asymp e^{2t}$ so that for all $y \in Y_0$ and $1 \le \rho \le 2$,*

$$\mathsf{A}_{t,\rho}\omega(y)^s \le c \cdot \omega(y)^s + D_2.$$

Proof. Let $t \ge t(c,s)$. We compare $\omega(yu_r a_t)$ and $\omega(y)$ for $r \in [-2,2]$. Setting

$$Q := \{a_\tau u_r : |r| \le 2, |\tau| \le t\},$$

we have $c_Q \asymp e^t$.

Let η_0 be as in Lemma 6.5. Fix $0 < \eta_X \le \min\{\varepsilon_X, \eta_0\}$ so that

$$\eta_X \asymp \varepsilon_X \quad \text{and} \quad \eta_X^{-1} \ge \sup_{y \in X_{\varepsilon_X} \cap Y_0} \omega(y);$$

We consider two cases.

Case 1: $\omega(y) \le 2c_Q/\eta_X$. In this case, for $h \in Q$ with $yh \in Y_0$,

$$\omega(yh) \le 2c_Q^2/\eta_X.$$

Hence, the claim in this case follows if we choose $D_2 = 2c_Q^2/\eta_X \asymp e^{2t}$.

Case 2: $\omega(y) > 2c_Q/\eta_X$. By the definition of ω, there exists $1 \le i \le \ell$ such that

$$\omega_i(y) > 2c_Q/\eta_X, \quad \text{and hence} \quad y \in \mathfrak{h}_i.$$

By the definition of c_Q, see (9.8), we have

$$\omega_i(yh) > 2/\eta_X, \quad \text{and hence} \quad yh \in \mathfrak{h}_i$$

for all $h \in Q$ with $yh \in Y_0$. Choose $g_0 \in G$ so that $y = [g_0]$. In view of Lemma 6.5, see in particular (6.6), and since $\eta_X \le \eta_0$ there exists $\gamma \in \Gamma$ such that simultaneously for all $h \in Q$ with $yh \in Y_0$,

$$\omega(yh) = \omega_i(yh) = \|v_i \gamma g_0 h\|^{-1}.$$

Since $v_i = e_1 g_i^{-1} \in e_1 G$ (see (6.1)), we may apply Lemma 5.13 (linear algebra lemma II) and deduce:

$$\begin{aligned}\mathsf{A}_{t,\rho}\omega(y)^s &= \frac{1}{\mu_y([-\rho,\rho])}\int_{-\rho}^{\rho} \frac{1}{\|v_i\gamma u_r a_t\|^s} d\mu_y(r)\\ &\le \frac{b_0 b_1 \mathsf{p}_Y^{\delta_Y} e^{-(\delta_Y - s)t/4}}{(\delta_Y - s)} \|v_i\gamma\|^{-s} \le c \cdot \omega(y)^s;\end{aligned}$$

in the last inequality we used the fact that $t \ge t(c,s)$. The proof is now complete. □

Log-continuity of $F_{s,\lambda}$. The following log-continuity lemma with a control on the multiplicative constant σ is the first hypothesis in Proposition 7.5.

Lemma 9.10 (Log-continuity lemma)**.** *There exists $2 \le \sigma \ll \mathsf{p}_Y^8$ so that the following holds: for every $\lambda \ge \tau_Z$, we have*

$$\sigma^{-1} F_{s,\lambda}(y) \le F_{s,\lambda}(yh) \le \sigma F_{s,\lambda}(y)$$

for all $y \in Y_0$ and all $h \in B_H(2)$ so that $yh \in Y_0$.

Let c_0 be as in Lemma 8.13. Recall from Theorem 3.3 that $\tau_Z \geq \varepsilon_X^2$, replacing c_0 by its multiple (which we continue to denote by c_0) if necessary we assume that $c_0\tau_Z \geq 1$.

We first obtain the following estimate for f on nearby points:

Lemma 9.11. *Let $Q \subset H$ be a compact subset. For any $y \in Y_0$ and $h \in Q$ such that $yh \in Y_0$, we have*

$$f_s(yh) \leq \sum_{v \in I_Z(y)} \|vh\|^{-s} + \left(c_0 c_Q d_Q D_1 \mathsf{p}_Y^6 \tau_Z\right) \omega(y)^s$$

where c_0 is as above and the sum is understood as 0 when $I_Z(y) = \emptyset$.

Proof. Let $y \in Y_0$ and $h \in Q$ with $yh \in Y_0$. If $I_Z(yh) = \emptyset$, then by (9.8), we have

$$f_s(yh) = \omega(yh)^s \leq c_Q^s \omega(y)^s$$

proving the claim; recall that $c_0\tau_Z \geq 1$.

Now suppose that $I_Z(yh) \neq \emptyset$. Setting

$$\varepsilon := (d_Q D_1 \omega(y))^{-1},$$

we write

$$f_s(yh) = \sum_{v \in I_Z(yh), \|v\| < \varepsilon} \|v\|^{-s} + \sum_{v \in I_Z(yh), \|v\| \geq \varepsilon} \|v\|^{-s}. \tag{9.12}$$

Since $\#I_Z(yh) \leq c_0 \mathsf{p}_Y^6 \tau_Z$ by Lemma 8.13, we have

$$\sum_{v \in I_Z(yh), \|v\| \geq \varepsilon} \|v\|^{-s} \leq \left(c_0 \mathsf{p}_Y^6 \tau_Z\right)\varepsilon^{-s} \leq \left(c_0 d_Q D_1 \mathsf{p}_Y^6 \tau_Z\right)\omega(y)^s. \tag{9.13}$$

Thus, if there is no $v \in I_Z(yh)$ with $\|v\| \leq \varepsilon$, then the lemma follows from (9.12).

If $v \in I_Z(yh)$ satisfies $\|v\| < \varepsilon$, then

$$\|vh^{-1}\| \leq d_Q \varepsilon = D_1^{-1}\omega(y)^{-1};$$

in particular, $vh^{-1} \in I_Z(y)$. Therefore, by setting $v' = vh^{-1}$,

$$\sum_{v \in I_Z(yh), \|v\| < \varepsilon} \|v\|^{-s} \leq \sum_{v' \in I_Z(y)} \|v'h\|^{-s}.$$

Together with (9.13), this finishes the proof. $\square$

Proof of Lemma 9.10. Since $B_H(2)^{-1} = B_H(2)$, it suffices to show the inequality $\leq$. By Lemma 9.11, applied with $Q = B_H(2)$, $c := c_{B_H(2)}$ and $d := d_{B_H(2)}$, we have that for all $h \in B_H(1)$ with $yh \in Y_0$, we have

$$\begin{aligned} f_s(yh) &\leq \sum_{v \in I_Z(y)} \|vh\|^{-s} + \left(c_0 c d D_1 \mathsf{p}_Y^6 \tau_Z\right) \omega(y)^s \\ &\leq d \sum_{v \in I_Z(y)} \|v\|^{-s} + c_0 c d D_1 \mathsf{p}_Y^6 \tau_Z \omega(y)^s. \end{aligned}$$

where we used the definition of d.

Recall from Theorem 3.3 that $\varepsilon_X^2 \le \tau_Z \le \lambda$ and that $D_1 \ll \mathsf{p}_Y^2$.

If $I_Z(y) = \emptyset$, then

$$\begin{aligned} F_{s,\lambda}(yh) &\ll \mathsf{p}_Y^8 \tau_Z \omega(y)^s + \lambda\omega(y)^s \ll \mathsf{p}_Y^8 \lambda \omega(y)^s \\ &\ll \mathsf{p}_Y^8 (f_s(y) + \lambda\omega(y)^s) \ll \mathsf{p}_Y^8 F_{s,\lambda}(y). \end{aligned}$$

If $I_Z(y) \neq \emptyset$, then

$$\begin{aligned} F_{s,\lambda}(yh) &\le d \cdot f_s(y) + c_0 c d D_1 \mathsf{p}_Y^6 \tau_Z \omega(y)^s + \lambda\omega(yh)^s \\ &\ll f_s(y) + \mathsf{p}_Y^8 \lambda\omega(y)^s \ll \mathsf{p}_Y^8 F_{s,\lambda}(y). \end{aligned}$$

This finishes the upper bound. The lower bound can be obtained similarly.

Main inequality. We will apply the following lemma to obtain the second hypothesis of Proposition 7.5 for $c := (8\sigma \mathsf{p}_Y^{\delta_Y})^{-1} < 1/2$.

Lemma 9.14 (Main inequality)**.** *Let $0 < c \le 1/2$. For $t \ge t(c/2, s)$ and $\lambda = \lambda(t, c, s)$, we have the following: for any $y \in Y_0$ and $1 \le \rho \le 2$, we have*

$$\mathsf{A}_{t,\rho} F_{s,\lambda}(y) \le c\, F_{s,\lambda}(y) + \lambda D_2$$

where $D_2 \ll e^{2t}$ is as in Lemma 9.9.

Proof. The following argument is based on comparing the values of $f_s(yu_r a_t)$ and $f_s(y)$ for $r \in [-2, 2]$ such that $yu_r a_t \in Y_0$.

Let $Q := \{a_\tau u_r : |r| \le 2, |\tau| \le t\}$. Then

$$c_Q \asymp e^t \quad \text{and} \quad d_Q \asymp e^t$$

where c_Q and d_Q are as in (9.6). Hence, by Lemma 9.11, we have that for any $|r| \le 2$ such that $yu_r a_t \in Y_0$,

$$f_s(yu_r a_t) \le \sum_{v \in I_Z(y)} \|v u_r a_t\|^{-s} + c_0 D_1 \mathsf{p}_Y^6 \tau_Z \omega(y)^s e^{2ts} \tag{9.15}$$

where c_0 is as in Lemma 9.11.

By averaging (9.15) over $[-\rho, \rho]$ with respect to μ_y, and applying (9.5), we get

$$\begin{aligned} \mathsf{A}_{t,\rho} f_s(y) &\le c \cdot f_s(y) + c_0 D_1 \mathsf{p}_Y^6 \tau_Z \omega(y)^s e^{2ts} \\ &\le c \cdot f_s(y) + \tfrac{\lambda c}{2} \omega(y)^s. \end{aligned} \tag{9.16}$$

Then by Lemma 9.9 and (9.16), we have

$$\begin{aligned} \mathsf{A}_{t,\rho} F_{s,\lambda}(y) &= \mathsf{A}_{t,\rho} f_s(y) + \mathsf{A}_{t,\rho} \lambda \omega(y)^s \\ &\le c \cdot f_s(y) + \tfrac{c\lambda}{2}\omega(y)^s + \tfrac{c\lambda}{2}\omega(y)^s + \lambda D_2 \\ &= c \cdot F_{s,\lambda}(y) + \lambda D_2. \end{aligned}$$

□

By Theorem 4.8, we have $\mathsf{s}_Y \asymp \mathsf{p}_Y$. For the sake of simplicity of notation, we put

$$\alpha_{Y,s} := \left(\frac{\mathsf{s}_Y}{\delta_Y - s}\right)^{1/(\delta_Y - s)} \asymp \left(\frac{\mathsf{p}_Y}{\delta_Y - s}\right)^{1/(\delta_Y - s)}. \tag{9.17}$$

We are now in a position to apply Proposition 7.5 to get the following estimate:

Theorem 9.18 (Margulis function on average)**.** *There exists $\lambda_s > 1$ such that*

$$m_Y(F_{s,\lambda_s}) \ll \alpha_{Y,s}^{\star}\tau_Z.$$

Proof. Let $1 \le \sigma \ll \mathsf{p}_Y^8$ be given by Lemma 9.10. Let $c := (8\sigma\mathsf{p}_Y^{\delta_Y})^{-1} < 1/2$, $t_s := t(c,s)$ and $\lambda_s := \lambda(t_s, c, s)$ be given by (9.4). Then in view of Lemmas 9.10 and 9.14, F_{s,λ_s} satisfies the conditions of Proposition 7.5 with $t = t_s$ and $D_0 = \lambda_s D_2$, where $D_2 \ll e^{2t_s}$ is given in Lemma 9.9. Therefore

$$m_Y(F_{s,\lambda_s}) \le 64\lambda_s \mathsf{p}_Y^{\delta_Y} D_2. \tag{9.19}$$

Since

$$e^{(\delta_Y - s)t_s} = \tfrac{(8\sigma b_0 b_1 \mathsf{p}_Y^{2\delta_Y})^4}{(\delta_Y - s)^4} \ll \left(\tfrac{\mathsf{p}_Y}{\delta_Y - s}\right)^{\star} \quad \text{and} \quad \lambda_s = \left(2c_0 D_1 p_Y^6 \tau_Z\right)\tfrac{e^{2t_s s}}{c},$$

we get

$$\lambda_s \mathsf{p}_Y^{\delta_Y} D_2 \ll \mathsf{p}_Y^{\star} e^{4t_s}\tau_Z \ll \alpha_{Y,s}^{\star}\tau_Z.$$

Combining this with (9.19) finishes the proof. $\square$

10. Quantitative isolation of a closed orbit

In this section, we deduce Theorem 1.5 from Theorem 9.18. Let Y, Z be non-elementary closed H-orbits in X. We allow the case $Y = Z$ as well. Let $\frac{\delta_Y}{3} \le s < \delta_Y$.

Recall the definitions of $f_s = f_{s,Y,Z}$ and $F_{s,\lambda} = F_{s,\lambda,Y,Z}$ from Definition 9.1. Let λ_s be given by Theorem 9.18. Using the log-continuity lemma for F_{s,λ_s} (Lemma 9.10), we first deduce the following estimate:

Proposition 10.1. *For any $0 < \varepsilon < \varepsilon_X$ and $y \in Y_0 \cap X_\varepsilon$, we have*

$$f_{s,Y,Z}(y) \le F_{s,\lambda_s}(y) \ll \frac{\alpha_{Y,s}^{\star}\tau_Z}{m_Y(B(y,\varepsilon))}.$$

Proof. Let $y \in Y_0 \cap X_\varepsilon$. Then $\operatorname{inj}(y) \ge \varepsilon$ and hence $yB_H(\varepsilon) = B(y,\varepsilon)$. For all $h \in B_H(\varepsilon_X)$, $F_{s,\lambda_s}(y) \le \sigma F_{s,\lambda_s}(yh)$ for some constant $\sigma \ll \mathsf{p}_Y^6$ by Lemma 9.10. By applying Theorem 9.18, we get

$$F_{s,\lambda_s}(y) \le \frac{\sigma\int_{x\in yB_H(\varepsilon)} F_{s,\lambda_s}(x)dm_Y(x)}{m_Y(B(y,\varepsilon))} \le \frac{\sigma\cdot m_Y(F_{s,\lambda_s})}{m_Y(B(y,\varepsilon))} \ll \frac{\alpha_{Y,s}^{\star}\tau_Z}{m_Y(B(y,\varepsilon))}.$$

$\square$

Recall from (6.8) that for all $x \in X_0$,

$$\frac{1}{2\alpha} \cdot \operatorname{inj}(x) \le \omega(x)^{-1} \le \frac{\alpha}{2} \cdot \operatorname{inj}(x). \tag{10.2}$$

Using the next lemma, we will be able to use the estimate for $f_{s,Y,Z}$ obtained in Proposition 10.1 to deduce a lower bound for $d(y, Z)$.

Lemma 10.3. (1) *Let $y \in Y_0$ and $z \in Z - B_Y(y, \operatorname{inj}(y))$. If $d(y, z) \le \frac{1}{2\alpha c_1 D_1} \operatorname{inj}(y)$, then*

$$d(y, z)^{-s} \le c_1 f_{s,Y,Z}(y)$$

where $c_1 \ge 1$ is as in (8.1).

(2) *If $Y \ne Z$, then for any $y \in Y_0$,*

$$d(y, Z)^{-s} \ll \mathsf{p}_Y^2 f_{s,Y,Z}(y).$$

Proof. As Z is closed and $d(y, z) \le \frac{1}{2\alpha c_1 D_1} \operatorname{inj}(y) < \frac{1}{2} \operatorname{inj}(y)$, the hypothesis $z \in Z - B_Y(y, \operatorname{inj}(y))$ and the choice of c_1 implies that z is of the form $y \exp(v) \exp(v')$ with $v \in i\mathfrak{sl}_2(\mathbb{R}) - \{0\}$ and $v' \in \mathfrak{sl}_2(\mathbb{R})$.

In particular $y \exp(v) = z \exp(-v') \in Z$. Moreover, by (8.1),

$$\|v\| \le \|v + v'\| \le c_1 d(y, z) \le D_1^{-1} \operatorname{inj}(y)/(2\alpha) \le (D_1 \omega(y))^{-1}.$$

It follows that $v \in I_Z(y, D_1)$. Therefore

$$d(y, z)^{-s} \le c_1^s \|v\|^{-s} \le c_1 \|v\|^{-s} \le c_1 f_s(y), \tag{10.4}$$

proving (1).

We now turn to the proof of (2); suppose thus that $Y \ne Z$. Then there exists $z \in Z$ such that $d(y, Z) = d(y, z)$. In view of (1), it suffices to consider the case when $d(y, z) > \frac{1}{2\alpha c_1 D_1} \operatorname{inj}(y)$.

Since $s \le 1$, $\omega(y)^s \le f_s(y)$, and $D_1 \ll \mathsf{p}_Y^2$, we get

$$d(y, z)^{-s} \le 2\alpha c_1 D_1 \operatorname{inj}(y)^{-s} \le 2\alpha^2 c_1 D_1 \omega(y)^s \ll \mathsf{p}_Y^2 f_{s,Y,Z}(y)$$

where we also used (10.2). The proof is complete. □

Theorem 1.5(1) is a special case of the following theorem:

Theorem 10.5 (Isolation in distance)**.** *For any $0 < \varepsilon < \varepsilon_X$, $y \in Y_0 \cap X_\varepsilon$, and $z \in Z$, at least one of the following holds:*

(1) *$z \in B_Y(y, \varepsilon) = y B_H(e, \varepsilon)$, or*

(2) *$d(y, z) \gg \alpha_{Y,s}^{-\star/s} m_Y(B(y, \varepsilon))^{1/s} {\tau_Z}^{-1/s}$, where $\alpha_{Y,s}$ is as given in* (9.17).

Proof. As $y \in X_\varepsilon$, $\operatorname{inj}(y) \ge \varepsilon$. Suppose that $z \notin B_Y(y, \varepsilon)$. We first observe that since $m_Y(B(y, \varepsilon))^{1/s} \ll \varepsilon$ and $\mathsf{p}_Y^{-2} \gg \alpha_{Y,s}^{-\star/s}$, we have

$$\frac{\varepsilon}{2\alpha c_1 D_1} \gg \mathsf{p}_Y^{-2} \varepsilon \gg \alpha_{Y,s}^{-\star/s} m_Y(B(y, \varepsilon))^{1/s}.$$

Therefore, if $d(y, z) \ge \frac{1}{2\alpha c_1 D_1} \varepsilon$, then (2) holds in view of the fact that $\tau_Z \ge \varepsilon_X^2$.

If $d(y,z) \le \frac{1}{2\alpha c_1 D_1}\varepsilon \le \frac{1}{2\alpha c_1 D_1}\,\mathrm{inj}(y)$, then by Lemma 10.3, $d(y,z)^{-s} \le c_1 f_s(y)$. Hence applying Proposition 10.1, we conclude

$$d(y,z)^{-s} \le c_1 f_s(y) \le c_1 \frac{\alpha^{\star}_{Y,s}\tau_Z}{m_Y(B(y,\varepsilon))}$$

which finishes the proof in this case as well. $\square$

The following theorem is Theorem 1.5(2):

Theorem 10.6 (Isolation in measure)**.** *Let $0 < \varepsilon \le \varepsilon_X$. Let $Y \ne Z$. We have*

$$m_Y\{y \in Y : d(y,Z) \le \varepsilon\} \ll \alpha^{\star}_{Y,s}\tau_Z \varepsilon^s.$$

Proof. Let λ_s be given by Theorem 9.18. By Lemma 10.3(2),

$$d(y,Z)^{-s} \le c f_{s,Y.Z}(y) \le C \cdot F_{s,\lambda_s}(y)$$

for some $1 < C \ll \mathsf{p}_Y^2$.

For $0 < \varepsilon < \varepsilon_X$, if we set

$$\Omega_\varepsilon := \{y \in Y_0 : F_{s,\lambda_s}(y) > C^{-1}\varepsilon^{-s}\},$$

then $\{y \in Y_0 : d(y,Z) \le \varepsilon\} \subset \Omega_\varepsilon$. On the other hand, we have

$$C^{-1}\varepsilon^{-s} m_Y(\Omega_\varepsilon) \le \int_{\Omega_\varepsilon} F_{s,\lambda_s}\, dm_Y \le m_Y(F_{s,\lambda_s}).$$

Since $m_Y(F_{s,\lambda_s}) \ll \alpha^{\star}_{Y,s}\tau_Z$ by Theorem 9.18, we get that

$$m_Y\{y \in Y_0 : d(y,Z) \le \varepsilon\} \le m_Y(\Omega_\varepsilon) \ll \alpha^{\star}_{Y,s}\tau_Z\varepsilon^s.$$

$\square$

Proof of Proposition 1.16. Let $F_s = F_{s,\lambda_s}$ be as in Theorem 9.18. Then F_s satisfies (1) in the proposition by Lemma 10.3. It satisfies (3) by Lemma 9.10.

Moreover, in view of Lemmas 9.10 and 9.14, F_s satisfies the conditions of Proposition 7.5. Hence, by Proposition 7.6, it also satisfies (2) in the proposition. $\square$

We remark that in both Theorems 10.5 and 10.6, the exponents $\star$ depend only on G, and the implied constants are respectively of the form $c\,\varepsilon_X^N$ and $c^{-1}\,\varepsilon_X^{-N}$ for some $c \le 1$ and $N \ge 1$ both depending only on G.

Number of properly immersed geodesic planes. When $\mathrm{Vol}(M) < \infty$, we record the following corollary of Theorem 10.5. Let $\mathcal{N}(T)$ denote the number of properly immersed totally geodesic planes P in M of area at most T.

We deduce the following upper bound from Theorem 10.5 using the pigeonhole principle:

Corollary 10.7. *Let* $\mathrm{Vol}(M) < \infty$. *There exists* $N \geq 1$ *(depending only on* G*) such that for any* $1/2 < s < 1$*, we have*

$$\mathcal{N}(T) \ll_s \mathrm{Vol}(M)\varepsilon_X^{-N} T^{\frac{6}{s}-1}$$

where the implied constant depends only on s.

Proof. We begin by recalling that $\alpha_{Y,s} = \alpha_s := (\frac{1}{1-s})^{1/(1-s)}$ for any closed H-orbit Y in X when $\mathrm{Vol}(M) < \infty$.

We obtain an upper bound for the number of closed H-orbits in X which yields the above result. The proof is based on applying Theorem 10.5.

If X is compact, let $\rho = 0.1\varepsilon_X$. If X is not compact, then the quantitative non-divergence of the action of U on X implies that there exists $\rho > 0$ so that for all $x \in X$ such that xU is not compact,

$$\frac{1}{T}\ell\{t \in [0,T] : xu_t \in X - X_\rho\} \leq 0.01$$

for all sufficiently large $T \gg 1$, e.g., see [9]. Moreover ρ can be taken to be $\asymp \varepsilon_X^k$ for some $k \geq 1$.

Since (Y, m_Y) is U-ergodic by the Moore's ergodicity theorem for every closed orbit $Y = xH$, the Birkhoff ergodic theorem says that for m_Y a.e. $y \in Y$,

$$\lim_{T\to\infty} \frac{1}{T}\ell\{t \in [0,T] : yu_t \in X - X_\rho\} = m_Y(X - X_\rho)$$

where ℓ denotes the Lebesgue measure on $\mathbb{R}$; therefore

$$m_Y(X - X_\rho) < 0.01. \tag{10.8}$$

For every $S > 0$ put

$$\mathcal{Y}(S) := \{xH : xH \text{ is closed and } S/2 < \mathrm{Vol}(xH) \leq S\}.$$

In view of the above choice of ρ, we have $\mathrm{Vol}(xH) \geq \rho^3 \gg 1$ for every closed orbit xH. Let $n_0 = \lfloor 3\log_2(\rho)\rfloor$ and for every $T > 1$, let $n_T = \lceil \log_2 T\rceil$. Then we have

$$\{xH : xH \text{ is closed and } \mathrm{vol}(xH) \leq T\} \subset \bigcup_{n_0}^{n_T} \mathcal{Y}(2^k).$$

Let $\eta \asymp \rho$ be so that the map $g \mapsto xg$ is injective for all $x \in X_\rho$ and all

$$g \in \mathrm{Box}(\eta) := \exp(B_{i\mathfrak{sl}_2(\mathbb{R})}(0,\eta))\exp(B_{\mathfrak{sl}_2(\mathbb{R})}(0,\eta)).$$

Fix some $1/2 < s < 1$ and some $z \in X$. We claim that

$$\#\big(\text{connected components of } \mathcal{Y}(2^k) \cap z.\mathrm{Box}(\eta)\big) \ll \alpha_s^{12/s} 2^{6k/s} \tag{10.9}$$

where the implied constant depends on ρ.

For any connected component C of $\mathcal{Y}(2^k) \cap z.\mathrm{Box}(\eta)$, there exists some $v \in i\mathfrak{sl}_2(\mathbb{R})$ so that

$$C = z\exp(v)\exp(B_{\mathfrak{sl}_2(\mathbb{R})}(0,\eta)).$$

Let us write $C = C_v$. Now in view of Theorem 10.5, for every two connected components $C_v \neq C_{v'}$, we have

$$\|v - v'\| \gg_\rho \alpha_s^{-4/s} 2^{-2k/s}. \tag{10.10}$$

Because $\dim(\mathfrak{r}) = 3$, the cardinality of an $\alpha_s^{-4/s}2^{-2k/s}$-separated set in $B_{i\mathfrak{sl}_2(\mathbb{R})}(0,\eta)$ is $\ll \alpha_s^{12/s}2^{6k/s}$, where the implied constant depends only on the choice of norm. The claim in (10.9) thus follows from (10.10).

Let $\{z_j.\mathrm{Box}(\eta) : 1 \leq j \leq R\}$ be a covering of X_ρ with sets of the form $z.\mathrm{Box}(\eta)$; we may find such a covering with $R = O(\mathrm{Vol}(X)\eta^{-6})$ the implied constant is absolute, see also the definition of c_1 in (8.1). Then we compute

$$\begin{aligned}
\mathcal{N}(2^k) &\leq 2^{-k+1} \sum_{\mathcal{Y}(2^k)} \mathrm{vol}(xH) && \text{by def. of } \mathcal{Y}(2^k)\\
&\ll 2^{-k} \sum_{j=1}^{M} \sum_{C_v \subset z_j.\mathrm{Box}(\eta)} \mathrm{vol}(C_v) && \text{by (10.8)}\\
&\ll \alpha_s^{12/s} \sum_{j=1}^{R} 2^{\frac{6k}{s}-k} && \text{by (10.9)}\\
&\ll \mathrm{Vol}(X)\alpha_s^{12/s}2^{\frac{6k}{s}-k} && \text{since } R = O(\mathrm{Vol}(X))
\end{aligned}$$

in the above we also used the fact that $\mathrm{vol}(C_v) \ll_\rho 1$.

Since $\rho \asymp \eta$ can be taken $\asymp \varepsilon_X^k$, we conclude that for some absolute constant $N_1, N_2 \geq 1$ and $c = c(s) \geq 1$,

$$\mathcal{N}(T) \leq c\ \mathrm{Vol}(X)\rho^{-N_1}\alpha_s^{12/s} \sum_{k=n_0}^{n_T} 2^{\frac{6k}{s}-k} \leq c\ \mathrm{Vol}(X)\varepsilon_X^{-N_2}T^{\frac{6}{s}-1}$$

which implies the claim (note here that $\mathrm{Vol}(X) = \mathrm{Vol}(M)$, since Γ is torsion-free.) $\square$

Remark 10.11. Let $\mathcal{N}_M(T)$ be the number of properly immersed geodesic planes of area at most T in a general geometrically finite manifold $M = \Gamma\backslash\mathbb{H}^3$. By [20] and [3], if $\mathrm{Vol}(M) = \infty$, there are only finitely many properly immersed geodesic planes of finite area (note that they are necessarily contained in the convex core of M); hence $\sup_T \mathcal{N}(T) < \infty$.

If Y is a closed H-orbit Y *of finite area* in $\Gamma\backslash G$, then $\mathsf{p}_Y \asymp \mathsf{s}_Y = 2$, $\tau_Y = \mathrm{Vol}(Y)$ and the non-divergence of the U-action as given in [6, Thm. 1.1] implies that (10.8) also holds in this setting.

In view of these, the proof of Corollary 10.7 works in the same way for the following: there exists $N \geq 1$ (depending only on G) such that for any $1/2 < s < 1$, we have

$$\mathcal{N}_M(T) \ll_s \mathrm{Vol}(\mathrm{core}M)\,\varepsilon_M^{-N}T^{\frac{6}{s}-1}$$

where the implied constant depends only on s. Note that this kind of upper bound is meaningful despite the finiteness result mentioned above, as the implied constant is independent of M.

11. Appendix: Proof of Theorem 1.1 in the compact case

In this section we present the proof of Theorem 1.1 when X is compact. As was mentioned in the introduction, this case is due to G. Margulis.

Let $Y \neq Z$ be two closed H-orbits in $X = \Gamma\backslash G$. Recall $\varepsilon_X = \min_{x\in X} \mathrm{inj}(x)$ where $\mathrm{inj}(x)$ is the injectivity radius measured in $\Gamma\backslash\mathbb{H}^3$.

Fix $0 < s < 1$, and define $f_s : Y \to [2,\infty)$ as follows: for any $y \in Y$,

$$f_s(y) = \begin{cases} \sum_{v\in I_Z(y)} \|v\|^{-s} & \text{if } I_Z(y) \neq \emptyset \\ \varepsilon_X^{-s} & \text{otherwise} \end{cases}$$

where

$$I_Z(y) = \{v \in i\mathfrak{sl}_2(\mathbb{R}) : 0 < \|v\| < \varepsilon_X,\ y\exp(v) \in Z\}.$$

Define $F_s = F_{s,Y,Z} : Y \to (0,\infty)$ as follows:

$$F_s(y) = f_s(y) + \mathrm{Vol}(Z)\varepsilon_X^{-s}.$$

Note that in the case at hand, F_s is a bounded Borel function on Y. We also note that in the case at hand ω, as defined in (6.3), is a bounded function on X (recall that $\omega = 2$ in this case), and hence F_s here and F_{s,λ_s} that we considered in the proof of Theorem 1.5 are essentially the same functions in this case.

We use the following special case of Lemma 5.6: for any $v \in i\mathfrak{sl}_2(\mathbb{R})$ with $\|v\| = 1$, $1/3 \le s < 1$ and $t > 0$, we have

$$\int_0^1 \frac{dr}{\|vu_ra_t\|^s} \le b_0 \frac{e^{(s-1)t/4}}{1-s} \tag{11.1}$$

where $vh = \mathrm{Ad}(h)(v)$ for all $h \in H$.

Remark 11.2. It is worth noting that the symmetric interval $[-1,1]$ was used in Lemma 5.6. We remark that this is necessary in the infinite volume setting; indeed the half interval $[0,1]$ may even be a null set for μ_y for some y, see (4.1) for the notation.

For a locally bounded function ψ on Y and $t > 0$, define

$$\mathsf{A}_t\psi(y) = \int_0^1 \psi(yu_ra_t)dr \quad \text{for } y \in Y. \tag{11.3}$$

Proposition 11.4. *Let $1/3 \le s < 1$. There exists $t = t(s) > 0$ such that for all $y \in Y$,*

$$\mathsf{A}_tF_s(y) \le \frac{1}{2}F_s(y) + c\,\varepsilon_X^{-4}\alpha_s^4\,\mathrm{Vol}(Z) \tag{11.5}$$

where $\alpha_s = (1-s)^{-1/(1-s)}$ and $c \ge 1$ is an absolute constant.

Proof. It suffices to show that $\mathsf{A}_t f_s(y) \le \frac{1}{2} f_s(y) + \alpha_s^4 \mathrm{Vol}(Z)$.

Let b_0 be as in (11.1), and let $t = t(s)$ be given by the equation

$$b_0 \frac{e^{(s-1)t/4}}{1-s} = 1/2.$$

We compare $f_s(yu_r a_t)$ and $f_s(y)$ for $r \in [0,1]$. Let $C_1 \asymp e^t$ be large enough so that $\|vh\| \le C_1\|v\|$ for all $v \in i\mathfrak{sl}_2(\mathbb{R})$ and all

$$h \in \{a_\tau u_r : |r| < 1, |\tau| \le t\}.$$

Let $v \in I_Z(yu_r a_t)$ be so that $\|v\| < \varepsilon_X/C_1$. Then $\|va_{-t}u_{-r}\| \le \varepsilon_X$; in particular, $va_{-t}u_{-r} \in I_Z(y)$.

In the following, if $I_Z(\cdot) = \emptyset$, the sum is interpreted as to equal to ε_X^{-s}. In view of the above observation and the definition of f_s, we have

$$\begin{aligned}
f_s(yu_r a_t) &= \sum_{v \in I_Z(yu_r a_t)} \|v\|^{-s} \\
&= \sum_{v \in I_Z(yu_r a_t), \|v\| < \varepsilon_X/C_1} \|v\|^{-s} + \sum_{v \in I_Z(yu_r a_t), \|v\| \ge \varepsilon_X/C_1} \|v\|^{-s} \\
&\le \sum_{v \in I_Z(y)} \|vu_r a_t\|^{-s} + \sum_{v \in I_Z(yu_r a_t), \|v\| \ge \varepsilon_X/C_1} \|v\|^{-s}. \qquad (11.6)
\end{aligned}$$

Moreover, note that $\#I_Z(y) \ll \varepsilon_X^{-3}\mathrm{Vol}(Z)$ (see the proof of Lemma 8.13). Hence,

$$\sum\nolimits_{\|v\| \ge \varepsilon_X/C_1} \|v\|^{-s} \ll C_1^s \varepsilon_X^{-4} \mathrm{Vol}(Z) \ll \varepsilon_X^{-4} e^{st} \mathrm{Vol}(Z). \qquad (11.7)$$

We now average (11.6) over $[0,1]$. Then using (11.7) and (11.1) we get

$$\mathsf{A}_t f_s(y) \le \tfrac{1}{2} f_s(y) + O(e^{st}\mathrm{Vol}(Z)).$$

As $(1-s)^{-1/(1-s)} \asymp e^{st/4}$, this proves (11.5). $\square$

Let m_Y be the H-invariant probability measure on Y:

Corollary 11.8. *We have*

$$m_Y(F_s) \le c\, \varepsilon_X^{-4} \alpha_s^4 \,\mathrm{Vol}(Z)$$

where $c \ge 1$ is an absolute constant.

Proof. Since m_Y is an H-invariant probability measure, $m_Y(\mathsf{A}_t f_s) = m_Y(f_s)$. Hence the claim follows by integrating (11.5) with respect to m_Y. $\square$

Proof of Theorem 1.1. There exists $\sigma > 0$ such that for any $h \in B_H(\varepsilon_X)$ and $y \in Y$, $F_s(y) \le \sigma F_s(yh)$ (cf. Lemma 9.10); $B_H(\varepsilon_X)$ denotes the ε_X-ball centered at the identity in H.

Hence, using Corollary 11.8, we deduce

$$\begin{aligned} f_s(y) \le F_s(y) &\le \frac{\sigma \int_{B_H(\varepsilon_X)} F_s(yh) dm_Y(yh)}{m_Y(B(y,\varepsilon_X))} \\ &\le \frac{\sigma \cdot m_Y(F_s)}{m_Y(B(y,\varepsilon_X))} \ll \alpha_s^4 \varepsilon_X^{-7} \operatorname{Vol}(Y) \operatorname{Vol}(Z) \end{aligned}$$

with an absolute implied constant. Since $d(y,Z)^{-s} \le c_1 f_s(y)$ for an absolute constant $c_1 \ge 1$ (see (10.4)), we have

$$d(y,Z) \gg \alpha_s^{-4/s} \varepsilon_X^{7/s} \operatorname{Vol}(Z)^{-1/s} \operatorname{Vol}(Y)^{-1/s}. \tag{11.9}$$

This shows Theorem 1.1(1). By Corollary 11.8 and the Chebyshev inequality, we get

$$m_Y\{y \in Y : d(y,Z) \le \varepsilon\} \le m_Y\{y \in Y : F_s(y) \ge c_1^{-1}\varepsilon^{-s}\} \le c_1 m_Y(F_s)\varepsilon^s.$$

Therefore

$$m_Y\{y \in Y : d(y,Z) \le \varepsilon\} \le c_1 c \varepsilon^s \varepsilon_X^{-4} \alpha_s^4 \operatorname{Vol}(Z), \tag{11.10}$$

which implies Theorem 1.1(2). □

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