Gelfand-Kirillov dimension

Let A be a f.g. k-algebra (not necessarily commutative).

Assume V is a k-subspace of A s.t.

(1) $\dim_k V < \infty$, (2) $1 \in V$ (3) $A = \bigcup_{n=1}^{\infty} V^{(n)}$, where

$V^{(n)} = V \cdot \dots \cdot V := \{ v_1 \cdot \dots \cdot v_n \mid v_i \in V \}$.

Let $D_V(n) := \dim_k V^{(n)}$ and

$$\mathrm{GKdim}(A) := \limsup_{n \to \infty} \frac{\log D_V(n)}{\log n}. \quad (\mathrm{I})$$

(a) Prove that that the value of (I) is independent of the choice of V, i.e if V and W satisfy (1), (2) and (3) then

$$\limsup_{n \to \infty} \frac{\log D_V(n)}{\log n} = \limsup_{n \to \infty} \frac{\log D_W(n)}{\log n}.$$

[**Remark**. $\mathrm{GKdim}(A) < \beta \iff$ for almost all n, $D_V(n) \le n^{\beta}$.]

(b) Prove the basic properties of Gelfand-Kirillov dimension

(1) $A \subseteq B$ f.g. k-algebras $\Rightarrow \mathrm{GKdim}(A) \le \mathrm{GKdim}(B)$.

(2) $B \to A \to 0$ " " $\Rightarrow \mathrm{GKdim}(A) \le \mathrm{GKdim}(B)$.

(3) $\mathrm{GKdim}(A[x]) = \mathrm{GKdim}(A) + 1$.

(c) Assume $A \subseteq B$ are two f.g. k-algebras and B is a f.g. A-module. Prove that $\mathrm{GKdim}(A) = \mathrm{GKdim}(B)$.

From this point on all the algebras are assumed to be f.g. commutative k-algebras.

(d) Using the fact that $\mathrm{Kdim}(k[x_1, \ldots, x_d]) = d$ (the Krull dimension), prove that $\mathrm{GKdim}(A) = \mathrm{Kdim}(A)$.

Hilbert-Poincare series.

1. (a) Let $\{d_n\}_{n=0}^{\infty}$ be a sequence of integers and let

$$P(t) := \sum_{n=0}^{\infty} d_n t^n \in \mathbb{Z}[\![t]\!]$$

be its generating function. Prove that $P(t)$ is a rational function, i.e. $\exists\, f(t), \underset{\neq 0}{g(t)} \in \mathbb{Z}[t]$, $g(t)\, P(t) = f(t)$, if and only if $\{d_n\}$ satisfies a linear recursive equation, i.e. $\exists\, (a_0, \ldots, a_k) \in \mathbb{Z}^k \setminus \{(0, \ldots, 0)\}$: $a_0 d_n + a_1 d_{n-1} + \cdots + a_k d_{n-k} = 0$ for any large enough integer n.

(b) What is the generating function of $d_n = \binom{n+d}{d}$?

[Here, $d \in \mathbb{Z}^{\geq 0}$ and $\binom{m}{i} = \frac{m(m-1)\cdots(m-i+1)}{i!}$.]

[Hint: $\binom{n+d}{d} = (-1)^n \binom{-d-1}{n}$.]

(c) Let $f \in \mathbb{Q}[t]$ s.t. (1) $\deg f \le D$

(2) $f(n) \in \mathbb{Z}$ for large enough integers.

Prove that $\exists!\, a_0, \dots, a_D \in \mathbb{Z}$ s.t.

$$f(t) = \sum_{i=0}^{D} a_i \binom{t+i}{i}$$

where $\binom{t+i}{i} = \dfrac{(t+i)(t+i-1)\cdots(t+1)}{i!}$.

[Hint: (1) $\deg\left(f(t+1) - f(t)\right) < \deg(f)$

(2) $\binom{t+1+d}{d} - \binom{t+d}{d} = \binom{t+d}{d-1}$.]

(d) Let $\{d_n\}$ be a sequence of integers. Prove that $d_n = f(n)$ for large enough n and a polynomial f of degree D if and only if $P(t) = \dfrac{h(t)}{(1-t)^{D+1}}$ where $h(t) \in \mathbb{Z}[t]$.

2. Let A be a f.g. k-algebra and $1 \in V \subseteq A$ be a finite-dimensional subspace such that $A = \bigcup_{i=0}^{\infty} V^{(i)}$ where $V^{(0)} = k$ and $V^{(i)} = \{v_1 \cdots v_i \mid v_j \in V\}$.

Let $P_{A,V}(t) := 1 + \sum_{n=1}^{\infty} \dim\left(V^{(n)}/V^{(n-1)}\right) t^n$.

(a) Find $P_{A,\bar{V}}(t)$ where $A = k[x_1, \dots, x_d]$ and

$\bar{V} = k\cdot 1 \oplus kx_1 \oplus \cdots \oplus kx_d$.

(b) Let $f \in k[x_1, \dots, x_d]$ be a non-constant homogeneous polynomial. Let $A = k[x_1, \dots, x_d]/\langle f \rangle$ and $\bar{V} = k\bar{1} + \sum_{i=1}^{d} k\bar{x}_i$. Show that there is a polynomial $g_f(t) \in \mathbb{Q}[t]$ of degree $d-1$ s.t. $g_f(n) = \dim(\bar{V}^{(n)})$ for large enough n.

(c) Find the leading coefficient of $g_f(t)$ from part (b).

3. Let $A = A_0 \oplus \bigoplus_{i=1}^{\infty} A_i$ and assume

(1) $A_0 = k$, (2) $A_i \cdot A_j \subseteq A_{i+j}$, (3) $A = k[A_1]$

(4) $\dim A_1 < \infty$.

Also assume that $\sum_{n=0}^{\infty} \dim(A_n) t^n = \dfrac{h(t)}{(1-t)^D}$ for some positive integer D and $h(t) \in \mathbb{Z}[t]$ s.t. $h(1) \neq 0$.

(a) Prove that $\mathrm{GKdim}(A) = D$.

(b) Let $f \in A_m$ $(m \neq 0)$ be a non-zero divisor. Let $\bar{A} = A/\langle f \rangle$ and $\bar{A}_n = A_n/(\langle f \rangle \cap A_n)$. Prove that

$$\sum_{n=0}^{\infty} \dim(\bar{A}_n) t^n = \frac{g(t)}{(1-t)^{D-1}}$$

where $g(t) \in \mathbb{Z}[t]$ and $g(1) \neq 0$. And conclude that

$$\operatorname{GKdim}(A/\langle f \rangle) = \operatorname{GKdim}(A) - 1.$$

4. Let k be an algebraically closed field and $f_1, \dots, f_{n-1} \in k[x_1, \dots, x_n]$.

Let $V(f_1, \dots, f_{n-1}) := \{ \vec{x} \in k^n \mid f_1(\vec{x}) = \dots = f_{n-1}(\vec{x}) = 0 \}$.

Prove that, if $V(f_1, \dots, f_{n-1}) \neq \emptyset$, then $|V(f_1, \dots, f_{n-1})| = \infty$.

5. Let A be a Noetherian domain. Prove that A is a UFD if and only if every prime ideal of height one is principal.