1. Let k be a finite Galois extension of $\mathbb{Q}$.

Let $\mathcal{O}_k$ be the integral closure of $\mathbb{Z}$ in k.

(a) Prove that $k/\mathcal{O}_k$ is a torsion group (with addition).

In particular,

$$\mathcal{C} := \{\mathcal{B} \subseteq \mathcal{O}_k \mid \mathcal{B} \text{ is a } \mathbb{Q}\text{-basis of } k\} \neq \emptyset.$$

(b) For any $\mathcal{B} \in \mathcal{C}$, let $\Delta(\mathcal{B}) := \det(\sigma_i(x_j))^2$ where $\mathrm{Gal}(k/\mathbb{Q}) = \{\sigma_1, \ldots, \sigma_d\}$. Prove that

$$\Delta(\mathcal{B}) \in \mathbb{Z} \setminus \{0\}.$$

(c) Let $\mathcal{B}_0 \in \mathcal{C}$ such that

$$|\Delta(\mathcal{B}_0)| = \min\{|\Delta(\mathcal{B})| \mid \mathcal{B} \in \mathcal{C}\}.$$

(Notice that because of (b), there is such a $\mathcal{B}_0$.)

Prove that $\mathcal{O}_k = \bigoplus_{x \in \mathcal{B}_0} \mathbb{Z}x$.

(d) Suppose $|\Delta(\mathcal{B}_1)| = |\Delta(\mathcal{B}_2)| = \min\{|\Delta(\mathcal{B})| \mid \mathcal{B} \in \mathcal{C}\}$.

$\Rightarrow \Delta(\mathcal{B}_1) = \Delta(\mathcal{B}_2)$

(The above common value just depends on k and is called the discriminant of $\underline{k}$.)

2. Let A be a f.g. k-algebra where k is a field.

An <u>A-sequence</u> is a sequence

$$x_1, \ldots, x_n \in A$$

such that, for any i, $x_i \notin D\left(A/\langle x_1, \ldots, x_{i-1}\rangle\right)$.

(a) Suppose that $x_1, \ldots, x_n$ is an A-sequence. Prove that for any $\mathfrak{p} \in \mathrm{Min}(\langle x_1, \ldots, x_n\rangle)$ (minimal prime ideal containing $\langle x_1, \ldots, x_n\rangle$.) We have $\mathrm{ht}(\mathfrak{p}) = n$.

(b) For an ideal $\mathfrak{a}$, let

$$\mathrm{grade}(\mathfrak{a}) = \max\{n \mid \exists \text{ an } A\text{-seq. } \{x_i\}_{i=1}^{n} \subseteq \mathfrak{a}\}.$$

Prove that $\mathrm{grade}(\mathfrak{a}) \leq \mathrm{ht}(\mathfrak{a})$.

3. Let me recall that $\mathrm{Pic}(A)$ is the set of f.g. projective A-mod M (up to an isomorphism) s.t. $\forall \mathfrak{p} \in \mathrm{Spec}(A)$,
$M_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$.

ⓐ Prove that $M_1, M_2 \in Pic(A) \Rightarrow M_1 \otimes_A M_2 \in Pic(A)$.

ⓑ Prove that $M \in Pic(A) \Rightarrow M^* = Hom_A(M, A) \in Pic(A)$

ⓒ $M^* \otimes_A M \simeq A$.

4. Let A be a Dedekind domain.

ⓐ Let M be a non-zero fractional ideal. Prove that M is projective. (Hint: $\exists\, x_i \in M$ and $y_i \in (A:M)$ s.t. $1 = \sum_{i=1}^{n} y_i x_i \Rightarrow 0 \to \ker \to A^n \to M \to 0$

$$(a_1, \dots, a_n) \mapsto \sum a_i x_i$$
$$(xy_1, \dots, xy_n) \leftarrow x.$$
)

ⓑ Let M be a non-zero fractional ideal. Prove that the isomorphism class of M is in $Pic(A)$.

ⓒ If M_1 and M_2 are two fractional ideals, then $M_1 \otimes_A M_2 \simeq M_1 M_2$. (Hint: Since M_1 is projective, it is flat $\Rightarrow M_1 \otimes_A M_2 \hookrightarrow M_1 \otimes_A F \simeq F$ where F is the field of fractions of A.)

ⓓ If $M \in Pic(A)$, then $M \hookrightarrow F$. And since M is f.g., it is isomorphic to a fractional ideal.

(Hint:

① $M \otimes_A F \simeq F$

② M is flat and $0 \to A \to F$.)

ⓔ Prove that $Cl(A) \simeq Pic(A)$.