1. (a) Let G be a group and X be a set. Assume $G \curvearrowright X$. Assume H is a subgroup of G which acts <u>transitively</u> on X. Prove that $G = \langle H, G_x \rangle$ for any $x \in X$, where $G_x = \{g \in G \mid g \cdot x = x\}$.

(b) Prove that $S_n = \langle (1\ 2), (1\ 2\ \cdots\ n) \rangle$.

(<u>Hint</u>. One way to do it is:

<u>Step 1.</u> Using induction hypothesis, conclude

$$S_{n-1} \subseteq \langle (1\ 2), (1\ 2\ \cdots\ n) \rangle$$

<u>Step 2.</u> Use part (a) for $H = \langle (1\ 2\ \cdots\ n) \rangle$, $S_n \curvearrowright \{1, 2, \ldots, n\}$ and $x = n$.)

(c) Prove that $S_p = \langle \tau, \sigma \rangle$ where τ is a transposition and $\operatorname{ord}(\sigma) = p$.

2. (a) Let G be a finite group. Assume $|G| = 2n$ where $2 \nmid n$. Prove that G has a normal subgroup of index 2.

(<u>Hint</u>. Consider $G \curvearrowright G$, its corresponding homo. $\phi: G \to S_G \simeq S_{2n}$, and think about A_{2n}.)

(b) Let G be a finite group. Assume that G has a cyclic Sylow 2-subgroup. Prove that G has a characteristic subgroup of index 2.

(Recall. $H \leq G$ is called a characteristic subgroup if $\forall \theta \in \mathrm{Aut}(G)$, $\theta(H) = H$. In particular, $H \triangleleft G$.)

(c) Let G be a finite group. Assume that G has a cyclic Sylow 2-subgroup. Prove that there is a characteristic subgroup N of G s.t.

$$2 \nmid |N| \quad \text{and} \quad |G/N| \text{ is a power of } 2.$$

(Hint: ① Use induction on $|G|$. ② If G_1 is a char. Subgp of G and G_2 is a char. subgp of G_1 then G_2 is a char. subgp of G.)

3.(a) Let P be a finite p-group and $N \triangleleft P$. Prove that $Z(P) \cap N \neq \{1\}$.

(Hint. Consider $P \curvearrowright N$ via conjugation.)

(b) Let P be a finite p-group. Suppose $A \lhd P$ is maximal among abelian normal subgroups of P. Prove that $C_P(A) = A$.

(Hint. ① $A \lhd P \Rightarrow C_P(A) \lhd P$.

② A abelian $\Rightarrow A \subseteq C_P(A)$.

③ Use part (a) for $C_P(A)/A \lhd P/A$ to show $\exists\, g \in C_P(A) \setminus A$ s.t. $\langle g \rangle A \lhd P$.)

4. Let G be a non-abelian group of order p^3. Prove that G has a quotient isomorphic to $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.

5. Let $n \geq 5$. Prove that there is no subgroup of A_n which is isomorphic to S_{n-1}.

(Hint. If $S_{n-1} \simeq H \leq A_n$, then consider $A_n \curvearrowright A_n/H$.)