1. (a) Prove that $\mathrm{Aut}\left(\left(\mathbb{Z}/n\mathbb{Z}\right)^{k}\right) \simeq GL_k\left(\mathbb{Z}/n\mathbb{Z}\right)$,

where $\left(\mathbb{Z}/n\mathbb{Z}\right)^{k} = \underbrace{\mathbb{Z}/n\mathbb{Z} \times \cdots \times \mathbb{Z}/n\mathbb{Z}}_{k\text{-times}}$ and

$$GL_k(\mathbb{Z}/n\mathbb{Z}) = \left\{ A \in M_k(\mathbb{Z}/n\mathbb{Z}) \,\middle|\, \exists\, B \in M_k(\mathbb{Z}/n\mathbb{Z}) \cdot AB = BA = I \right\}$$

(b) Prove that $\theta : GL_k(\mathbb{Z}/mn\mathbb{Z}) \longrightarrow GL_k(\mathbb{Z}/m\mathbb{Z}) \times GL_k(\mathbb{Z}/n\mathbb{Z})$ is an isomorphism where $m$ and $n$ are coprime positive integers and

$$\theta(X) = (X \pmod m,\ X \pmod n).$$

2. Prove that there is no group $G$ s.t. $G^{(1)} \simeq S_4$.

(Hint: Show that $S_4$ does not have an element of order 6.

- Notice that $N = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is a normal subgroup of $S_4$.
- Use the previous steps to conclude that, if $H = S_4$, then $H^{(1)} = A_4$, $H^{(2)} = N$, $H^{(3)} = 1$. And

so $H/H^{(1)} \simeq \mathbb{Z}/2\mathbb{Z}$, $H^{(1)}/H^{(2)} \simeq \mathbb{Z}/3\mathbb{Z}$ and $H/H^{(2)} \simeq \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}/3\mathbb{Z}$ $(\simeq S_3)$. In particular, $Z(H/H^{(2)}) = \{\bar{1}\}$.

• Notice that, if $M \trianglelefteq K$ and $M$ is abelian, then $K/M$ acts on $M$ via conjugation, i.e. $(kM) \cdot x := kxk^{-1}$ is well-defined for $k \in K$ and $x \in M$.)

3. Suppose $A, B \trianglelefteq G$. Prove that, if $G/A$ and $G/B$ are abelian, then $G/A \cap B$ is abelian.

4. ⓐ Find $|GL_n(\mathbb{Z}/p\mathbb{Z})|$. [Hint: If the first $i$ columns are fixed, we have $p^n - p^i$ choice for the $(i+1)^{th}$ column.]

ⓑ Prove that, if $P$ is a $p$-subgroup of $GL_n(\mathbb{Z}/p\mathbb{Z})$, then $\exists\, g \in GL_n(\mathbb{Z}/p\mathbb{Z})$ such that

$$gPg^{-1} \subseteq \left\{ \begin{bmatrix} 1 & & a_{ij} \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \,\middle|\, a_{ij} \in \mathbb{Z}/p\mathbb{Z} \text{ for } i<j \right\}$$

5. Let $G$ be a finite group. Suppose

$$n \mid |G| \Rightarrow |\{g \in G \mid g^n = 1\}| \leq n.$$

Prove that $G$ is cyclic.

[Hint: Let $\Psi(m) := |\{g \in G \mid \operatorname{ord}(g) = m\}|$.

- Show that, if $\Psi(m) \neq 0$, then $\Psi(m) = \Phi(m)$.
- Use $\sum_{m \mid n} \phi(m) = n$.]

6. Suppose $N \triangleleft G$ and $P \in \operatorname{Syl}_p(N)$. Prove that

$$G = N_G(P)\,N.$$

7. Suppose $p < q < r$ are primes. If $G$ is a finite group of order $pqr$, then a Sylow $r$-subgroup is normal.

Chapter 7.1, Problem 26, 28.

Chapter 7.2, Problem 3, 5.

Chapter 7.3, Problem 33.

Let $R$ be a unital ring, and $J \subseteq M_n(R)$. Then

$$J \triangleleft M_n(R) \iff \exists\, I \triangleleft R,\ J = M_n(I).$$

(Hint. $E_{ii} A E_{jj} = a_{ij} E_{ij}$ where $E_{ij} = \begin{array}{cc} & \phantom{0 \cdots}\ j \\ i & \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \end{array}$)