# LECTURE 11.

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## 1. Review of an equivalence relation.

**Definition 1** (Equivalence relation)**.** A relation $\sim$ on a set $X$ is called an equivalence relation if it satisfies the following properties:

(1) For any $x \in X$, $x \sim x$.
(2) If $x \sim y$, then $y \sim x$.
(3) If $x \sim y$ and $y \sim z$, then $x \sim z$.

**Example 2.** (1) *The congruent relation on the triangles.*
(2) *The mod $m$ congruent relation on the set of integers, i.e. $a \sim b$ if and only if $m|a-b$.*
(3) *If $H$ is a subgroup of $G$, then $g_1 \sim_H g_2$ if $g_1^{-1}g_2 \in H$ is an equivalence relation on $G$.*

**Definition 3** (Equivalence class)**.** Let $\sim$ be an equivalence relation on $X$. The equivalence class of $x \in X$ is

$$\{y \in X |\ x \sim y\}$$

and it is denoted by $[x]_\sim$.

**Lemma 4.** *Let $\sim$ be an equivalence relation on $X$. Then*

(1) *$X/\sim := \{[x]_\sim |\ x \in X\}$ is a partition of $X$.*
(2) *$[x]_\sim = [x']_\sim$ if and only if $x \sim x'$.*

**Remark 5.** One can view $X/\sim$ as a set created after gluing points which are equivalent to each other (with respect to $\sim$). In some sense, we are putting some new glasses on and from this new point of view there is no distinction between two elements that are equivalent to each other.

**Example 6.** (1) *$G/H$ is the same as $G/\sim_H$, where $\sim_H$ is defined in the previous example.*
(2) *Let $\sim$ be the following relation on $[0,1]$, for any $x \in [0,1]$ $x \sim x$ and $0 \sim 1$ and $1 \sim 0$. Then $\sim$ is an equivalence relation and $[0,1]/\sim$ can be identified with a circle.*

## 2. Field of quotients.

Any unital subring of a field is an integral domain. Its inverse is also correct:

**Theorem 7.** *Let $R$ be an integral domain. Then there is a field $F$ (called the field of quotients of $R$) that contains $R$ and moreover, if $R$ is a subring of a field $E$, then $F$ can be also embedded into $E$.*

*Proof.* **Construction of the set:** Let $\mathcal{F} = R \times (R \setminus \{0\})$ and $\simeq$ be the following equivalence relation on $\mathcal{F}$:

$$(a,b) \sim (c,d) \text{ if and only if } ad = bc.$$

Now let $F = \mathcal{F}/\sim$. (You have to view $[(a,b)]_\sim$ as $a/b$ in the set of rational numbers!)

**Operations:** We define the following operations on $F$:

$$[(a,b)]_\sim + [(c,d)]_\sim := [(ad+bc, bd)]_\sim,$$

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and

$$[(a,b)]_\sim \cdot [(c,d)]_\sim := [(ac,bd)]_\sim.$$

One can check that these are well-defined and $(F,+,\cdot)$ is a ring. Here we just check why the addition is well-defined:

Assume that $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$. We have to show $(ad+bc,bd) \sim (a'd'+b'c',b'd')$. So we have $ab'=ba'$ and $cd'=c'd$,

$$(ad+bc)(b'd') = ab'dd' + bb'cd' = a'bdd' + bb'c'd = (a'd'+b'c')(bd).$$

**Field:** $F$ is a unital ring and any non-zero element is a unit. First we notice that $[(0,1)]_\sim$ is the zero in $F$ and $[(1,1)]_\sim$ is the one in $F$. Now one can check that $[(a,b)]_\sim$ is zero if and only if $a=0$. Hence if $[(a,b)]_\sim$ is not zero, then $[(b,a)]_\sim$ is also in $F$. On the other hand, $[(a,b)]_\sim \cdot [(b,a)]_\sim = [(ab,ab)]_\sim = [(1,1)]_\sim$.

$F$ **contains** $R$**:** It is easy to show that the map $f : R \to F$ given by

$$a \longmapsto [(a,1)]_\sim,$$

is an injective ring homomorphism.

$F$ **is the smallest field which contains** $R$**:** Let $E$ be a field which contains $R$. It is enough to check the map $g : F \to E$ given by

$$g([(a,b)]_\sim) := ab^{-1},$$

is an injective ring homomorphism. □

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