LECTURE 19.

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Definition 1. Let D be an integral domain and a be a non-zero and non-unit element of D.

(1) a is called irreducible if $a = bc$ implies either b is unit or c is unit.
(2) a is called prime if $a|bc$ implies either $a|b$ or $a|c$.
(3) b and c are called associates if there is a unit u such that $a = bu$.

Lemma 2. *Let D be an integral domain.*

(1) *a and b are associates if and only if $\langle a\rangle = \langle b\rangle$.*
(2) *a is prime if and only if $\langle a\rangle$ is a non-zero prime ideal.*
(3) *a is irreducible if and only if it is a non-zero ideal which is maximal among proper principal ideals, i.e. $0 \neq \langle a\rangle \neq D$ and $\langle a\rangle \subseteq \langle b\rangle$ implies that either $\langle b\rangle = \langle a\rangle$ or $\langle b\rangle = D$.*

Proof. 1. If a and b are associates, then there is a unit u such that $a = bu$. Thus $a \in \langle b\rangle$. On the other hand, $b = au^{-1}$. So $b \in \langle a\rangle$. Thus $\langle a\rangle = \langle b\rangle$.

If $\langle a\rangle = \langle b\rangle$, then there are $c, d \in D$ such that $a = bc$ and $b = ad$. Thus $a = acd$. If $a = 0$, then clearly $b = 0$ and we are done. If $a \neq 0$, then $cd = 1$. So c is unit and therefore a and b are associates.

2. If a is prime, then $\langle a\rangle$ is a non-zero proper ideal.

$$\begin{aligned} bc \in \langle a\rangle &\Rightarrow a|bc \\ &\Rightarrow a|b \text{ or } a|c \\ &\Rightarrow b \in \langle a\rangle \text{ or } c \in \langle a\rangle. \end{aligned}$$

Hence $\langle a\rangle$ is a non-zero prime ideal.

Now, let's assume that $\langle a\rangle$ is a nonzero prime ideal. Hence it is a proper ideal. So a is a non-zero and non-unit element.

$$\begin{aligned} a|bc &\Rightarrow bc \in \langle a\rangle \\ &\Rightarrow b \in \langle a\rangle \text{ or } c \in \langle a\rangle \\ &\Rightarrow a|b \text{ or } a|c. \end{aligned}$$

3. If a is an irreducible, then $\langle a\rangle$ is a non-zero proper ideal.

$$\begin{aligned} \langle a\rangle \subseteq \langle b\rangle &\Rightarrow a = bc \\ &\Rightarrow \text{either } b \text{ or } c \text{ is unit} \\ &\Rightarrow \langle b\rangle = D \text{ or } a \text{ and } b \text{ are associates} \\ &\Rightarrow \langle b\rangle = D \text{ or } \langle b\rangle = \langle a\rangle. \end{aligned}$$

Now, let's assume that $\langle a\rangle$ is a nonzero ideal which maximal among proper principal ideals. Then a is a non-zero and non-unit element.

$$\begin{aligned} a = bc &\Rightarrow \langle a\rangle \subseteq \langle b\rangle \\ &\Rightarrow \langle b\rangle = \langle a\rangle \text{ or } \langle b\rangle = D \\ &\Rightarrow \text{either } b \text{ and } a \text{ are associates or } b \text{ and } 1 \text{ are associates} \\ &\Rightarrow \text{there is unit } u \text{ such that either } a = bu \text{ or } b = u \\ &\Rightarrow \text{by the cancellation property, either } c \text{ or } b \text{ is unit.} \end{aligned}$$

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□

Corollary 3. (1) *In a PID, a is irreducible if and only if $\langle a\rangle$ is a non-zero maximal ideal.*
(2) *In a PID, every irreducible is a prime.*

Lemma 4. *In any integral domain, every prime is an irreducible.*

Proof. Let a be a prime. So it is a non-zero and no-unit element.

$$\begin{aligned} a=bc &\Rightarrow bc\in\langle a\rangle \\ &\Rightarrow b\in\langle a\rangle \text{ or } c\in\langle a\rangle \\ &\Rightarrow \langle b\rangle=\langle a\rangle \text{ or } \langle c\rangle=\langle a\rangle \text{ (here we are using the fact that } a\in\langle b\rangle\cap\langle c\rangle.\text{)} \\ &\Rightarrow \text{either } a \text{ and } b \text{ are associates or } a \text{ and } c \text{ are associates} \\ &\Rightarrow \text{again by cancellation we have that either } c \text{ or } b \text{ is unit.} \end{aligned}$$

□

Corollary 5. (1) *In a PID, a is a prime if and only if it is an irreducible.*
(2) *In a PID, a non-zero prime ideal is maximal.*

Proof. 1. It is a clear corollary of Corollary 3 and Lemma 4.

2. Let P be a non-zero prime ideal. Since D is a PID, there is $a\in D$ such that $P=\langle a\rangle$. By Lemma 2, Part 2, we have that a is a prime. By Lemma 4, a is an irreducible. By Corollary 3, $P=\langle a\rangle$ is maximal. □

We will see that 2 and 5 are irreducible in $\mathbb{Z}[\sqrt{10}]$ but they are not primes.

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