LECTURE 20.

ALIREZA SALEHI GOLSEFIDY

1. RECALL

Definition of a prime and irreducible elements.

Lemma 1. *Let D be an integral domain, $a \in D$ and $I = \langle a \rangle$.*

1. *a is prime if and only if I is a non-zero prime ideal.*
2. *a is irreducible if and only if I is maximal among principle ideals.*
3. *a and b are associates if and only if $\langle a \rangle = \langle b \rangle$.*

Corollary 2. *In a PID, every irreducible is prime.*

Lemma 3. *In any integral domain, every prime is irreducible.*

2. UFD

Definition 4. An integral domain is called a *Unique Factorization Domain* if it satisfies the following properties:

1. Every non-zero and non-unit element can be written as product of irreducible elements.
2. This decomposition is unique up to associates and the order of its irreducible factors.

Lemma 5. *Let D be a PID and $\{I_i\}$ be an ascending chain of ideals of D, i.e.*

$$I_1 \subseteq I_2 \subseteq I_2 \subseteq \cdots .$$

Then there is a positive integer n such that

$$I_n = I_{n+1} = \cdots .$$

Proof. Let $I = \bigcup_{i=1}^{\infty} I_i$. We showed that I is an ideal. Since D is a PID, there is $a \in D$ such that $I = \langle a \rangle$. So there is n such that $a \in I_n$. Then we argued that $I_n = I$ and so for any $i \geq n$, $I_i = I$. □

Remark 6. If any ideal in D is finitely generated, then a similar argument implies that there is no distinct ascending chain of ideal of D.

Lemma 7. *Let D be a PID. If $a \in D$ is a non-zero, non-unit element, then there is an irreducible p and a nonzero element b such that $a = pb$.*

Proof. If not, we construct two sequences $\{b_i\}_{i=1}^{\infty}$ and $\{b'_i\}_{i=1}^{\infty}$ of non-zero and non-unit elements of D such that

$$a = b_n \cdot b'_n \cdot b'_{n-1} \cdot \cdots \cdot b'_1,$$

and

$$b_n = b_{n+1} \cdot b'_{n+1},$$

for any n.

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By our contrary assumption, a is not irreducible so there are non-zero, non-unit elements b_1 and b_1' such that $a = b_1 \cdot b_1'$. If b_1 is irreducible, then we have found an irreducible factor of a, so it is not and it can be written as product of two non-zero, non-unit elements b_2 and b_2'. Repeating this argument, we get the desired sequences.

But this implies that

$$\langle b_1 \rangle \subsetneq \langle b_2 \rangle \subsetneq \langle b_3 \rangle \subsetneq \cdots ,$$

which contradicts Lemma 5. □

Lemma 8. *Any non-zero, non-unit element in a PID can be written as a product of irreducibles.*

Proof. If not, then there is a non-zero and non-unit element $a \in D$ which cannot be written as product of irreducibles. We will construct a sequence $\{p_i\}_{i=1}^{\infty}$ of irreducibles and a sequences $\{a_i\}_{i=1}^{\infty}$ of non-zero elements in D such that

$$a = p_1 \cdot p_2 \cdot \cdots \cdot p_n \cdot a_n,$$

and

$$a_n = p_{n+1} \cdot a_{n+1}.$$

By Lemma 7, we can find an irreducible p_1 and a nonzero element a_1 such that $a = p_1 \cdot a_1$. If a_1 is unit, a is written as product of irreducibles. So by the contrary assumption, a_1 is not unit. So again by Lemma 7, there is an irreducible p_2 and a non-zero element a_2 such that $a_1 = p_2 \cdot a_2$. Repeating this argument, we get the desired sequences. But this implies that

$$\langle a_1 \rangle \subsetneq \langle a_2 \rangle \subsetneq \langle a_3 \rangle \subsetneq \cdots ,$$

which contradicts Lemma 5. □

Mathematics Dept, University of California, San Diego, CA 92093-0112

E-mail address: golsefidy@ucsd.edu