Naïve view towards linear algebraic groups

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Many groups that one faces in mathematics can be described as solutions of certain polynomial equations, e.g.

- $SL_n(A) = \{ g \in M_n(A) \mid \det(g) = 1 \}$ for any commutative ring A.
- $O(n) := \{ g \in M_n(\mathbb{R}) \mid g g^t = I \}$

  These are linear maps which preserve the bilinear form $f(\vec{v}, \vec{w}) := \sum_{i=1}^{n} v_i w_i$. Of course similar groups can be written for other forms and other rings.

  Here are two other examples of this type:

- $O_n(A) := \{ g \in M_n(A) \mid g \begin{bmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{bmatrix} g^t = \begin{bmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{bmatrix} \}$
- $Sp_{2n}(A) := \{ g \in M_{2n}(A) \mid g \begin{bmatrix} & I_n \\ -I_n & \end{bmatrix} g^t = \begin{bmatrix} & I_n \\ -I_n & \end{bmatrix} \}$.

  (Symplectic group.)

Naïve but useful view towards algebraic groups:

They are common solutions of a family of polynomials.

As we can in the above examples, the family of polynomials are the driving force (and not much the solutions). In the

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sense for a given family of polynomials we can ask for their common solutions in various rings; for example

$$\begin{array}{ccc} \text{Rings} & \longrightarrow & \text{Groups} \\ A & \rightsquigarrow & SL_n(A) \end{array}$$

Notice that, any ring homomorphism (which sends 1 to 1) $A_1 \to A_2$ induces a group homomorphism $SL_n(A_1) \to SL_n(A_2)$. This means, we can view SL_n as a functor from the category of unital commutative rings to the category of groups.

Let's try to make these a bit formal:

Let $\{f_\ell(\underline{x}_{ij})\}_{\ell\in I}$ be a family of polynomials in $k[x_{11},\dots,x_{nn}]$ (where k is a unital commutative ring).

For any k-algebra A (i.e. $k \to A$), we can look at the common solutions of $f_\ell(g)$ in A:

$$\{g \in M_n(A) \mid f_\ell(g)=0 \quad \forall \ell\in I\}.$$

It means to each variable $\underline{x}_{ij}$ we want to assign $g_{ij} \in A$

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such that $f_\ell(g_{ij})=0$. So

$\underline{x}_{ij} \longmapsto g_{ij}$ induces an k-algebra homomorphism

$$k[\underline{x}_{11},\dots,\underline{x}_{nn}] \xrightarrow{\tilde{\phi}_g} A \quad \text{s.t.} \quad \tilde{\phi}_g(\underline{x}_{ij}) = g_{ij}$$

and $\tilde{\phi}_g(f_\ell) = f_\ell(g) = 0$.

Therefore $f_\ell \in \ker \tilde{\phi}_g \Rightarrow \mathfrak{a} = \langle f_\ell \mid \ell \in I\rangle \subseteq \ker \tilde{\phi}_g$.

Hence $\exists$ a k-algebra homomorphism

$$\phi_g : k[\underline{x}_{11},\dots,\underline{x}_{nn}]/\mathfrak{a} \longrightarrow A.$$

So any common solution of $\{f_\ell\}_{\ell\in I}$ gives us an element of $\mathrm{Hom}_{k\text{-alg.}}(k[\underline{x}_{11},\dots,\underline{x}_{nn}]/\mathfrak{a}, A)$.

Note that, if $\phi \in \mathrm{Hom}_{k\text{-alg.}}(k[\underline{x}_{11},\dots,\underline{x}_{nn}]/\mathfrak{a}, A)$, then $g=[\phi(\underline{x}_{ij}+\mathfrak{a})] \in M_n(A)$ is a common solution of f_ℓ's :

$$f_\ell(g) = f_\ell([\phi(\underline{x}_{ij}+\mathfrak{a})]) \overset{\boxed{\phi \text{ is a } k\text{-alg. homo.}}}{=} \phi(f_\ell(\underline{x}_{ij})+\mathfrak{a})$$

$$\underset{\boxed{f_\ell \in \mathfrak{a}}}{=} 0.$$

There is a bijection between common solutions of f_ℓ's in A and $\mathrm{Hom}_{k\text{-alg}}(k[\underline{x}_{ij}]/\langle f_\ell\rangle, A)$.

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So far we described <u>the set</u> of common solutions using the k-algebra $k[G]=k[\underline{x}_{11},\dots,\underline{x}_{nn}]/\mathfrak{a}$ where $\mathfrak{a}=\langle f_\ell \mid \ell\in I\rangle$.

Q. Under what conditions this set could be a <u>subgroup</u> of $GL_n(A)$?

- Can we make these conditions independent of A?

Let's use the 2^{nd} part of the question to find a good answer:

Let $\phi_1, \phi_2 : k[G] \longrightarrow A$ be k-algebra homomorphisms.

Then $g_1=[\phi_1(x_{ij})]$ and $g_2=[\phi_2(x_{ij})]\in M_n(A)$ are two common solutions (where $x_{ij}=\underline{x}_{ij}+\mathfrak{a}$). The i,j entry of $g_1\cdot g_2$ is $\sum_{s=1}^{n}\phi_1(x_{is})\phi_2(x_{sj})$. And it is suppose to be a common solution, which means

$$(*) \qquad x_{ij} \longmapsto \sum_{s=1}^{n}\phi_1(x_{is})\phi_2(x_{sj})$$

is supposed to be a <u>well-defined k-algebra homomorphism</u> $k[G]\longrightarrow A$.

- Can we get rid of A in the above condition?

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Let's take $A = k[G] \otimes_k k[G]$ and $k[G] \xrightarrow{\phi_1} k[G] \otimes k[G]$, $f \longmapsto f \otimes 1$

and $k[G] \xrightarrow{\phi_2} k[G] \otimes k[G]$, $f \longmapsto 1 \otimes f$. Then $\circledast$ implies

$$x_{ij} \longmapsto \sum_{s=1}^{n} (x_{is} \otimes 1)(1 \otimes x_{sj}) = \sum_{s=1}^{n} x_{is} \otimes x_{sj}$$

is a well-defined k-algebra homomorphism

$$m^* : k[G] \longrightarrow k[G] \otimes k[G].$$

<u>Claim</u>. If $m^* : k[G] \longrightarrow k[G] \otimes k[G]$,

$$m^*(x_{ij}) := \sum_{s=1}^{n} x_{is} \otimes x_{sj}$$

is a well-defined k-algebra homomorphism,

then for any $\phi_1, \phi_2 \in \mathrm{Hom}_{k\text{-alg}}(k[G], A)$

$$x_{ij} \longmapsto \sum_{s=1}^{n} \phi_1(x_{is})\, \phi_2(x_{sj})$$

is a well-defined k-algebra homomorphism.

<u>Pf of claim</u>. $\phi_1, \phi_2 \in \mathrm{Hom}_{k\text{-alg}}(k[G], A) \Longrightarrow$

$\phi_1 \otimes \phi_2 \in \mathrm{Hom}_{k\text{-alg}}(k[G] \otimes k[G], A)$, $(\phi_1 \otimes \phi_2)(f_1 \otimes f_2) := \phi_1(f_1)\phi_2(f_2)$.

$\Rightarrow (\phi_1 \otimes \phi_2) \circ m^* \in \mathrm{Hom}_{k\text{-alg}}(k[G], A)$ and

$$(\phi_1 \otimes \phi_2) \circ m^*(x_{ij}) = (\phi_1 \otimes \phi_2)\Big(\sum_{s=1}^{n} x_{is} \otimes x_{sj}\Big) = \sum \phi_1(x_{is})\phi_2(x_{sj}).$$

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What we are doing is giving a polynomial rule for multiplying two "general" elements which satisfy the desired relations, and then "specialize" it via ϕ_i's.

- Of course identity should be a common solution for any A, in particular for $\underline{\underline{k}}$: $\exists\, e_I : k[G] \to k$, $e_I(x_{ij}) = [i=j]$.

Now, for any k-algebra A, i.e. $k \xrightarrow{c} A$, we get

$c \circ e_I \in \mathrm{Hom}_{k\text{-alg}}(k[G], A)$, $[c \circ e_I(x_{ij})] = I$.

- <u>How about taking inverse?</u>

Following the above philosophy, it is enough to have a description of inverse for a "general" element.

A "general" element is $[x_{ij}] = [\underline{x_{ij}} + \mathfrak{a}] \in M_n(k[G])$

where $\mathfrak{a} = \langle f_j \mid j \in I \rangle$. In view of identifying A points of G (which we denote by $G(A)$) with $\mathrm{Hom}_{k\text{-alg}}(k[G], A)$, we have that $[x_{ij}] \in G(k[G])$ is in fact

$$\mathrm{id}_{k[G]} \in \mathrm{Hom}_{k\text{-alg}}(k[G], k[G]).$$

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Suppose $i^* \in G(k[G]) := \mathrm{Hom}_{k\text{-alg.}}(k[G], k[G])$ is the inverse of $id_{k[G]}$ as an element of $G(k[G]) \hookrightarrow GL_n(k[G])$.

Then $(i^* \otimes id_{k[G]}) \circ m^* = c \circ e_I$ and

$(id_{k[G]} \otimes i^*) \circ m^* = c \circ e_I$.

This can be written as in e following commuting diag.

$$\begin{tikzcd}
& k[G]\otimes k[G] \arrow[r, "i^*\otimes id_{k[G]}"] & k[G]\otimes k[G] \arrow[dr] & \\
k[G] \arrow[ur, "m^*"] \arrow[dr, "m^*"'] \arrow[r, "e_I"] & k \arrow[rr, "c"] & & k[G] \\
& k[G]\otimes k[G] \arrow[r, "id_{k[G]}\otimes i^*"'] & k[G]\otimes k[G] \arrow[ur] &
\end{tikzcd}$$

- Now, for any k-algebra A and $g \in G(A)$, there is $\phi : k[G] \to A$ s.t. $[\phi(x_{ij})] = g$.

By assumption $[x_{rs}]^{-1} = [x'_{rs}] \in GL_n(k[G])$; so (given by $i^*(x_{rs})$)

$[\phi(x'_{rs})][\phi(x_{rs})] = I \rightsquigarrow g$ has an inverse in $G(A)$.

- Alternatively: $\forall\, \phi : k[G] \to A$, $(\phi \otimes \phi\circ i^*)\circ m^*$

$= \phi\circ(id_{k[G]} \otimes i^*)\circ m^*$

$= \phi\circ c\circ e_I \rightsquigarrow$ the identity in $G(A)$.

Lecture 01: Summary

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We showed that, if $R = k[X_{ij}]/\mathfrak{A}$ satisfies the following conditions:

$\exists\ R \xrightarrow{m^*} R\otimes R$ (coproduct), (where $m^*(x_{ij}) = \sum_{r=1}^{n} x_{ir}\otimes x_{rj}$)

$\exists\ R \xrightarrow{i^*} R$ (coinverse),

$\exists\ R \xrightarrow{e} k$ (coidentity), s.t.

$$\begin{tikzcd}
 & R\otimes R \arrow[rr, "i^*\otimes id_R"] & & R\otimes R \arrow[dr, "p"] & \\
R \arrow[ur, "m^*"] \arrow[rr, "e"] \arrow[dr, "m^*"'] & & k \arrow[rr, "c"] & & R \\
 & R\otimes R \arrow[rr, "id_R\otimes i^*"'] & & R\otimes R \arrow[ur, "p"'] &
\end{tikzcd}$$

Then, for any k-algebra A,

$$\begin{aligned} G(A) := \mathrm{Hom}_{k\text{-alg}}(R, A) &\hookrightarrow GL_n(A) \\ \phi &\longmapsto [\phi(x_{ij})] \end{aligned}$$

gives us a subgroup of $GL_n(A)$.

The importance of this fact is that the mentioned conditions are on $\underline{R}$ and independent of A. We can push this philosophy further and ask the following question:

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$\boxed{Q}$ Describe representable functors $\underline{k\text{-algebras}} \longrightarrow \underline{\text{Groups}}$; i.e.

Find the necessary and sufficient conditions on R s.t.

$$A \overset{G_R}{\longmapsto} G_R(A) := \mathrm{Hom}_{k\text{-alg}}(R, A)$$

$$A_1 \xrightarrow{f} A_2 \longmapsto G_R(A_1) \longrightarrow G_R(A_2)$$

$$\phi \longmapsto f \circ \phi$$

defines a functor from the category of $\underline{k\text{-algebras}}$ to the category of $\underline{\text{Groups}}$.

<u>Exercise</u>. Answer the above question. Here are the steps:

- Use multiplication in $G_R(R \otimes R)$ to define coproduct $m^* : R \longrightarrow R \otimes R$.
- Use associativity of product in $G_R(R \otimes R \otimes R)$ to a get certain commuting conditions for m^*.
- Show that these conditions are enough to get a <u>semigroup</u> structure on $G_R(A)$ for any k-alg. A in a "natural" way.
- Use identity element of $G_R(k)$ to get $R \xrightarrow{e^*} k$.
- Use inverse of $id_R \in G_R(R)$ to get $R \xrightarrow{\imath^*} R$
- Get a commuting diag. for $\imath^*$, c.
- Show that these conditions are enough to get a group

structure on $G_R(A)$ for any k-alg. A in a "natural" way.

Lecture 01: Hopf algebra

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<u>Def.</u> A k-algebra R is called a Hopf algebra (commutative) if

$$A \longmapsto G_R := \mathrm{Hom}_{k\text{-alg.}}(R, A)$$

defines a functor from the category of k-algebras to the category of groups.

Equivalently if it satisfies the conditions that you can find as part of the previous exercise.

- In this course we study representable group functors G_R where R is

① of finite type i.e. finitely generated k-algebra.

② k is algebraically closed and $\mathrm{char}(k) = 0$.

③ R is smooth; for an algebraically closed field k, it is equivalent to say R is reduced, i.e. it has no non-zero nilpotent element.