Regular representation

Thursday, April 6, 2017 11:25 AM

Let G be a representable functor from the category of k-algebras to the category of groups. Then as we mentioned in the previous lecture there is a commutative Hopf k-algebra $k[G]$ s.t. $G(A) = \mathrm{Hom}_{k\text{-alg}}(k[G], A)$.

Proposition (Regular representation) For any $g \in G(A)$, let

$$\underbrace{k[G] \otimes A \xrightarrow{m^* \otimes id_A} k[G] \otimes k[G] \otimes A \xrightarrow{id_{k[G]} \otimes g \otimes id_A} k[G] \otimes A \otimes A \xrightarrow{id \otimes p} k[G] \otimes A}_{\pi(g)}$$

Then π is a group homomorphism from $G(A)$ to A-mod automorphism of $k[G]$.

Pf. • It is clear that $\pi(g) \in \mathrm{End}_A(k[G] \otimes A)$.

• Why $\pi(e) = id_{k[G] \otimes A}$? We know that

$$\underbrace{k[G] \xrightarrow{m^*} k[G] \otimes k[G] \xrightarrow{id \otimes e^*} k[G] \otimes k \longrightarrow k[G]}_{id_{k[G]}}$$

So, if $m^*(r) = \sum l_i \otimes r_i$, then $\sum e^*(r_i) l_i = r$.

Hence $r \otimes a \longmapsto \sum l_i \otimes r_i \otimes a \longmapsto \sum l_i \otimes e^*(r_i) a$

$$= (\sum e^*(r_i) l_i) \otimes a = r \otimes a,$$

which implies $\pi(e) = id_{k[G] \otimes A}$.

Regular representation

Thursday, April 6, 2017 3:06 PM

• Why $\pi(gh) = \pi(g) \circ \pi(h)$?

By the coassociativity law we have

$$\begin{tikzcd}
k[G] \arrow[r, "m^*"] \arrow[d, "m^*"'] & k[G] \otimes k[G] \arrow[d, "\mathrm{id} \otimes m^*"] \\
k[G] \otimes k[G] \arrow[r, "m^* \otimes \mathrm{id}"'] \arrow[ur, phantom, "\circlearrowleft"] & k[G] \otimes k[G] \otimes k[G]
\end{tikzcd}$$

So, if $m^*(r) = \sum l_i \otimes r_i$,

then $\sum l_i \otimes m^*(r_i) = \sum m^*(l_i) \otimes r_i$.

$$\pi(g)(\pi(h)(r \otimes a)) = \pi(g)\left(\sum_i l_i \otimes h(r_i)\, a\right)$$

$$= \sum_i \pi(g)(l_i \otimes h(r_i)\, a)$$

$$= \sum_{i,j} l_{ij} \otimes g(r_{ij})\, h(r_i)\, a$$

where $m^*(l_i) = \sum_j l_{ij} \otimes r_{ij}$.

On the other hand, $\pi(gh)(r \otimes a) = \sum_i l_i \otimes (gh)(r_i)\, a$

$$= \sum_i l_i \otimes p(g \otimes h)(m^*(r_i))\, a$$

$$= (\mathrm{id} \otimes p \otimes p)(\mathrm{id} \otimes g \otimes h)\underbrace{\left(\sum l_i \otimes m^*(r_i)\right)}_{\sum m^*(l_i) \otimes r_i} \otimes a$$

$$= \sum_{i,j} l_{ij} \otimes g(r_{ij})\, h(r_i)\, a .$$

Therefore $\pi(g) \circ \pi(h) = \pi(gh)$, which implies π is a group homomorphism. ∎

Regular representation is locally finite

Friday, April 7, 2017 6:03 PM

<u>Theorem.</u> Suppose G is a representable functor from k-algebras to groups where k is a field. Suppose the associated Hopf k-algebra $k[G]$ is finitely gener. Then there is a finite dimensional k-vector subspace V of $k[G]$, such that

① $G(A) \curvearrowright V\otimes A$ through regular representation.

② $G(A) \curvearrowright V\otimes A$ faithfully, i.e. $\pi(g) = \mathrm{id}_{V\otimes A} \Rightarrow g = e \in G(A)$.

To prove this theorem, first we prove $G \curvearrowright \underline{k[G]}$ is locally finite.

<u>Lemma.</u> For any $r \in k[G]$, there is a finite-dimensional subspace V of $k[G]$ s.t. $r \in V$ and $V\otimes A \subseteq k[G]\otimes A$ is $G(A)$-invariant under the regular action.

<u>Pf.</u> Let $\{r_i\}_{i\in I}$ be a k-basis of $k[G]$. Then there are unique $\{l_i\}_{i\in I}$ s.t. $m^*(r) = \sum l_i \otimes r_i$ and $l_i = 0$ except for finitely many i's.

Let $V = \sum_i k\, l_i$. As we have seen before, $r = \sum e(r_i) l_i$

Regular representation is locally finite

Tuesday, April 11, 2017 9:46 PM

So $r \in V$.

Now we have to show $\pi(g)(l_s \otimes 1) \in V \otimes A$ for any i.

By the definition of π, we have

$$\pi(g)(l_s \otimes 1) = \sum l_{si} \otimes g(r_i) \quad \text{where } m^*(l_s) = \sum l_{si} \otimes r_i .$$

Let $c_{ij}^{(s)} \in k$ be s.t. $m^*(r_s) = \sum_{i,j} c_{ij}^{(s)} r_i \otimes r_j$. By coassociativity we have

$$\begin{tikzcd}
r \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \sum l_s \otimes r_s \arrow[ddd, mapsto] \\
& k[G] \arrow[r, "m^*"] \arrow[d, "m^*"'] & k[G] \otimes k[G] \arrow[d, "m^* \otimes id"] & \\
& k[G] \otimes k[G] \arrow[r, "id \otimes m^*"'] & k[G] \otimes k[G] \otimes k[G] & \\
\sum l_s \otimes r_s \arrow[rrr, mapsto] & & & \sum l_{si} \otimes r_i \otimes r_s = \sum_{i,j,s} c_{ij}^{(s)} l_s \otimes r_i \otimes r_j
\end{tikzcd}$$

$$\Rightarrow \sum_{i,s} l_{si} \otimes r_i \otimes r_s = \sum_{i,j} \Big(\sum_s c_{ij}^{(s)} l_s \Big) \otimes r_i \otimes r_j$$

$$\Rightarrow \quad l_{si} = \sum_j c_{is}^{(j)} l_j \in V .$$

So $\pi(g)(l_s \otimes 1) \in V \otimes A$. ∎

<u>Pf of Theorem.</u> Suppose $k[G]$ is generated by $f_1, \dots, f_n$ as a k-algebra. By Lemma, there are finite-dimensional k-vector

Linearity of representable group functors

Tuesday, April 11, 2017 11:58 PM

spaces $V_i \subseteq k[G]$ s.t. ① $f_i \in V_i$ ② $G(A) \curvearrowright V_i \otimes A$.

Let $V := \sum V_i$. So $\dim_k V < \infty$ and $G(A) \curvearrowright V \otimes A$.

Suppose $\overline{\pi}(g) = \mathrm{id}_{V\otimes A}$. Then $\overline{\pi}(g)(f_i \otimes 1) = f_i \otimes 1$,

which implies $\pi(g)(f_i \otimes 1) = f_i \otimes 1$.

Notice that $\pi(g) \in \mathrm{Aut}_{A\text{-alg.}}(k[G]\otimes A)$ and $k[G]\otimes A$ is generated by $f_i \otimes 1$ as an A-algebra. Hence

$$\pi(g) = \mathrm{id}_{k[G]\otimes A}.$$

<u>Exercise</u> Prove that $g = e \in G(A)$

<u>Outline.</u> Let $\{r_i\}_{i\in I}$ be a k-basis of $k[G]$, and

$$m^*(r_i) = \sum r_j \otimes r_{ij}.$$

<u>Step 1.</u> Show that $g(r_{ij}) = [i=j] = e(r_{ij})$

<u>Step 2.</u> Show that $r_i = \sum e(r_j)\, r_{ij}$, and deduce

$g(r_i) = e(r_i)$ for any i. ∎

<u>Remark.</u> Going through the proof, one can see that the assumption that k is a field can be replaced with ① $k[G]$ is a free k-module. ② A f.g. k-submod of $k[G]$ is a free k-mod.

Linearity of representable group functors

Wednesday, April 12, 2017 1:07 AM

<u>Def.</u> • Any k-vector space V defines a functor $\underline{V}$ from k-algebras to sets: $A \mapsto \underline{V}(A) := V \otimes A$.

• V also defines a group functor $GL(\underline{V})$

$$A \longmapsto GL(\underline{V})(A) := \mathrm{Aut}_{A\text{-mod}}(V \otimes A).$$

• We say a group functor $G \curvearrowright \underline{V}$ if $G(A) \curvearrowright \underline{V}(A)$ in a natural way; which is equival. to say there are natural homomorphism $G(A) \longrightarrow GL(\underline{V})(A)$.

<u>So</u> we have proved that

- If G is a representable group functor, $G \curvearrowright k[G]$.
- If G is a repr. group functor, k is a field, and $k[G]$ is a f.g. k-algebra, then $\exists$ a k-vector space V s.t. $\dim V < \infty$ and $G \hookrightarrow GL(\underline{V})$.