Background on AG: algebraic sets

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<u>Definition</u>. Let k be an algebraically closed set. An <u>algebraic set</u> is the set of common solutions of a family of polynomials $f_i \in k[x_1, \dots, x_n]$.

And we denote it by $V(\{f_i\}_{i\in I}) := \{\vec{x} \in k^n \mid \forall i \in I, f_i(\vec{x}) = 0\}$.

<u>Basic properties</u>.

① $V(\{f_i\}_{i\in I}) = V(\langle f_i \rangle_{i\in I})$ where $\langle f_i \rangle_{i\in I}$ is the ideal generated by f_i's.

<u>Pf</u>. • $\vec{x} \in V(\{f_i\}_{i\in I})$.

$\forall f \in \langle f_i \rangle \Rightarrow f = \sum f_i g_i$ for some $g_i \in k[x_1, \dots, x_n]$.

$\Rightarrow f(\vec{x}) = \sum f_i(\vec{x}) g_i(\vec{x}) = 0$.

$\Rightarrow \vec{x} \in V(\langle f_i \rangle)$.

• $\left.\begin{array}{l} \vec{x} \in V(\langle f_i \rangle) \\ f_i \in \langle f_i \rangle \end{array}\right\} \Rightarrow f_i(\vec{x}) = 0 \Rightarrow \vec{x} \in V(\{f_i\})$.

So it is all about common zeros of elements of an ideal.

② For $\mathfrak{a} \subseteq \mathfrak{b}$ two ideals of $k[x_1, \dots, x_n]$, $V(\mathfrak{a}) \supseteq V(\mathfrak{b})$

<u>Pf</u>. $\left.\begin{array}{l} \vec{x} \in V(\mathfrak{b}) \\ f \in \mathfrak{a} \Rightarrow f \in \mathfrak{b} \end{array}\right\} \Rightarrow f \in V(\mathfrak{b})$.

③ For any ideal $\mathfrak{a}$ of $k[x_1, \dots, x_n]$, $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

<u>Pf</u>. We already know $V(\mathfrak{a}) \supseteq V(\sqrt{\mathfrak{a}})$ as $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$.

$\vec{x} \in V(\mathfrak{a})$ $\Big\} \Rightarrow f(\vec{x})^m = 0 \Big\} \Rightarrow f(\vec{x}) = 0$

$$\left.\begin{array}{l}\vec{x}\in V(\mathfrak{a}) \\ f\in\sqrt{\mathfrak{a}} \Rightarrow \exists m\in\mathbb{Z}^+, f^m\in\mathfrak{a}\end{array}\right\} \Rightarrow \left.\begin{array}{l} f(\vec{x})^m=0 \\ f(\vec{x})\in k\end{array}\right\} \Rightarrow f(\vec{x})=0$$

$$\Downarrow$$

$$\vec{x}\in V(\sqrt{\mathfrak{a}}).$$

Definition. For $X\subseteq k^n$, let $I(X):=\{f\in k[\underline{x}_1,\dots,\underline{x}_n] \mid \forall \vec{x}\in X, f(\vec{x})=0\}$;

in particular, $I(\emptyset)=k[\underline{x}_1,\dots,\underline{x}_n]$.

4. For any $X\subseteq k^n$, $I(X)$ is a radical ideal of $k[\underline{x}_1,\dots,\underline{x}_n]$

(radical ideal means: $\sqrt{I(X)}=I(X)$.) (Check it on your own.)

5. $$\begin{array}{rcl} k[\underline{x}_1,\dots,\underline{x}_n] & \longrightarrow & \mathrm{Fun}(X,k) \\ f & \longmapsto & f|_X \end{array}$$ is a k-algebra homomorphism.

Its kernel is exactly $I(X)$. So its image, which we denote by $k[\mathbf{X}]$ is isomorphic to $k[\underline{x}_1,\dots,\underline{x}_n]/I(X)$, and we call it the coordinate ring or ring of regular functions of X.

$$\boxed{\begin{array}{lccl} \text{Subsets of } k^n & \underset{V}{\overset{I}{\rightleftarrows}} & & \text{Subsets of } k[\underline{x}_1,\dots,\underline{x}_n] \\ \quad X & \longmapsto & & I(X) \text{ radical ideal} \\ \text{algebraic subset} \quad V(A) & \longleftarrow\!\!\shortmid & & A \end{array}}$$

Theorem (Hilbert's Nullstellensatz theorem)

For any ideal $\mathfrak{a} \lhd k[\underline{x}_1, \dots, \underline{x}_n]$, $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(Recall that k is algebraically closed.)

<u>Exercise</u>. Here are some of the steps of one method of proving this result:

<u>Step 1</u>. There is a bijection between the set $\mathrm{Max}(k[\underline{x}_1, \dots, \underline{x}_n])$ of maximal ideals of $k[\underline{x}_1, \dots, \underline{x}_n]$ and k^n.

$$\begin{array}{ccc} k^n & & \mathrm{Max}(k[\underline{x}_1, \dots, \underline{x}_n]) \\ \vec{x} & \longmapsto & \mathfrak{m}_{\vec{x}} := \langle \underline{x}_1 - x_1, \dots, \underline{x}_n - x_n \rangle \end{array}$$

And we have $I(\{\vec{x}\}) = \mathfrak{m}_{\vec{x}}$, $|V(\mathfrak{m})| = 1$ for any $\mathfrak{m} \in \mathrm{Max}(k[\underline{x}_1, \dots, \underline{x}_n])$.

<u>Almost Proof.</u> $k[\underline{x}_1, \dots, \underline{x}_n]/\mathfrak{m}$ is a f.g. k-algebra and a field. Hence it is a finite extension of k. (This is a non-trivial fact which is true for any field k. It is also known as a Nullstellensatz theorem. In Atiyah–Macdonald it is proved using valuation theory (Corollary 5.24).) Since k is algebraically closed, we get $k[\underline{x}_1, \dots, \underline{x}_n]/\mathfrak{m} = k$. Hence $\exists$ $x_i \in k$ s.t. $\underline{x}_i + \mathfrak{m} = x_i + \mathfrak{m} \Rightarrow \langle \underline{x}_1 - x_1, \dots, \underline{x}_n - x_n \rangle \subseteq \mathfrak{m}$. ⊛

On the other hand, $f \longmapsto f(x_1, \dots, x_n)$ is an onto k-algebra homomorphism from $k[\underline{x}_1, \dots, \underline{x}_n]$ to k and its kernel is $\mathfrak{m}_{\vec{x}} = \langle \underline{x}_1 - x_1, \dots, \underline{x}_n - x_n \rangle$. Since k is a field, we get that $\mathfrak{m}_{\vec{x}}$ is a maximal ideal. Hence $\mathfrak{m} = \mathfrak{m}_{\vec{x}}$ (because of ⊛). ∎

$\mathfrak{m}_{\vec{x}}$ is a maximal ideal. Hence $\mathfrak{m} = \mathfrak{m}_{\vec{x}}$ (because of $\circledast$). ∎

<u>Step 2.</u> $V(\mathfrak{a}) = \emptyset \Rightarrow \mathfrak{a} = k[\underline{x}_1, \dots, \underline{x}_n]$.

<u>Pf.</u> If not, then there is a maximal ideal $\mathfrak{m}$ of $k[\underline{x}_1, \dots, \underline{x}_n]$ which contains $\mathfrak{a}$. So, by step 1, $\mathfrak{a} \subseteq I(\{\vec{x}\})$ for some $\vec{x}$.

$\Rightarrow \vec{x} \in V(\mathfrak{a})$, which contradicts the assumption that $V(\mathfrak{a}) = \emptyset$. ∎

<u>Step 3.</u> $\sqrt{\mathfrak{a}} \subseteq I(V(\mathfrak{a}))$ (very easy).

<u>Step 4.</u> Finishing the proof.

<u>Almost proof.</u> Suppose $f \in I(V(\mathfrak{a})) \setminus \sqrt{\mathfrak{a}}$.

Since $\sqrt{\mathfrak{a}} = \bigcap\limits_{\substack{\mathfrak{p} \supseteq \mathfrak{a} \\ \text{prime}}} \mathfrak{p}$, there is a prime ideal $\mathfrak{p}$ s.t.

$\mathfrak{a} \subseteq \mathfrak{p}$ and $f \notin \mathfrak{p}$. So $k[\underline{x}_1, \dots, \underline{x}_n]/\mathfrak{p}$ is an integral domain and

$\bar{f} := f + \mathfrak{p}$ is a non-zero element of $k[\underline{x}_1, \dots, \underline{x}_n]/\mathfrak{p}$. Hence

$$\left(k[\underline{x}_1, \dots, \underline{x}_n]/\mathfrak{p}\right)\left[\tfrac{1}{\bar{f}}\right] \simeq \left(k[\underline{x}_1, \dots, \underline{x}_n]/\mathfrak{p}\right)[\underline{y}]/\langle \underline{y}\bar{f} - 1\rangle \qquad \text{(why ?)}$$

$$\simeq k[\underline{x}_1, \dots, \underline{x}_n, \underline{y}]/\langle \mathfrak{p}, \underline{y}f - 1\rangle \qquad \text{(why ?)}$$

<u>Claim</u> $V(\mathfrak{p}, \underline{y}f - 1) = \emptyset$.

<u>Pf of claim</u>. Suppose $(\vec{x}, y) \in V(\mathfrak{p}, \underline{y}f - 1)$.

$$\Rightarrow \left.\begin{matrix} \forall g \in \mathfrak{a} \\ \mathfrak{a} \subseteq \mathfrak{p} \end{matrix}\right\} \Rightarrow g(\vec{x}, y) = g(\vec{x}) = 0 \Rightarrow \vec{x} \in V(\mathfrak{a})$$

$$\Rightarrow f(\vec{x}) = 0$$

$$\Rightarrow y f(\vec{x}) - 1 = -1 \neq 0$$

which is a contradiction. ■

By Step 2 and the above claim, we get $1 \in \langle \mathfrak{p}, \underline{y} f - 1 \rangle$, which implies $1 = g\, h_1 + (\underline{y} f - 1)\, h_2$ for $g \in \mathfrak{p}$, $h_1, h_2 \in k[\underline{x}_1, \ldots, \underline{x}_n, \underline{y}]$. Let $\underline{y} = \dfrac{1}{f(\underline{x}_1, \ldots, \underline{x}_n)}$ to get $1 = g(\underline{x}_1, \ldots, \underline{x}_n)\, h_1(\underline{x}_1, \ldots, \underline{x}_n, \frac{1}{f})$. After multiplying both sides by a large power of f, we get

$f^m(\underline{x}_1, \ldots, \underline{x}_n) \in \langle g \rangle \subseteq \mathfrak{p}$; hence $f \in \mathfrak{p}$, which is a contradiction. ■

<u>Summary</u>.

$$\text{Points of } k^n \xleftrightarrow{\text{bijection}} \operatorname{Max}(k[\underline{x}_1, \ldots, \underline{x}_n])$$

$$\{\vec{x}\} \longmapsto I(\{\vec{x}\})$$

$$V(\mathfrak{m}) \longleftarrow\!\!| \ \mathfrak{m}$$

$V(\mathfrak{m})$ has one point; inverse of each other.

$$\text{Algebraic sets} \xleftrightarrow{\text{bijection}} \text{Radical ideals}$$

$$X \longmapsto I(X)$$

$$V(\mathfrak{a}) \longleftarrow\!\!| \ \mathfrak{a}.$$

Inverse of each other.

$$V(\mathfrak{a}) \longleftrightarrow \{\mathfrak{m} \in \operatorname{Max}(k[\underline{x}_1, \ldots, \underline{x}_n]) \mid \mathfrak{a} \subseteq \mathfrak{m}\}.$$

<u>Pf</u>. Only one step should be checked.

$V(I(X)) = X$ if X is an algebraic set.

In fact we show:

Lemma. For any $X \subseteq k^n$, $V(I(X))$ is the smallest algebraic subset of k^n which contains X.

Pf. Clearly $V(I(X)) \supseteq X$ and $V(I(X))$ is an algebraic set. Now suppose $V(\mathfrak{a}) \supseteq X$; then

$$\mathfrak{a} \subseteq I(V(\mathfrak{a})) \subseteq I(X).$$

So $V(\mathfrak{a}) \supseteq V(I(X))$. ∎ ∎

Corollary. For any algebraic set X, there is a bijection between X and $\mathrm{Max}(k[X])$, where $k[X] = k[x_1, \dots, x_n]/I(X)$.

$$\{\vec{x}\} \longmapsto \mathfrak{m}_{\vec{x}} := \{f \in k[X] \mid f(\vec{x}) = 0\}$$

- $k[X]$ is a f.g., reduced k-algebra. (A k-algebra is called reduced if it has no non-zero nilpotent element.)

(Easy).

Zariski-topology on k^n: closed sets are algebraic subsets of k^n.

Why is it a topology?

- $\emptyset = V(1)$, $X = V(\emptyset)$
- (Finite union) $V(\mathfrak{a}_1) \cup \cdots \cup V(\mathfrak{a}_n) = V(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n)$
- (Intersection) $\bigcap_{i \in I} V(\mathfrak{a}_i) = V(\sum_{i \in I} \mathfrak{a}_i)$.

Pf. • $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n \subseteq \mathfrak{a}_i \Rightarrow V(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n) \supseteq V(\mathfrak{a}_i)$

$$\Rightarrow V(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n) \supseteq \bigcup V(\mathfrak{a}_i).$$

$\cdot$ $\vec{x} \in V(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n) \Rightarrow \underbrace{\mathfrak{m}_{\vec{x}}}_{\text{prime}} \supseteq \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n$

$$\Rightarrow \exists i, \ \mathfrak{m}_{\vec{x}} \supseteq \mathfrak{a}_i \Rightarrow \vec{x} \in V(\mathfrak{a}_i)$$

The rest are straightforward. ∎

<u>Zariski-topology on Max(A)</u>. Inspired by the above def. and the mentioned correspondence, one can define a topology on

$\mathrm{Max}(A) := \{ \mathfrak{m} \lhd A \mid \mathfrak{m} : \text{maximal ideal} \}$:

closed sets are $V_m(\mathfrak{a}) := \{ \mathfrak{m} \in \mathrm{Max}(A) \mid \mathfrak{a} \subseteq \mathfrak{m} \}$.

<u>Zariski-topology on Spec(A)</u>. One can go further and consider all the prime ideals of A, and define the Zariski-topology on that set:

$\mathrm{Spec}(A) := \{ \mathfrak{p} \lhd A \mid \mathfrak{p} \text{ is a prime ideal} \}$.

Closed sets are $V(\mathfrak{a}) := \{ \mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p} \}$.

<u>Exercise</u> • Max(A) consists of closed points of spec(A).

• Max(A) is a dense subset of Spec(A) if A is a f.g. k-algebra. (It is enough to assume that A is a Jacobson ring.)

(<u>Hint</u>. 1^{st} part is clear. For the 2^{nd} part, here is an outline of an argument for algebraically closed field k:

$$\operatorname{Max}(A) \subseteq V(\mathfrak{a}) \Rightarrow \mathfrak{a} \subseteq \bigcap_{\mathfrak{m} \in \operatorname{Max}(A)} \mathfrak{m}.$$

So it is enough to show $\bigcap_{\mathfrak{m}\in \operatorname{Max}(A)} \mathfrak{m} = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(A)} \mathfrak{p} = \operatorname{Nil}(A)$.

Going to $A/\operatorname{Nil}(A)$, we can assume A is a reduced, f.g. k-algebra.

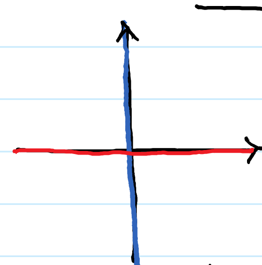
So $A = k[X]$ for some algebraic set X. Hence

$$\bigcap_{\mathfrak{m}\in \operatorname{Max}(k[X])} \mathfrak{m} = \bigcap_{x\in X} \mathfrak{m}_x = \{f \in k[X] \mid f(x)=0 \ \forall x \in X\} = 0.)$$

<u>Zariski-topology on algebraic sets</u>. For $X \subseteq k^n$, we can take the induced topology from k^n. So Zariski-closed subsets are $V(\mathfrak{a}) \cap X$. In view of the bijection between an algebraic set X and $\operatorname{Max}(k[X])$, one can see that the mentioned topology is exactly the Zariski-topology on $\operatorname{Max}(k[X])$.

<u>Def</u>. A topological space is called <u>irreducible</u> if X cannot be written as a union of two <u>proper</u>, <u>closed</u> subsets.

<u>Ex</u>. $\{(x,y)\in \mathbb{R}^2 \mid xy=0\}$ is connected, but it is NOT <u>irreducible</u>.



<u>Lemma</u>. Let X be a topological space and $Y \subseteq X$ a subset.

Then Y is irreducible (w.r.t. the induced topology) $\Leftrightarrow$

$\overline{Y}$ is irreducible.

<u>Pf</u>. Suppose C_1 and C_2 are two closed subsets of X.

$$Y \subseteq C_1 \cup C_2 \Leftrightarrow \overline{Y} \subseteq C_1 \cup C_2 .$$

$(\Rightarrow)$ $\overline{Y} \subseteq C_1 \cup C_2 \Rightarrow Y \subseteq C_1 \cup C_2 \Rightarrow \exists i,\ Y \subseteq C_i$

$$\Rightarrow \overline{Y} \subseteq C_i .$$

$(\Leftarrow)$ $Y \subseteq C_1 \cup C_2 \Rightarrow \overline{Y} \subseteq C_1 \cup C_2 \Rightarrow \exists i,\ \overline{Y} \subseteq C_i$

$$\Rightarrow Y \subseteq C_i . \qquad \blacksquare$$

<u>Corollary</u>. $\mathrm{Spec}(A)$ is irreducible $\Leftrightarrow$ $\mathrm{Max}(A)$ is irreducible.

if A is a f.g. k-algebra.

<u>Lemma</u> (a) Suppose A is Noetherian. Then

$\mathrm{Spec}(A)$ is irreducible $\Leftrightarrow$ A has only <u>one</u> minimal prime ideal

(b) Suppose A is reduced. Then

$\mathrm{Spec}(A)$ is irreducible $\Leftrightarrow$ A is an integral domain.

$(\Leftarrow)$

<u>Pf</u>. (a) Let $\mathfrak{p}_0$ be the unique minimal prime ideal of A. Then

0 is a $\mathfrak{p}_0$-primary ideal. So $\mathfrak{p}_0^{n_0} = 0 \Rightarrow \overline{\{\mathfrak{p}_0\}} = \mathrm{Spec}(A)$

Since $\{\mathfrak{p}_0\}$ is irreducible, $\overline{\{\mathfrak{p}_0\}} = \mathrm{Spec}(A)$ is irreducible.

$(\Rightarrow)$ Since A is Noetherian, 0 has primary decomposition:

($\Rightarrow$) Since A is Noetherian, 0 has primary decomposition:

$0 = \bigcap_{i=1}^{m} \mathfrak{q}_i$. Let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Then

$$\left.\begin{array}{l} \mathrm{spec}(A) = \bigcup_{i=1}^{n} V(\mathfrak{p}_i) \\ \text{irreducible} \end{array}\right\} \Rightarrow \exists\, i_0 : \mathrm{Spec}(A) = V(\mathfrak{p}_{i_0})$$

$\Rightarrow$ $\mathfrak{p}_{i_0}$ is the unique minimal prime ideal of A.

(Under Noeth.)

(b) Let $\mathfrak{p}_0$ be the unique prime ideal of A. By the above argument $\mathfrak{p}_0$ is nilpotent. Since A is reduced, $\mathfrak{p}_0 = 0$.

$\Rightarrow$ A is an integ. domain.

($\Leftarrow$) 0 is the unique minimal prime ideal of A as A is an integ. domain.

(b) [Due to Justin] ($\Rightarrow$) If $ab = 0$, then $V(\langle a \rangle) \cup V(\langle b \rangle) = \mathrm{spec}(A)$.

$\Rightarrow$ either $V(\langle a \rangle) = \mathrm{Spec}(A)$ or $V(\langle b \rangle) = \mathrm{Spec}(A)$

$\Rightarrow$ either $a \in \mathrm{Nil}(A)$ or $b \in \mathrm{Nil}(A)$ $\Rightarrow$ $a = 0$ or $b = 0$.

($\Leftarrow$) $\{0\}$ is irreducible $\Rightarrow$ $\overline{\{0\}} = \mathrm{spec}(A)$ is irreducible. ■

$$\boxed{\begin{array}{ccccl} \text{Irreducible Algebraic Subsets} & \xleftrightarrow{\text{bijection}} & \mathrm{Spec}(k[\underline{x_1}, \dots, \underline{x_n}]) & & \\ V(\mathfrak{p}) & \longleftarrow & \mathfrak{p} & \big\} & \text{inverse} \\ X & \longmapsto & I(X) & \big\} & \text{of each other.} \end{array}}$$

Corollary. Any algebraic set X can be written as a finite union of some closed irreducible subsets of X.

Pf. $\mathrm{Spec}(k[X]) = \bigcup V(\mathfrak{p}_i)$ where $\mathfrak{p}_i \in \mathrm{Ass}(\sigma)$.

$$\Rightarrow \mathrm{Max}(k[X]) = \bigcup \Big(\underbrace{\mathrm{Max}(k[X]) \cap V(\mathfrak{p}_i)}_{\text{closed and irreducible.}}\Big).$$

$$\updownarrow$$

$$X$$

$\blacksquare$

Morphisms (of algebraic sets)

Def. Let $X \subseteq k^n$ and $Y \subseteq k^m$ be two algebraic subsets. Then $\phi: X \to Y$ is a morphism if $\phi(\vec{x}) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$ where $f_i \in k[\underline{x_1}, \dots, \underline{x_n}]$.

Proposition. ① Suppose $X \subseteq k^n$ and $Y \subseteq k^m$ are algebraic subsets. And $X \xrightarrow{\phi} Y$ is a morphism. Then

$$\phi^*: k[Y] \to k[X], \quad \phi^*(f) := f \circ \phi$$

is a k-algebra homomorphism.

② Let A and B be two k-algebras, and $f: A \to B$ be a k-algebra homomorphism. Then

ⓐ f^{-1} induces a continuous map $\mathrm{Spec}(B) \xrightarrow{f^*} \mathrm{Spec}(A)$.

ⓑ In general $f^*(\mathrm{Max}(B))$ is NOT a subset of $\mathrm{Max}(A)$, but if we assume A and B are of finite type, i.e. f.g. k-algebras, then $f^*(\mathrm{Max}(B)) \subseteq \mathrm{Max}(A)$.

③ After identifying X with $\mathrm{Max}(k[X])$ and Y with $\mathrm{Max}(k[Y])$, ϕ gets identified with $(\phi^*)^*$.

Pf. ① $\phi^*(f)(\vec{x}) = f(f_1(\vec{x}), \dots, f_m(\vec{x}))$

Since $f_i \in k[\underline{x}_1, \dots, \underline{x}_n]$ and $f \in k[\underline{y}_1, \dots, \underline{y}_m]$, $f(f_1(\underline{x}_1, \dots, \underline{x}_n), \dots, f_m(\underline{x}_1, \dots, \underline{x}_n))$
$\in k[\underline{x}_1, \dots, \underline{x}_n]$.

So $\phi^*(f) \in k[X]$.

It is clear that ϕ^* is a k-algebra homomorphism.

② (a) $\mathfrak{p} \lhd B \implies f^{-1}(\mathfrak{p})$ is an ideal of

and $A/f^*(\mathfrak{p}) \hookrightarrow B/\mathfrak{p}$

Since $\mathfrak{p} \in \mathrm{Spec}(B)$, $B/\mathfrak{p}$ is an integral domain.

$\Rightarrow A/f^*(\mathfrak{p})$ is an integral domain $\implies f^*(\mathfrak{p}) \in \mathrm{Spec}(A)$

<u>Why is it continuous?</u>

Let $\mathfrak{a} \lhd A$. We need to understand $(f^*)^{-1}(V(\mathfrak{a}))$.

$$(f^*)^{-1}(V(\mathfrak{a})) = \{\mathfrak{p} \in \mathrm{Spec}(B) \mid f^*(\mathfrak{p}) \in V(\mathfrak{a})\}$$
$$= \{\mathfrak{p} \in \mathrm{Spec}(B) \mid \mathfrak{a} \subseteq f^*(\mathfrak{p})\}$$

$= \{\mathfrak{p} \in \mathrm{Spec}(B) \mid \mathfrak{a} \subseteq f^{-1}(\mathfrak{p})\}$

$= \{\mathfrak{p} \in \mathrm{Spec}(B) \mid f(\mathfrak{a}) \subseteq \mathfrak{p}\}$

$= \{\mathfrak{p} \in \mathrm{Spec}(B) \mid \langle f(\mathfrak{a})\rangle \subseteq \mathfrak{p}\}$

$= V(\langle f(\mathfrak{a})\rangle)$ which is closed.

(b) Ex. $\mathbb{Q}[x] \hookrightarrow \mathbb{Q}(x)$, $0 \in \mathrm{Max}(\mathbb{Q}(x))$, $0 \notin \mathrm{Max}(\mathbb{Q}[x])$.

Now suppose A and B are of finite type and $\mathfrak{m} \in \mathrm{Max}(B)$.

Then $A/_{f^*(\mathfrak{m})} \hookrightarrow B/_{\mathfrak{m}}$.

$B/_{\mathfrak{m}}$ is a f.g. k-algebra which is a field. So (by weak Nullstellensatz) $B/_{\mathfrak{m}}$ is an algebraic extension of k. So $A/_{f^*(\mathfrak{m})}$ is a <u>f.g.</u> <u>algebraic extension</u> of k which is an integral domain. Hence it is a field. Therefore $f^*(\mathfrak{m}) \in \mathrm{Max}(A)$.

③ Exercise. ■

<u>Product (of two algebraic sets)</u>

<u>Lemma</u> Let A and B be two f.g. k-algebras where k is an algebraically closed field. Then

① If A and B are reduced, then $A \otimes_k B$ is reduced.

② If A and B are integral domains, then $A \otimes_k B$ is an integral domain.

Pf. ① Suppose $\sum a_i \otimes b_i$ is nilpotent and b_i's are k-linearly independent. Then for any k-alg. homomorphism $g: A \to k$ we have $\sum g(a_i) b_i \in B$ is nilpotent. Hence $\sum g(a_i) b_i = 0$ as B is reduced, which implies $g(a_i) = 0$ $\forall i$ and $\forall g \in \mathrm{Hom}_{k\text{-alg}}(A, k)$. Therefore, by Nullstellensatz, $a_i = 0$ $\forall i \Rightarrow \sum a_i \otimes b_i = 0$.

② Suppose $(\sum a_i \otimes b_i)(\sum c_i \otimes d_i) = 0$ and b_i's and d_i's are k-linearly independent. Then for any $g \in \mathrm{Hom}_{k\text{-alg}}(A, k)$ we have $(\sum g(a_i) b_i)(\sum g(c_i) d_i) = 0$. As B is an integral domain, either $\sum g(a_i) b_i = 0$ or $\sum g(c_i) d_i = 0$. Thus either $\forall i$, $g(a_i) = 0$ or $\forall i$, $g(c_i) = 0$. So $\forall i, j$, $g(a_i c_j) = 0$. Therefore $\forall i, j$, $a_i c_j = 0$. Since A is an integral domain, either $\forall i$, $a_i = 0$ or $\forall i$, $c_i = 0$. Hence either

$$\sum a_i \otimes b_i = 0 \quad \text{or} \quad \sum c_i \otimes d_i = 0. \qquad \blacksquare$$

Proposition. Let $X \subseteq k^n$ and $Y \subseteq k^m$ be two algebraic subsets. Then

① $X \times Y \subseteq k^{n+m}$ is an algebraic subset.

② $k[X] \otimes k[Y] \to k[X \times Y]$

$f \otimes g \mapsto \phi(f \otimes g)$ where $\phi(f \otimes g)(x, y) := f(x) g(y)$

is an isomorphism.

③ X, Y are irreducible $\Leftrightarrow X \times Y$ is irreducible.

Pf. ① $X \times Y = V(\langle I(X), I(Y)\rangle)$ where

$I(X) \subseteq k[\underline{x_1}, \ldots, \underline{x_n}] \subseteq k[\underline{x_1}, \ldots, \underline{x_n}, \underline{y_1}, \ldots, \underline{y_m}]$ and

$I(Y) \subseteq k[\underline{y_1}, \ldots, \underline{y_m}] \subseteq k[\underline{x_1}, \ldots, \underline{x_n}, \underline{y_1}, \ldots, \underline{y_m}]$.

② (i) ϕ is a well-defined k-algebra homomorphism.

(ii) "monomials" are in the image of ϕ. So ϕ is onto

(iii) Suppose $\phi(\sum f_i \otimes g_i) = 0$ and $g_i \in k[Y]$ are k-linearly independent. So $\forall x \in X, y \in Y$, $\sum f_i(x) g_i(y) = 0 \Rightarrow$ $\sum f_i(x) g_i = 0$ in $k[Y] \Rightarrow f_i(x) = 0 \ \forall x \in X \ \underline{\forall i}$ $\Rightarrow f_i = 0 \ \forall i \Rightarrow \sum f_i \otimes g_i = 0$.

③ • X, Y : irreducible $\Rightarrow k[X]$ and $k[Y]$ are integral domains

$\Rightarrow k[X] \otimes k[Y]$ is an integral domain

$\Rightarrow k[X \times Y]$ is an integral domain

$\Rightarrow X \times Y$ is irreducible.

• $pr: X \times Y \to X$ and Y are continuous (why ?)

So, if $X \times Y$ is irreducible, then X and Y are irreducible. ■