Affine algebraic groups: def, morphism, ker, image.

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<u>Def</u>. An algebraic set $G \subseteq k^n$ is called an <u>affine algebraic group</u> if $g \mapsto g^{-1}$ and $(g_1, g_2) \mapsto g_1 \cdot g_2$ give two morphisms (of algebraic sets) $i: G \to G$ and $m: G \times G \to G$, respectively.

<u>Lemma</u>. If G is an affine algebraic group, then $k[G]$ is a Hopf algebra.

<u>Pf</u>. Let $i^*: k[G] \to k[G]$ and $m^*: k[G] \to k[G \times G] \simeq k[G] \otimes k[G]$ and

$e^*: k[G] \to k$ be the associated k-alg. homomorphism.

Since (G, e, i, m) is a group, one can check all the Hopf assumptions of $k[G]$. ∎

<u>Corollary</u>. Any affine alg. group $G \subseteq k^n$ defines a functor from k-algebras to groups

$$A \longmapsto G(A) := \mathrm{Hom}_{k\text{-alg}}(k[G], A),$$

and $G(k) \simeq G$.

Pf. Exercise. ∎

<u>Prop</u>. Let G be an affine algebraic group. Then

① There is a unique irreducible component G° of G which contains e.

G° is a closed, normal subgroup of finite-index in G.

② Cosets of G° are connected components of G.

③ If H is a closed subgroup of finite-index in G, then $G^\circ \subseteq H$.

Pf. ① Suppose X and Y are irreducible components of G which contain e.

$X \subseteq G$ irredu. comp. $\Rightarrow \overline{X}$ is irre $\Rightarrow \overline{X} = X$ is affine and irre.

Similarly Y is affine and irreducible. So $X \times Y$ is irreducible.

$\Rightarrow m(X \times Y) = X \cdot Y$ is irreducible.

$\Rightarrow \overline{X \cdot Y}$ is irreducible.

$$\Rightarrow \left.\begin{array}{l} X \subseteq \overline{X \cdot Y} \\ Y \subseteq \overline{X \cdot Y} \\ \overline{X \cdot Y} : \text{irred.} \end{array}\right\} \Rightarrow X = Y = \overline{X \cdot Y} \Rightarrow \left\{\begin{array}{l} \text{ⓐ There is a unique} \\ \text{irreducible comp. } G^\circ \\ \text{which contains } e. \\ \text{ⓑ } G^\circ = \overline{G^\circ} \\ \text{ⓒ } G^\circ \text{ is a semigp.} \end{array}\right.$$

$$\bullet \left.\begin{array}{l} i(G^\circ) \text{ is irreducible} \\ e \in i(G^\circ) \end{array}\right\} \Rightarrow \underbrace{i(G^\circ) \subseteq G^\circ \Rightarrow G^\circ \subseteq i(G^\circ)}_{\Downarrow}$$

$$i(G^\circ) = G^\circ$$

$\Rightarrow G^\circ$ is a subgroup.

• $c_g : G \to G$, $c_g(x) = g x g^{-1}$ is a morphism (of algeb. sets)

$$\left.\begin{array}{l} \Rightarrow c_g(G^\circ) \text{ is irred.} \\ e \in c_g(G^\circ) \end{array}\right\} \Rightarrow c_g(G^\circ) \subseteq G^\circ \quad \left.\begin{array}{l} \\ \text{Similarly } c_{g^{-1}}(G^\circ) \subseteq G^\circ \end{array}\right\} \Rightarrow$$

$c_g(G^\circ) = G^\circ \Rightarrow G^\circ$ is a normal subgp of G.

$$\left.\begin{array}{l} \cdot\ G = \bigcup g G^\circ \text{ and} \\ m(\{g\} \times G^\circ) = g G^\circ \text{ is irredu.} \\ G \text{ is Noeth.} \end{array}\right\} \Rightarrow G = \bigsqcup_{i=1}^{m} g_i G^\circ \Rightarrow G^\circ \text{ is of finite-index.}$$

(2) $G^\circ = G \setminus \bigsqcup_{\substack{i=1 \\ g_i \notin G^\circ}}^{m} g_i G^\circ \Rightarrow G^\circ$ is open.

$$\left.\begin{array}{l} \Rightarrow G^\circ \text{ is open and closed} \\ G^\circ \text{ is irreducible} \end{array}\right\} \Rightarrow G^\circ \text{ is a connected comp. of } G.$$

And $g_i G^\circ$ are disjoint. So $\{g_i G^\circ\}$ are connected comp. of G.

(3) $\underset{\substack{\text{closed} \\ \text{f.i.}}}{H^\circ} \subseteq \underset{\substack{\text{closed} \\ \text{f.i.}}}{H} \subseteq G \Rightarrow \underset{\substack{\text{closed} \\ \text{f.i}}}{H^\circ} \subseteq G \Rightarrow H^\circ$ is open and closed and irreducible

$\Rightarrow G^\circ = H^\circ \Rightarrow G^\circ \subseteq H$. ■

$\boxed{Q}$ Let $G \subseteq k^n$ be an affine algebraic group, and $H \leq G$ be an abstract subgroup of finite-index (k is algebraically closed).

Can we deduce H is a Zariski-closed subgroup and contains G°?

(Due to Francois).

<u>Outline of a solution</u> (We have not covered all the needed tools.)

<u>Step 1.</u> If $G^\circ \subseteq H$, then H is a finite union of cosets of G°. And so H is Zariski-closed.

<u>Step 2.</u> $H \cap G^\circ$ is a subgroup of finite index in G°. So it is enough to show:

$$\left.\begin{array}{l} k^n \supseteq G : \text{affine algebraic group; Zariski-connected} \\ \mathbf{H} \leq G \ \text{an abstract subgroup of finite-index} \end{array}\right\} \overset{?}{\Rightarrow} G = H$$

$\boxed{2a}$ Suppose G is Zariski-connected. Show that

$$g \mapsto g^m \text{ is a dominant map if } m \in \mathbb{Z} \setminus \{0\}$$

(Its image is Zariski-dense.)

$\boxed{2b}$ Using (2a) and Chevalley's theorem conclude H contains a non-empty open subset of G. Then deduce $H = G$. $\square$

We often use algebraic groups to study (abstract) linear groups. The following is the connection:

Lemma. Let H be an abstract subgroup of an affine algebraic group $G \subset k^n$. Let $\overline{H}$ be the Zariski-closure of H in G. Then $\overline{H}$ is a subgroup of G. (And so it is an affine algebraic subgroup of G)

<u>Pf</u>. $\forall h \in H$, $H \subseteq h\overline{H}$ and $h\overline{H}$ is closed $\Rightarrow$ $\overline{H} \subseteq h\overline{H}$.

$$\Rightarrow \forall h \in H,\ h^{-1}\overline{H} \subseteq \overline{H} \qquad \Rightarrow \qquad H\overline{H} = \overline{H}.$$

- So $\forall x \in \overline{H}$, $\quad Hx \subseteq \overline{H} \Rightarrow H \subseteq \overline{H}\cdot x^{-1} \Rightarrow \overline{H} \subseteq \overline{H}\cdot x^{-1}$

$$\Rightarrow \overline{H}x \subseteq \overline{H} \Rightarrow \overline{H}\cdot\overline{H} = \overline{H}.$$

- $H \subseteq \overline{H}^{-1} \Rightarrow \overline{H} \subseteq \overline{H}^{-1} \Rightarrow \overline{H} = \overline{H}^{-1}$. ■

Our next task is proving an isomorphism theorem. So we need to

① define a morphism of affine algebraic groups.

② Understand $\ker \phi$ and $\operatorname{Im} \phi$.

③ Understand the quotient space

④ Prove an isomorphism theorem.

We will see that this is NOT a trivial task and we need more tools from algebraic geometry.

<u>Def</u>. Let $G_1 \subseteq k^{n_1}$ and $G_2 \subseteq k^{n_2}$ be two affine algebraic groups.

$\phi: G_1 \to G_2$ is called an affine alg. gp. homomorphism if

① ϕ is a morphism of algebraic sets.

② ϕ is a group homomorphism.

Def. ① [Kernel of affine algebraic group morphism]

$\phi : G_1 \longrightarrow G_2$ affine algebraic group morphism.

Let $\ker(\phi) := \{ g \in G_1 \mid \phi(g) = e \}$.

② [Kernel of a morphism between two group functors]

Let G_1 and G_2 be two functors from k-algebras to groups. $G_1 \xrightarrow{\phi} G_2$ is a morphism if for any k-algebra A, we get natural homomorphisms $G_1(A) \xrightarrow{\phi(A)} G_2(A)$. Then $\ker(\phi)$ is the following group functor:

$$\ker(\phi)(A) := \ker\left(G_1(A) \xrightarrow{\phi(A)} G_2(A)\right).$$

Remark.

- If $\phi : G_1 \longrightarrow G_2$ is an affine alg. gp homomorphism, then it induces a homomorphism for the associated group functors. But the functor given by $\ker(\phi)$ as an affine alg. gp. homo. is NOT the same as the group functor $\ker(\phi)$.

The latter is called the scheme theoretic kernel of ϕ. Over fields of char. 0 they are the same.

$\ker(\phi)$ is again a representable group functor, but the associated ring is not smooth in general.

<u>Ex.</u> Let k be an alg. closed field of char. p.

- Let $\phi : SL_p(k) \longrightarrow GL(M_p(k))$,

$$\phi(g)(X) := gXg^{-1}.$$

Then
$$\begin{aligned} g \in \ker(\phi) &\iff gXg^{-1} = X \quad \forall X \in M_p(k) \\ &\iff g = cI_p \text{ and } \det g = 1 \\ &\iff g = cI_p \text{ and } c^p = 1. \\ &\iff g = cI_p \text{ and } (c-1)^p = 0 \\ &\iff g = I_p. \end{aligned}$$

So $\ker(\phi) = \{I\}$ (as an affine alg. gp. homomo.)

- For any k-algebra A, ϕ induces

$$\phi_A : SL_p(A) \longrightarrow GL(M_p(A))$$
$$\phi_A(g)(X) = gXg^{-1}.$$

Then as above

$$g \in \ker \phi_A \iff g = cI_p \text{ and } c^p = 1.$$

So $\underbrace{(\ker \phi)}_{\substack{\text{as kernel of} \\ \text{group functors}}}(A) = \mu_p(A) := \{c \in A \mid c^p = 1\}$

$= \mathrm{Hom}_{k\text{-alg.}}(k[t]/\langle t^p - 1\rangle, A)$

which is a non-trivial, representable group functor.

<u>What goes wrong?</u> $k[t]/\langle t^p-1\rangle = k[t]/\langle (t-1)^p\rangle$ is NOT reduced. So the k-points of this group scheme loses lots of information.

<u>Remark</u>. We will work with affine alg. gp. homo. and, because of the problems that can arise as above, we get an isomorphism theorem with some additional geometric conditions on ϕ.

<u>Lemma</u>. $\ker \phi$ is a closed subgroup of G_1, and so it is an affine algebraic subgp of G_1.

<u>Pf</u>. $\ker \phi = \phi^{-1}(e)$. ∎

To understand the image we need the following special case of Chevalley's theorem.

<u>Theorem</u>. Let $X \subseteq k^n$ and $Y \subseteq k^m$ be two alg. sets and $\phi : X \to Y$ is a morphism. Then $\phi(X)$ contains a

non-empty open subset of $\overline{\phi(X)}$.

<u>Pf.</u> Changing Y to $\overline{\phi(X)}$, we can assume $\phi(X)$ is dense in Y. Let X_i's be irreducible components of X. So $X = \bigcup_{i=1}^{s} X_i$, $\phi(X) = \bigcup_{i=1}^{s} \phi(X_i)$ and $Y_i := \overline{\phi(X_i)}$ are irreducible algebraic sets. Droping some of, if needed, we can assume Y_i's are irreducible components of Y.

If $\exists\, \emptyset \neq O_i \subseteq \phi(X_i)$ which is open in Y_i, then $\exists$ $\tilde{O}_i \subseteq Y$ open s.t. $\tilde{O}_i \cap Y_i = O_i \Rightarrow$

$$\emptyset \neq \Big(\underbrace{\tilde{O}_i \setminus \bigcup_{j\neq i} Y_j}_{\tilde{U}_i \text{ open in } Y}\Big) \cap Y_i =: U_i \subseteq O_i \subseteq \phi(X_i)$$

$$\Rightarrow \emptyset \neq \bigcup U_i = \bigcup \big(\tilde{U}_i \cap Y_i\big) = \bigcup \big(\tilde{U}_i \cap Y\big) = \big(\bigcup \tilde{U}_i\big) \cap Y \subseteq \phi(X)$$

is open in Y.

So we can assume X and Y are irreducible and $\phi(X)$ is dense. So $\phi^*: k[Y] \longrightarrow k[X]$ is injective (why?), and $k[Y]$ and $k[X]$ are integral domains.

Recall a theorem from 200C

$$\left.\begin{array}{l} A \subseteq B \ \text{integral domains} \\ B : \text{f.g. } A\text{-alg.} \\ k : \text{alg. closed field.} \end{array}\right\} \Rightarrow \begin{array}{l} \forall\, b \in B,\ \exists\, 0 \neq a \in A,\ \forall \\ y : A \to k,\ y(a) \neq 0 \\ \text{can be extended to} \\ \tilde{y} : B \to k,\ \tilde{y}(b) \neq 0 \end{array}$$

By the above result $\exists\, a \in k[Y]$ s.t.

for any $y \in Y := \operatorname{Hom}_{k\text{-alg}}(k[Y], k)$ s.t. $a(y) \neq 0$

can be extended to

$$\tilde{y} : k[X] \to k \text{ . i.e.}$$

$\tilde{y}(\phi^*(f)) = f(y)$ for any $f \in k[Y]$.

$\tilde{y}$ gives us a point x in X and $\tilde{y}(\phi^*(f)) = \phi^*(f)(x)$

$\Rightarrow \phi^*(f)(x) = f(y) \Rightarrow f(\phi(x)) = f(y) \quad \forall f \in k[Y]$

$\Rightarrow y = \phi(x)$.

So $a(y) \neq 0 \Rightarrow y \in \operatorname{Im}\phi$.

$$\hookrightarrow \underbrace{Y \setminus V(a)}_{\text{open in } Y} \subseteq \operatorname{Im}\phi .$$

∎

Proposition. Let $\phi : G_1 \to G_2$ be a morphism of two affine alg.

groups. Then

① $\mathrm{Im}\,\phi := \phi(G_1)$ is a closed subgp of G_2.

② $\phi(G_1^\circ) = \phi(G_1)^\circ$

Pf. By Chevalley's theorem, $\phi(G_1^\circ)$ has a non-empty open $\mathcal{O}$ subset of $\overline{\phi(G_1^\circ)}$. Since G_1° is irreducible, $\phi(G_1^\circ)$ and $\overline{\phi(G_1^\circ)}$ are irreducible. So any two non-empty open subsets intersect. $\Rightarrow \forall g \in \overline{\phi(G_1^\circ)}$, $g\mathcal{O} \cap \mathcal{O} \neq \emptyset$

$$\Rightarrow g \in \mathcal{O}\mathcal{O}^{-1} \subseteq \phi(G_1^\circ)$$

Hence $\overline{\phi(G_1^\circ)} = \phi(G_1^\circ)$. Therefore $\phi(G_1^\circ)$ is a closed irreducible subgroup of G_2. And

$$\phi(G_1) = \bigcup_{i=1}^{s} \phi(g_i)\,\phi(G_1^\circ) \quad \text{if} \quad G_1 = \bigcup_{i=1}^{s} g_i\, G_1^\circ.$$

Hence $\phi(G_1)$ is a closed subgp of G_2.

Since $[\phi(G_1) : \phi(G_1^\circ)] < \infty$, $\phi(G_1^\circ)$ is a closed irred. subgp, we get $\phi(G_1)^\circ = \phi(G_1^\circ)$. ∎