Background on AG: sheaf of regular functions; varieties

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To understand quotient spaces G/H, orbits, etc., we need to go beyond algebraic sets. To do so

1st. We go "local": get algebraic structures on open sets.

2nd. We "glue" finitely many of algebraic sets according to the additional local data.

<u>Def.</u> • Let $X \subseteq k^n$ be an algebraic set and U be a nbhd of $x_0 \in X$. We say a function $f: U \to k$ is regular at x_0 if

$\exists\, V \subseteq U$ open nbhd of x_0 and $g, h \in k[X]$ s.t.

① $\forall x \in V$, $h(x) \neq 0$

② $\forall x \in V$, $f(x) = g(x)/h(x)$.

• We say $f: U \to k$ is regular if $\forall x \in U$, f is regular at x. Let $\mathcal{O}_X(U)$ be the set of all regular k-valued functions on U.

<u>Lemma</u>. • $\mathcal{O}_X(U)$ is a k-algebra

• If $U_1 \overset{\text{open}}{\subseteq} U_2$, then $f \mapsto f|_{U_1}$ induces a k-alg. injection $\mathcal{O}_X(U_2) \hookrightarrow \mathcal{O}_X(U_1)$

$\big|_{U_1}$

injection $\quad \mathcal{O}_X(U_2) \hookrightarrow \mathcal{O}_X(U_1)$

Pf. (Sheaf). $U = \bigcup_{i \in I} U_i$ is an open covering

$f_i \in \mathcal{O}_X(U_i)$ and $f_i\big|_{U_i \cap U_j} = f_j\big|_{U_i \cap U_j}$, $\forall i, j$,

then $\exists!\ f \in \mathcal{O}_X(U)$ s.t.

$$f\big|_{U_i} = f_i .$$

Def. For any topological space X, a sheaf of k-valued functions on X is $U \to \mathcal{O}_X(U)$ where $\mathcal{O}_X(U)$'s satisfy the above lemma.

Lemma. Suppose X is algebraic set, and U is a nbhd of x_0. Then $f : U \to k$ is regular at x_0 if $\exists\, g, h \in k[X]$ s.t.

① $h(x_0) \neq 0$

② either $f(x) = g(x)/h(x)$ or $h(x) = 0$.

Pf. By def., $\exists$ a nbhd V of x_0 and $g_1, h_1 \in k[X]$ s.t.

$D_{h_1} := X \setminus V(h_1) \supseteq V = X \setminus V(\mathfrak{a})$ and $f(x) = g_1(x)/h_1(x)$ $\forall x \in V$.

$x_0 \notin V(\mathfrak{a}) = V(\langle f_1, \dots, f_m \rangle) \Rightarrow \exists j,\ x_0 \notin V(f_j)$

$$\Rightarrow D_{h_1} \supseteq V \supseteq X \setminus V(f_j) =: D_{f_j} \ni x_0$$

$$\Rightarrow \quad D_{h_1} \supseteq V \supseteq X \setminus V(f_j) =: D_{f_j} \ni x_0$$

$$\Rightarrow \quad V(h_1) \subseteq V(f_j)$$

$$\Rightarrow \quad \sqrt{\langle h_1 \rangle} \supseteq \sqrt{\langle f_j \rangle}$$

$$\Rightarrow \quad f_j^m = h_1 \cdot h_2$$

So $f(x) = g_1(x)\, h_2(x) \Big/ h_1(x)\, h_2(x) = g_1(x)\, h_2(x) \Big/ f_j^m(x)$

$\forall x \in V \supseteq D_{f_j} \Longrightarrow$ $\boxed{f_j^m(x)\ f(x) = g_1(x)\, h_2(x) \quad \forall x \in D_{f_j},\ \text{and } x_0 \in D_{f_j}.}$ ∎

<u>Def.</u> The stalk of $\mathcal{O}_X$ at x_0 is $\mathcal{O}_{X,x_0} := \varinjlim \mathcal{O}_X(U)$ i.e.

$[f, U]$ where $f \in \mathcal{O}_X(U)$, $x_0 \in U$, and

$$(f_1, U_1) \sim (f_2, U_2) \Longleftrightarrow \exists\ x_0 \in U \subseteq U_1 \cap U_2 \text{ s.t. } f_1|_U = f_2|_U.$$

(It is a kind of union!)

<u>Thm</u>. Let X be an algebraic set. Then

ⓐ $\forall x_0 \in X$, $\mathcal{O}_{X,x_0} \simeq k[X]_{\mathfrak{m}_{x_0}}$.

ⓑ $\mathcal{O}_X(X) = k[X]$.

<u>Pf.</u> ⓐ Let $\phi : k[X]_{\mathfrak{m}_{x_0}} \longrightarrow \mathcal{O}_{X,x_0}$ be

$$\phi(g/h) := [g/h, D_h].$$

① It is well-defined.

$g_1/h_1 = g_2/h_2 \Rightarrow \exists\, s \notin \mathfrak{m}_{x_0}$ s.t. $s(h_2 g_1 - h_1 g_2) = 0$

$\Rightarrow s(x_0) h_1(x_0) h_2(x_0) \neq 0$ and in $D_{sh_1h_2}$ we have

$g_1(x)/h_1(x) = g_2(x)/h_2(x) \mapsto [g_1/h_1, D_{h_1}] = [g_2/h_2, D_{h_2}]$.

② k-alg. homo. is clear.

③ Injective: $[g/h, D_h] = 0 \Rightarrow$

$\exists\, x_0 \in U$ nbhd, $\forall x \in U$, $g(x) = 0$ and $h(x) \neq 0$.

$\Rightarrow \exists\, h_1 \in k[X] \setminus \mathfrak{m}_{x_0}$ s.t. $g(x) h_1(x) = 0 \quad \forall x \in X$.

$\Rightarrow g/h = gh_1/hh_1 = 0$.

④ Surjective:

$[f, U] = [g/h, D_h]$ (by the above lemma)

ⓑ $\phi: k[X] \longrightarrow \mathcal{O}_X(X)$, $\phi(f) = f$.

The only question is why ϕ is surjective.

$f \in \mathcal{O}_X(X) \Rightarrow \forall x \in X$, $\exists\, g_x^{(1)}, h_x^{(1)} \in k[X]$ s.t.

① $h_x^{(1)}(x) \neq 0$

② $\forall x \in X$, either $f(x) = g_x^{(1)}(x)/h_x^{(1)}(x)$ or $h_x^{(1)}(x) = 0$.

$\Rightarrow {h_x^{(1)}}^2(x) f(x) = h_x^{(1)}(x)\, g_x^{(1)}(x)$

So $\exists\, h_x, g_x \in k[X]$ s.t.

① $h_x(x) \neq 0$ ② $h_x(x) f(x) = g_x(x) \quad \forall x \in X$.

So $\exists\, x_1, \dots, x_m \in X$ and $q_1, \dots, q_m \in k[X]$ s.t.

$$\sum q_i h_{x_i} = 1 \in k[X].$$

$$\Rightarrow \sum_i q_i(x) h_{x_i}(x) f(x) = \sum_i q_i(x)\, g_{x_i}(x)$$

$$\Rightarrow f(x) = \sum_i q_i(x)\, g_{x_i}(x) \in \phi(k[X]). \quad \blacksquare$$

Def. A prevariety is a topological space X with a sheaf of k-valued functions $\mathcal{O}_X$; with an open covering X_i s.t. $(X_i, \mathcal{O}_X|_{X_i})$ is an algebraic set with its sheaf of regular k-valued functions.

- Let X and Y be two prevarieties.

  $\phi : X \to Y$ is called a morphism if

  ① It is continuous

  ② $\forall\, U \subseteq Y$ open, $f \in \mathcal{O}_Y(U)$, $f \circ \phi \in \mathcal{O}_Y(\phi^{-1}(U))$.

  [k-alg. homo. $\mathcal{O}_Y(U) \xrightarrow{\phi^*} \mathcal{O}_Y(\phi^{-1}(U))$.]

- A variety is a prevariety X s.t.

$$\Delta_X := \{(x,x) \mid x \in X\} \text{ is closed in } X.$$

(Separation axiom)

Ex. Give an example of a prevariety which is NOT a variety.

Ex. A product of two varieties exist and is unique up to isomorphism.

The following lemma shows why the separation axiom is important:

Lemma. X : variety ; Y: prevariety.

① $Y \xrightarrow{\phi} X$ morphism $\Rightarrow$ $\Gamma_\phi := \{(y, \phi(y)) \mid y \in Y\}$ is closed in $Y \times X$.

② $Y \xrightarrow{\psi, \phi} X$ morphisms $\Rightarrow$ $\{y \in Y \mid \phi(y) = \psi(y)\}$ is closed.

Pf. ① $\begin{array}{rcl} Y \times X & \xrightarrow{\Phi} & X \times X \\ (y, x) & \longmapsto & (\phi(y), x) \end{array}$ a morphism $\Big\} \Rightarrow \Phi^{-1}(\Delta_X) = \{(y, \phi(y)) \mid y \in Y\}$ is closed.

② $\begin{array}{rcl} Y & \xrightarrow[\Theta]{} & X \times X \\ y & \longmapsto & (\phi(y), \psi(y)) \end{array}$ $\Big\} \Rightarrow \Theta^{-1}(\Delta_X) = \{y \mid \phi(y) = \psi(y)\}$ is closed. ∎

Propos. Let X be an algebraic set, U be a non-empty open subset, and $\mathcal{O}_X$ be the sheaf of k-valued regular functions on X. Then $(U, \mathcal{O}_X|_U)$ is a variety. [These are called quasi-affine varieties.]

Pf. • For any $x \in U$, $\exists f_x \in k[X]$ s.t. $x \in D_{f_x} \subseteq U$

$$\Rightarrow V(\langle f_x \mid x \in U \rangle) \subseteq X \setminus U = V(\underbrace{\langle g_1, \dots, g_m \rangle}_{\text{radical ideal}})$$

$$\Rightarrow \sqrt{\langle f_x \rangle_{x \in U}} \supseteq \langle g_1, \dots, g_m \rangle$$

$$\Rightarrow \exists x_i \in U \text{ s.t. } \sqrt{\langle f_{x_i} \mid 1 \le i \le k \rangle} \supseteq \langle g_1, \dots, g_m \rangle.$$

$$\Rightarrow V(\langle f_{x_i} \mid 1 \le i \le k \rangle) \subseteq X \setminus U$$

$$\Rightarrow \left.\begin{array}{l} \bigcup D_{f_{x_i}} \supseteq U \\ D_{f_{x_i}} \subseteq U \end{array}\right\} \Rightarrow U = \bigcup D_{f_{x_i}} \text{ is a finite open cover.}$$

• $(D_f, \mathcal{O}_X|_{D_f}) \simeq (X_f, \mathcal{O}_{X_f})$ where X_f is the algebraic set $\{(x, a) \in k^n \times k \mid \forall g \in I(X),\ g(x) = 0,\ f(x) \cdot a = 1\}$.

Pf. • $\phi : D_f \to X_f$, $\phi(x) := (x, 1/f(x))$.

• ϕ is a bijection (?)

• $\phi^{-1} = pr_X|_{X_f}$ is continuous. (?)

• $C \subseteq X_f$ closed $\Rightarrow C = V(f_i(x, a))$

$$\Rightarrow \phi^{-1}(C) := \{x \in D_f \mid f_i(x, 1/f(x)) = 0 \ \forall i \in I\}$$

$$\boxed{f_i(x, 1/f(x)) = \frac{g_i(x)}{f(x)^{n_i}} \text{ for some } g_i \in k[X]}$$

$$\Rightarrow \phi^{-1}(C) = \{x \in D_f \mid g_i(x) = 0 \ \forall i \in I\}$$

closed. $\Rightarrow$ ϕ is continuous.

• Let U be a non-empty open subset of X_f. So $X_f \setminus U = V(f_i(x,a))$. Let $g_i \in k[X]$ be s.t. $f_i(x, 1/f(x)) = g_i(x)/f(x)^{n_i}$. Then $\phi^{-1}(U) = X \setminus V(f, g_i; i \in I)$.

If $r \in \mathcal{O}_{X_f}(U)$, then $\forall (x, 1/f(x)) \in U$, there are $g_x, h_x \in k[X_f]$ s.t.

① $h_x(x, 1/f(x)) \neq 0$

② $\forall (x', a') \in U$, $h_x(x',a')\, r(x',a') = g_x(x',a')$. ⊛

There are $\bar{g}_x, \bar{h}_x \in k[X]$ s.t. $g_x(x', 1/f(x')) = \dfrac{\bar{g}_x(x')}{f(x')^N}$

and $h_x(x', 1/f(x')) = \dfrac{\bar{h}_x(x')}{f(x')^N}$ for any $x' \in D_f$.

$\overset{⊛}{\Rightarrow} \forall x' \in D_f$, $\bar{h}_x(x')\, \phi^*(r)(x') = \bar{g}_x(x')$.

and $\bar{h}_x(x) \neq 0$.

So $\phi^*(r) \in \mathcal{O}_{D_f}(\phi^{-1}(U))$ (as $x \in U$ was arbitrary.)

• Since ϕ is surjective, $\phi^*: \mathcal{O}_{X_f}(U) \to \mathcal{O}_{D_f}(\phi^{-1}(U))$ is injective.

• ϕ^* is clearly a k-alg. homo.

• $\bar{r} \in \mathcal{O}_{D_f}(\phi^{-1}(U)) \Rightarrow \exists\, \tilde{r} \in \mathcal{O}_X(\phi^{-1}(U))$ s.t. $\tilde{r}|_{D_f} = \bar{r}$.

Let $r: U \to k$, $r(x, 1/f(x)) := \tilde{r}(x)$. Since $\tilde{r}$ is regular at any point of $\Phi^{-1}(U)$, r is regular at any point of U. (?)

So $\Phi^*: \mathcal{O}_{X_f}(U) \to \mathcal{O}_{D_f}(\Phi^{-1}(U))$ is surjective.

- Separation axiom. $\Delta_U \subseteq U \times U$ is closed.

The topology on $U \times U$ is induced by $X \times X$ (why ?)

And in $X \times X$, Δ_X is closed (?). So $\Delta_U = U \times U \cap \Delta_X$ is closed in $U \times U$.

∎

Proposition. $\left.\begin{array}{l}(X, \mathcal{O}_X) \text{ variety} \\ U \subseteq X \text{ open}\end{array}\right\} \Rightarrow (U, \mathcal{O}_X|_U)$ is a variety.

Pf. • Let $X = \bigcup X_i$ be an open affine covering. Then $U = \bigcup (U \cap X_i)$ is an open quasi-affine covering, and by the above result $U \cap X_i$ has open affine covering $\bigcup X_{ij}$.

$\Rightarrow U$ is a prevariety.

- $\Delta_U = \Delta_X \cap U \times U$ is closed in $U \times U$ as the topology on $U \times U$ is the induced topology from $X \times X$. ∎

Ex. $i: X \to \Delta_X$, $x \mapsto (x, x)$ is a homeomorphism if X is a prevariety.

Lemma. Suppose X is a variety. Then $(\Delta_X, \mathcal{O}_{X\times X}|_{\Delta_X})$ is a variety and $i: X \to \Delta_X$ is an isomorphism of varieties.

Pf. Ex.

$\boxed{\text{Useful criterion}}$

ⓐ X: variety ; U, V: affine open sets in X.

$\Rightarrow$ • $U \cap V$: affine open set

• $\langle \mathcal{O}_X(U)|_{U\cap V}, \mathcal{O}_X(V)|_{U\cap V} \rangle = \mathcal{O}_X(U\cap V)$.

ⓑ X: prevariety ; $X = \bigcup_{i=1}^{m} X_i$: open affine covering

X : variety $\Leftrightarrow$ $\forall i,j$, $\langle \mathcal{O}_X(X_i)|_{X_i \cap X_j}, \mathcal{O}_X(X_j)|_{X_i\cap X_j} \rangle$

$$= \mathcal{O}_X(X_i \cap X_j)$$

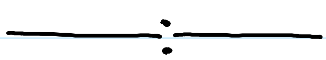
(as k-alg.).

Pf. ⓐ $\mathcal{O}_X(U\cap V) \simeq \mathcal{O}_{\Delta_X}(\Delta_X \cap (U\times V)) \simeq \mathcal{O}_{U\times V}|_{\Delta_X \cap (U\times V)}(\Delta_X \cap (U\times V))$

$\simeq$ restriction of $k[U] \otimes k[V]$ to $\Delta_X \cap (U\times V)$.

$\Rightarrow \mathcal{O}_X(U\cap V) = \langle \mathcal{O}_X(U)|_{U\cap V}, \mathcal{O}_X(V)|_{U\cap V} \rangle$.

(b) ex. $\blacksquare$

Ex.  is NOT a variety.

$\phi_i : k \to X$ and $\phi_i^{-1}(\mathrm{Im}\,\phi_1 \cap \mathrm{Im}\,\phi_2) = k \setminus \{0\}$.

$X_i := \mathrm{Im}\,\phi_i$.

$X = X_1 \cup X_2$ is an open affine cover.

Observe. $\mathcal{O}_X(X_1 \cap X_2) \simeq k[x][1/x]$

$\langle \mathcal{O}_X(X_1)|_{X_1\cap X_2}, \mathcal{O}_X(X_2)|_{X_1 \cap X_2} \rangle \simeq k[x]$.

Projective space $\mathbb{P}(k^{n+1})$

$$\mathbb{P}(k^{n+1}) := \{ [x_0 : x_1 : \cdots : x_n] \mid x_i \in k \, ; \exists i, x_i \neq 0 \}.$$

Let $X_i := \{ [x_0 : x_1 : \cdots : x_n] \in \mathbb{P}(k^{n+1}) \mid x_i \neq 0 \}$.

Then $(x_0, \ldots, \widehat{x_i}, \ldots, x_n) \longmapsto [x_0 : x_1 : \cdots : 1 : \cdots : x_n]$ is a bijection between k^n and X_i. We view $\mathbb{P}(k^{n+1})$ as a prevariety given by open affine covering $\bigcup_{i=0}^{n} X_i$.

Ex. $X \subseteq \mathbb{P}(k^{n+1})$ is closed $\Leftrightarrow$ $\exists$ homogen. poly. f_i s.t.

$$X = \{ [x] \mid f_i(x) = 0 \;\; \forall i \}.$$

. A closed subset of $\mathbb{P}(k^{n+1})$ gives us a variety. This type of variety is called projective variety.

- An open subset of a projective variety gives us a variety. This type of variety is called quasi-projective.

Ex. $\mathbb{A}^n \setminus \{0\}$ is NOT an affine variety if $n \geq 2$.

[Outline: Let $X_i := \mathbb{A}^n \setminus \{\vec{x} \mid x_i = 0\}$. Then $X_i = D_{\underline{x}_i}$ is open affine.

Let $X = \mathbb{A}^n \setminus \{0\}$.

$$f \in \mathcal{O}_X(X) \Rightarrow f|_{X_i} \in \mathcal{O}_X(X_i) = \mathcal{O}_{X_i}(X_i) = k[\underline{x}_1, \ldots, \underline{x}_n][1/\underline{x}_i]$$

$$\Rightarrow f|_{X_1 \cap \cdots \cap X_n} \in \bigcap_{i=1}^{n} k[\underline{x}_1, \ldots, \underline{x}_n][1/\underline{x}_i] \underset{\boxed{n \geq 2}}{=} k[\underline{x}_1, \ldots, \underline{x}_n]$$

Since $X_1 \cap \cdots \cap X_n$ is dense in X, $f \longmapsto f|_{X_1 \cap \cdots \cap X_n}$ is injective.

Hence $\mathcal{O}_X(X) = k[\underline{x}_1, \ldots, \underline{x}_n]$.

If X is affine, then the coordinate ring of X is isomorphic to $\mathcal{O}_X(X) = k[\underline{x}_1, \ldots, \underline{x}_n]$

Notice that, given a reduced f.g. k-algebra $k[\xi_1, \ldots, \xi_m]$, we get the unique (up to isomorphism) algebraic set Y with coordinate ring $k[\xi_1, \ldots, \xi_m]$ as follows:

$$\mathfrak{a} := \ker\left(k[\underline{x}_1, \ldots, \underline{x}_m] \rightarrow k[\xi_1, \ldots, \xi_m]\right) \rightsquigarrow Y \simeq V(\mathfrak{a}).$$

So $X = \mathbb{A}^n$, which is a contradiction. (why?) ∎]

So $X = \mathbb{A}^n$, which is a contradiction. (why?) $\blacksquare$]