Background on AG: Module of differentials

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Def. Let X be a variety. We say $x \in X$ is a simple point if $\dim T_x X = \dim X$.

Alternatively we might say X is smooth at x or X is non-singular at x.

To understand simple points we have to study the module of differentials.

Recall • A: R-algebra. Then

$$A\text{-mod.} \longrightarrow R\text{-mod}.$$
$$M \longmapsto \mathrm{Der}_R(A, M)$$

• $A_1 \xrightarrow{\phi} A_2$ R-algebra homomorphism induces the following exact sequence

$$0 \to \mathrm{Der}_{A_1}(A_2, M) \to \mathrm{Der}_R(A_2, M) \to \mathrm{Der}_R(A_1, M)$$

• $M_1 \xrightarrow{\phi} M_2$ A-mod. Then

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "\phi_*(\delta)"'] & M_1 \arrow[d, "\phi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & M_2 \end{tikzcd}$$

$$, \quad \phi_*(\delta)(a_1 a_2) = \phi(a_1 \delta(a_2) + a_2 \delta(a_1))$$
$$= a_1 \phi_*(\delta)(a_2) + a_2 \phi_*(\delta)(a_1)$$

So we get $\mathrm{Der}_R(A, M_1) \xrightarrow{\phi_*} \mathrm{Der}_R(A, M_2)$.

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and ϕ_* is R-mod. homomorphism.

So it is a functor.

$\boxed{Q}$ Is it a representable functor, i.e. is there an A-mod. Ω

$$M \longmapsto \operatorname{Hom}_A(\Omega, M)$$

is naturally the same as $\operatorname{Der}_R(A, M)$?

<u>Theorem</u>. $\exists !$ (Ω, d) s.t. ① Ω : A-mod.

② $d \in \operatorname{Der}_R(A, \Omega)$

and ③ (universal property)

$\forall$ M: A-mod,

$\delta \in \operatorname{Der}_R(A, M)$

$\exists$ $\phi_\delta \in \operatorname{Hom}_A(\Omega, M)$ s.t.

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "\delta"'] & \Omega \arrow[d, dashed, "\phi_\delta"] \arrow[dl, phantom, "\circlearrowleft"] \\ & M \end{tikzcd}$$

And
$$\operatorname{Hom}_A(\Omega, M) \longrightarrow \operatorname{Der}_R(A, M)$$
$$\phi \longmapsto \phi_*(d)$$

is an R-mod. isomorphism.

One can use the above universal property to construct Ω.

But we will take another route here:

Let $A \otimes_R A \xrightarrow{m} A$ be the multiplication homomorphism.
$a \otimes b \longmapsto ab$

Let $I := \ker(m)$, and $\Omega_{A/R} := I/I^2$.

Clearly $\Omega_{A/R}$ is an $A \otimes_R A$-mod. Notice that I annihilates $\Omega_{A/R}$. So $\Omega_{A/R}$ is an A-mod.

<u>Lemma</u>. I is generated by $a \otimes 1 - 1 \otimes a$.

<u>Pf</u>. Let $J = \langle a \otimes 1 - 1 \otimes a \mid a \in A \rangle$. It is clear that $J \subseteq I$.

Suppose $\sum_{i=1}^{n} a_i b_i = 0$. Then

$$\left(\sum_{i=1}^{n} a_i \otimes b_i\right) + J = \left(\sum_{i=1}^{n} a_i b_i\right) \otimes 1 + J = J$$

$$\boxed{\begin{array}{l} \text{in } A \otimes A/J \text{ we have } \overline{b \otimes 1} = \overline{1 \otimes b} \\ \text{and so } \overline{a \otimes b} = \overline{ab \otimes 1}. \end{array}}$$

$\Rightarrow \sum_{i=1}^{n} a_i \otimes b_i \in J$. So $I \subseteq J$. ■

Let $d : A \longrightarrow \Omega_{A/R}$, $da := (a \otimes 1 - 1 \otimes a) + I^2$.

So by the above lemma, $\Omega_{A/R}$ is generated by <u>da</u> as an A-mod.

<u>Lemma</u>. $d : A \longrightarrow \Omega_{A/R}$, $da := (a \otimes 1 - 1 \otimes a) + I^2$ is an R-derivative.

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<u>Pf.</u> • H is clearly R-mod. homomorphism.

• $d(a_1a_2) = [a_1a_2 \otimes 1 - 1 \otimes a_1a_2]$

$$a_1\, da_2 + a_2\, da_1 = a_1[a_2 \otimes 1 - 1 \otimes a_2] + a_2[a_1 \otimes 1 - 1 \otimes a_1]$$

$$= [a_1a_2 \otimes 1 - a_1 \otimes a_2] + [a_1 \otimes a_2 - 1 \otimes a_1a_2]$$

$$= [a_1a_2 \otimes 1 - 1 \otimes a_1a_2]. \quad \blacksquare$$

<u>Lemma</u>. $(\Omega_{A/R}, d)$ satisfies the universal property.

<u>Pf.</u> Let $\delta : A \to M$ be an R-derivation. To define ϕ_δ, we look at the following diagram

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "\delta"'] & \Omega_{A/R} \arrow[d, dashed] \arrow[dl, phantom, "\circlearrowleft"] \\ & M \end{tikzcd}$$

So $\phi_\delta(da)$ should be $\delta(a)$.

We have to show $\phi_\delta(da) := \delta(a)$ is well-defined and A-linear. So we have to show $\phi_\delta(a_1\, da_2) := a_1\, \delta(a_2)$ is well-defined (and R-bilinear on (a_1, a_2)).

Let $\Psi : A \otimes A \to M$, $\Psi(a_1 \otimes a_2) := a_2 \phi(a_1)$. Using properties of tensor prod. we can show Ψ is a well-defined A-mod. homomorphism (A acts on the 2^{nd} factor of $A \otimes A$).

<u>Claim</u>. $\Psi(I^2) = 0$.

Pf. $\Psi((a_1\otimes 1 - 1\otimes a_1)(a_2\otimes 1 - 1\otimes a_2))$

$$= \Psi(a_1a_2\otimes 1 - a_2\otimes a_1 - a_1\otimes a_2 + 1\otimes a_1a_2)$$

$$= \underbrace{a_1a_2\,\delta(1)}_{0} - \underbrace{a_2\,\delta(a_1) - a_1\,\delta(a_2) + \delta(a_1a_2)}_{0}$$

$$= 0. \quad \blacksquare$$

So Ψ induces an A-mod. homomorphism $\phi_\delta : I/I^2 = \Omega_{A/R} \longrightarrow M$,

and $\phi_\delta(da) = \delta(a) - a\,\delta(1) = \delta(a)$.

So $\mathrm{Hom}_A(\Omega_{A/R}, M) \longrightarrow \mathrm{Der}_R(A, M)$ is surjective.

$$\phi \longmapsto \phi_*(d)$$

<u>Why is it injective?</u>

$$\phi_*(d) = 0 \Rightarrow \phi(da) = 0 \quad \forall a \in A$$

$$\Rightarrow \phi = 0 \quad \text{as } \Omega_{A/R} \text{ is generated by } da\text{'s}.$$

$\blacksquare$

<u>Lemma</u> $(\Omega_{A/R}, d)$ is the unique pair which satisfies the mentioned properties in the above theorem.

Pf. Let (Ω, δ) be another such pair. Then by universal properties $\exists\ \phi_d \in \mathrm{Hom}_A(\Omega_{A/R}, \Omega)$ and $\phi_\delta \in \mathrm{Hom}_A(\Omega, \Omega_{A/R})$ s.t. $\circledast$ is a commutative diag.

$$\circledast \qquad \begin{tikzcd} & \Omega_{A/R} \arrow[d, "\phi_d"] \\ A \arrow[ur, "d"] \arrow[r, "\delta"] \arrow[dr, "d"'] & \Omega \arrow[d, "\phi_\delta"] \\ & \Omega_{A/R} \end{tikzcd}$$

$\Rightarrow$ by universal property of $\Omega_{A/R}$

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$\Phi_\delta \circ \Phi_d = id_{\Omega_{A/R}}$. Similarly, $\Phi_d \circ \Phi_\delta = id_{\Omega}$. So we get the uniqueness. ∎

<u>Can we compute $\Omega_{A/R}$ in certain examples?</u>

<u>Lemma</u>. $\Omega_{R[\underline{x}_1,\dots,\underline{x}_n]/R} = R[\underline{x}_1,\dots,\underline{x}_n]\, d\underline{x}_1 \oplus \cdots \oplus R[\underline{x}_1,\dots,\underline{x}_n]\, d\underline{x}_n$.

And $\forall f \in R[\underline{x}_1,\dots,\underline{x}_n]$, $df = \partial_1 f\, d\underline{x}_1 + \partial_2 f\, d\underline{x}_2 + \cdots + \partial_n f\, d\underline{x}_n$.

<u>Pf</u>. It is enough to prove that

$(R[\underline{x}_1,\dots,\underline{x}_n] \oplus \cdots \oplus R[\underline{x}_1,\dots,\underline{x}_n], d)$ where

$d : R[\underline{x}_1,\dots,\underline{x}_n] \longrightarrow R[\underline{x}_1,\dots,\underline{x}_n] \oplus \cdots \oplus R[\underline{x}_1,\dots,\underline{x}_n]$,

$d(f) := (\partial_1 f, \dots, \partial_n f)$ satisfies the universal property.

Let $\delta : R[\underline{x}_1,\dots,\underline{x}_n] \longrightarrow M$ be an R-derivation.

Let $\Phi_\delta : R[\underline{x}_1,\dots,\underline{x}_n]^n \longrightarrow M$, $\Phi_\delta(f_1,\dots,f_n) := f_1\,\delta(\underline{x}_1) + \cdots + f_n\,\delta(\underline{x}_n)$.

So Φ_δ is an A-mod. homomorphism, and

$$\Phi_\delta(d(f)) = \partial_1 f\ \delta(\underline{x}_1) + \cdots + \partial_n f\ \delta(\underline{x}_n).$$

So $\Phi_\delta(d(\underline{x}_i)) = \delta(\underline{x}_i)$. Since $R[\underline{x}_1,\dots,\underline{x}_n]$ is gener. by $\underline{x}_i$'s, we have $\Phi_\delta \circ d = \delta$. ∎

To go to f.g. R-algebras, first let's observe the following:

- $A \xrightarrow{\Phi} B$ R-algebra homomorphism.

$\Rightarrow$ $0 \longrightarrow I_A \hookrightarrow A \otimes A \longrightarrow A \longrightarrow 0$

$$\Rightarrow \quad \begin{tikzcd} 0 \arrow[r] & I_A \arrow[r, hook] \arrow[d, dashed] & A \otimes A \arrow[r] \arrow[d, "\phi\otimes\phi"] & A \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & I_B \arrow[r, hook] & B \otimes B \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

So we get $\phi^0 : \Omega_{A/R} \longrightarrow \Omega_{B/R}$,

$$\phi^0(d_{A/R}\, a) = d_{B/R}\, \phi(a)$$

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "d"'] & B \arrow[d, "d"] \\ \Omega_{A/R} \arrow[r, "\phi^0"] & \Omega_{B/R} \end{tikzcd}$$

Moreover we get the following commuting diagrams for a B-mod.

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ \mathrm{Hom}_B(\Omega_{B/A}, M) \arrow[r, "\sim"] \arrow[d] & \mathrm{Der}_A(B, M) \arrow[d] \\ \mathrm{Hom}_B(\Omega_{B/R}, M) \arrow[r, "\sim"] \arrow[d, "(\phi^0)^*"'] & \mathrm{Der}_R(B, M) \arrow[d] \\ \mathrm{Hom}_A(\Omega_{A/R}, M) \arrow[r, "\sim"] & \mathrm{Der}_R(A, M) \end{tikzcd}$$

<u>Proposition</u> (Finitely presented algebras)

Let $A = R[\underline{x_1}, \dots, \underline{x_n}] / \langle f_1, \dots, f_m \rangle$ and

$\phi : R[\underline{x_1}, \dots, \underline{x_n}] \longrightarrow A$ be the quotient R-algebra homomorphism. Then

$$0 \longrightarrow \langle \textstyle\sum \partial_i f_j\, d\underline{x_i} \otimes 1 \rangle \longrightarrow \Omega_{R[\underline{x_1}, \dots, \underline{x_n}]/R} \otimes_{R[\underline{x_1}, \dots, \underline{x_n}]} A \longrightarrow \Omega_{A/R} \longrightarrow 0$$

$$\omega \otimes 1 \longmapsto \phi^0(\omega)$$

is an exact sequence.

(Alternatively we can say

$$\psi : A^n \longrightarrow \Omega_{A/R}, \ \psi(a_1, \dots, a_n) = \sum_{i=1}^{n} a_i \, d[x_i]$$

is onto and its kernel is generated by

$$([\partial_1 f_j], \dots, [\partial_n f_j]) \ .)$$

<u>Pf.</u> It is enough to prove

$$\langle ([\partial_1 f_j], \dots, [\partial_n f_j]) \rangle \quad \text{and}$$

$$d : A \longrightarrow \Omega, \ d([f]) := [([\partial_1 f], \dots, [\partial_n f])]$$

satisfy the universal property (justify this).

<u>Why is d well-defined</u> ?

$$[f] = 0 \iff f \in \langle f_1, \dots, f_m \rangle \iff f = \sum_r p_r f_r$$

$$\Rightarrow \partial_i f = \sum_r \partial_i p_r \cdot f_r + p_r \cdot \partial_i f_r$$

$$\Rightarrow [\partial_i f] = \sum_r [\partial_i p_r][f_r] + [p_r] \cdot [\partial_i f_r]$$

$$= \sum_r [p_r][\partial_i f_r]$$

$$\Rightarrow ([\partial_1 f], \dots, [\partial_n f]) = \sum_r [p_r] ([\partial_1 f_r], \dots, [\partial_n f_r])$$

$$\Rightarrow [([\partial_1 f], \dots, [\partial_n f])] = 0 \ .$$

- Check that d is an R-derivation.

- Suppose $\delta : A \longrightarrow M$ is an R-derivation.

Let $\tilde{\Phi}_\delta : A^n \longrightarrow M$ be $\tilde{\Phi}_\delta(a_1, \dots, a_n) := \sum^n a_i \, \delta([x_i])$.

Let $\tilde{\Phi}_\delta : A^n \to M$ be $\tilde{\Phi}_\delta(a_1, \dots, a_n) := \sum_{i=1}^{n} a_i \, \delta([\underline{x}_i])$.

<u>Claim.</u> $\tilde{\Phi}_\delta([\partial_1 f_j], \dots, [\partial_n f_j]) = 0$.

<u>Pf.</u>

We know that $\hat{\Phi}_\delta : R[\underline{x}_1, \dots, \underline{x}_n]^n \to M$,

$$\hat{\Phi}_\delta(g_1, \dots, g_n) = \sum [g_i] \, \delta([\underline{x}_i]) \text{ is a well-defined}$$

$R[\underline{x}_1, \dots, \underline{x}_n]$-mod. homomorphism and $\hat{\Phi}_\delta(\partial_1 f, \dots, \partial_n f) = \delta([f])$.

So $\hat{\Phi}_\delta(\partial_1 f_j, \dots, \partial_n f_j) = \delta([f_j]) = 0$.

$\Rightarrow \tilde{\Phi}_\delta([\partial_1 f_j], \dots, [\partial_n f_j]) = 0$. $\square$

So $\tilde{\Phi}_\delta$ induces an A-mod. homomorphism $\Phi_\delta : \Omega \to M$ s.t.

$\Phi_\delta(a_1, \dots, a_n) = \sum a_i \, \delta([\underline{x}_i])$. In particular,

$$\Phi_\delta(d([\underline{x}_i])) = \delta([\underline{x}_i]). \text{ So } \Phi_\delta \circ d = \delta. \quad \blacksquare$$

<u>Ex.</u> Let $A = k[x]/\langle f(x) \rangle$. Then $\Omega_{A/k} \simeq k[x]/\langle f'(x), f(x) \rangle$

And so $\Omega_{A/k} = 0 \iff \langle f(x), f'(x) \rangle = k[x]$.

When k is a field, it is equ. to say $\gcd(f, f') = 1$.

<u>Lemma.</u> A: R-algebra and integral domain

F: the quotient field of A.

$$\Rightarrow \Omega_{F/R} \simeq \Omega_{A/R} \otimes_A F.$$

<u>Pf.</u> It is enough to check the universal property for

$\Omega_{A/R} \otimes_A F$ and $d : F \to \Omega_{A/R} \otimes F$.

$$d(a_1/a_2) := (a_2\, d(a_1) - a_1\, d(a_2)) \otimes a_2^{-2}.$$

Let $\delta: F \to M$ be an R-derivation. Then $\exists\ \overline{\Phi}_\delta : \Omega_{A/R} \to M$ which is A-mod and $\overline{\Phi}_\delta(da) = \delta(a)$.

Let $\phi_\delta : \Omega_{A/R} \otimes_A F \to M$, $\phi_\delta(\omega \otimes c) := c\, \overline{\Phi}_\delta(\omega)$.

So ϕ_δ is an F-mod homomorphism and

$$\phi_\delta(d(a_1/a_2)) = a_2^{-2}\,(a_2\, \delta(a_1) - a_1\, \delta(a_2))$$
$$= \delta(a_1/a_2). \qquad \blacksquare$$

<u>Corollary</u>. $\dim \Omega_{k(\underline{x}_1, \dots, \underline{x}_n)/k} = n$.

<u>Corollary</u>. $\dim_{k(\alpha)} \Omega_{k(\alpha)/k} \leq 1$ and $\Omega_{k(\alpha)/k} = 0 \iff k(\alpha)/k$ is separably algebraic over k.

Let's focus on field extensions as they are typically easier to study (and birationality condition is about $k(X)$ as we will see later.)

- Suppose $F \subseteq E' \subseteq E$ are field extensions, and E/F and E'/F are of finite type.

Then
$$\begin{array}{ccccccc} 0 \to & \mathrm{Der}_{E'}(E,E) & \to & \mathrm{Der}_F(E,E) & \to & \mathrm{Der}_F(E',E) \\ & \wr\wr & & \wr\wr & & \wr\wr \\ 0 \to & \mathrm{Hom}_E(\Omega_{E/E'}, E) & \to & \mathrm{Hom}_E(\Omega_{E/F}, E) & \to & \mathrm{Hom}_{E'}(\Omega_{E'/F}, E) \end{array}$$

And so $\quad E \otimes_{E'} \Omega_{E'/F} \xrightarrow{\alpha} \Omega_{E/F} \longrightarrow \Omega_{E/E'} \longrightarrow 0$

<u>Lemma</u>. If E/E' is an algebraic separable extension, then

$$E \otimes_{E'} \Omega_{E'/F} \hookrightarrow \Omega_{E/F}.$$

<u>Pf.</u> It is equivalent to say $\mathrm{Der}_F(E,E) \longrightarrow \mathrm{Der}_F(E',E)$ is onto, which means, any F-derivation

$$\delta : E' \longrightarrow E$$

can be extended to an F-derivation

$$\tilde{\delta} : E \longrightarrow E.$$

It is enough to prove it for a simple extension (why?)

So let $E = E'[a]$. So $\min_{E'}(a; \underline{x}) = f(\underline{x})$ is irred. in $E'[\underline{x}]$ and $f'(a) \neq 0$.

Suppose $\tilde{\delta}$ is a function on E s.t. $\tilde{\delta}(a) = b$ and $\tilde{\delta}|_{E'} = \delta$

For $\tilde{\delta}$ to be an F-derivation, we should have:

- By induction on m, $\tilde{\delta}(a^m) = m\, a^{m-1}\, \tilde{\delta}(a)$

  $[\tilde{\delta}(a^{m+1}) = a\, \tilde{\delta}(a^m) + a^m\, \tilde{\delta}(a) = m\, a^m\, \tilde{\delta}(a) + a^m\, \tilde{\delta}(a)$
  $= (m+1)\, a^m\, \tilde{\delta}(a).]$

- $\tilde{\delta}(c_m a^m) = a^m\, \delta(c_m) + m\, a^{m-1}\, \tilde{\delta}(a)$ for any $c_m \in E'$.

So $\tilde{\delta}(g(a)) = \delta(g)(a) + g'(a)\,\tilde{\delta}(a)$

where $\delta(g)(\underline{x}) = \sum \delta(c_i)\,\underline{x}^i$ if $g(\underline{x}) = \sum c_i\,\underline{x}^i$.

Hence we define

$$\tilde{\delta}(g(a)) := \delta(g)(a) + g'(a)\,b$$

for some $b \in E$ to be determined later.

We will choose b in a way that makes $\tilde{\delta}$ a well-defined function:

$$g_1(a) = g_2(a) \iff g_2(\underline{x}) = g_1(\underline{x}) + f(\underline{x})\,p(\underline{x}).$$

$$\begin{aligned}
\text{So } \tilde{\delta}(g_1(a)) = \tilde{\delta}(g_2(a)) \iff \delta(g_1)(a) + g_1'(a)\,b &= \delta(g_2)(a) + g_2'(a)\,b \\
&= \delta(g_1)(a) + \delta(f\cdot p)(a) \\
&\quad + g_1'(a)\,b + (f'(a)p(a) + \underbrace{f(a)p'(a)}_{0})\,b
\end{aligned}$$

$$\iff \delta(f\cdot p)(a) + f'(a)\,p(a)\,b = 0$$

for any $p(\underline{x}) \in E'[\underline{x}]$. $\circledast$

Notice that $\delta(f_1\cdot f_2) = f_2\,\delta(f_1) + f_1\,\delta(f_2)$ (why?)

$$\text{So } \circledast \text{ holds} \iff \delta(f)(a)\,p(a) + \underbrace{f(a)}_{0}\,\delta(p)(a) + f'(a)p(a)\,b = 0$$

$$\iff \big(\delta(f)(a) + f'(a)\,b\big)\,p(a) = 0$$

for any $p(x) \in E'[x]$.

$$\iff \quad b = -\frac{\delta(f)(a)}{f'(a)} \quad \text{which is possible}$$

as $f'(a) \neq 0$.

So $\delta : E' \to E$ has a <u>unique</u> extension to an F-derivation.

$\tilde{\delta} : E \to E$, where $\tilde{\delta}(a) = -\dfrac{\delta(f)(a)}{f'(a)}$. ■

<u>Corollary</u> If E/F and E'/F are of finite type and E/E' is separably algebraic, then

$$\dim_{E'} \Omega_{E'/F} = \dim_E \Omega_{E/F}.$$

<u>Pf.</u> Since E/E' is algebraic and of finite type, it is a finite separable extension. So E is a simple, separable, algebraic extension of E'. Hence $\Omega_{E/E'} = 0$. On the other hand, by the above lemma

$$0 \to E \otimes_{E'} \Omega_{E'/F} \to \Omega_{E/F} \to \Omega_{E/E'} \to 0$$

is an exact sequence. So $E \otimes_{E'} \Omega_{E'/F} \simeq \Omega_{E/F}$, which implies

$$\dim_{E'} \Omega_{E'/F} = \dim_E \Omega_{E/F}. \quad ■$$

<u>Def.</u> We say E is <u>separably generated</u> over F if

$\exists\ F \subseteq E' \subseteq E$ s.t. E/E' is separably algebraic and E'/F is purely transcendental.

<u>Corollary</u>. If E/F is separably generated, then

$$\dim_E \Omega_{E/F} = \mathrm{tr.deg}_F E.$$

<u>Pf</u>. $\exists$ a purely transcendental extension E'/F s.t. E/E' is separable and finite. So $\dim_E \Omega_{E/F} = \dim_{E'} \Omega_{E'/F} = \mathrm{tr.deg}_F E$. ∎

<u>Theorem</u> Suppose E/F is of finite type

(a) $\dim_E \Omega_{E/F} \geq \mathrm{tr.deg}_F E$

(b) Equality holds $\Leftrightarrow$ E/F is separably generated.

<u>Pf</u>. We proceed by induction on $\dim_E \Omega_{E/F}$ (and number of generators of E).

. $\underline{\dim_E \Omega_{E/F} = 0}$ and $\underline{E = F(\alpha_1, \dots, \alpha_n)}$

We know that $\Omega_{E/F} \twoheadrightarrow \Omega_{E/\underbrace{F(\alpha_1,\dots,\alpha_{n-1})}_{E'}}$. So $\Omega_{E'(\alpha_1)/E'} = 0$.

$\Rightarrow$ $E'(\alpha_1)/E'$ is a finite separable extension.

$\Rightarrow$ $\dim_E \Omega_{E/F} = \dim_{E'} \Omega_{E'/F} \Rightarrow \Omega_{E'/F} = 0$.

$\Rightarrow$ by the induction hypothesis on the number of generators,

E'/F is a finite separable extension.

$\Rightarrow$ E/F is a finite separable extension.

So $\dim_E \Omega_{E/F} = 0 \iff E/F$ is a finite separable extension.

- $\underline{\dim_E \Omega_{E/F} = d > 0}$

Since $d > 0$, $\exists\, \alpha \in E$ s.t. $d_{E/F}(\alpha) \neq 0$. Hence $d_{F(\alpha)/F}(\alpha) \neq 0$.

So $\Omega_{F(\alpha)/F} \neq 0$, which implies that α is transcendental over F.

We have
$$E \otimes_{F(\alpha)} \Omega_{F(\alpha)/F} \longrightarrow \Omega_{E/F} \longrightarrow \Omega_{E/F(\alpha)} \longrightarrow 0$$

Since $d_{E/F}(\alpha) \neq 0$ and $\Omega_{F(\alpha)/F} = F(\alpha)\, d_{F(\alpha)/F}(\alpha)$, we have that
$$0 \longrightarrow E \otimes_{F(\alpha)} \Omega_{F(\alpha)/F} \longrightarrow \Omega_{E/F} \longrightarrow \Omega_{E/F(\alpha)} \longrightarrow 0$$
is an exact sequence. So
$$\dim_E \Omega_{E/F} = 1 + \dim_E \Omega_{E/F(\alpha)}.$$

Notice that
$$\begin{aligned} \operatorname{tr.deg}_F E &= \operatorname{tr.deg}_{F(\alpha)} E + \operatorname{tr.deg}_F F(\alpha) \\ &= \operatorname{tr.deg}_{F(\alpha)} E + 1. \end{aligned}$$

So by the induction hypothesis we get
$$\begin{aligned} \dim_E \Omega_{E/F} = 1 + \dim_E \Omega_{E/F(\alpha)} &\geq 1 + \operatorname{tr.deg}_{F(\alpha)} E \\ &= \operatorname{tr.deg}_F E; \end{aligned}$$

this finishes the proof of part (a).

Suppose $\dim_E \Omega_{E/F} = \operatorname{tr.deg}_F E$. Then we get
$$\dim_{F(\alpha)} \Omega_{E/F(\alpha)} = \operatorname{tr.deg}_{F(\alpha)} E = d-1.$$

So by the induction hypothesis, $E/F(\alpha)$ is separably generated.

$\Rightarrow \exists\ E'/_{F(\alpha)}$ which is purely transcendental and

$E/_{E'}$ is separably algebraic.

$\Rightarrow E'/_F$ is purely transc. $\Rightarrow E/_F$ is separa. genera.

The other direction of part (b) is proved in the previous corollary. ■

<u>Def.</u> We say $E/_F$ is a separable$'$ extension if either

$\text{char } F = 0$, or $\text{char } F = p > 0$ and

$\forall\, x_1, \dots, x_n \in E$, $\dim_F \sum F x_i = \dim_F \sum F x_i^p$.

<u>Lemma</u>. Suppose $\text{char } F = p > 0$. Then

$$E/_F \text{ is separable}' \iff \begin{array}{ccc} E^p \otimes_{F^p} F & \hookrightarrow & E \\ a^p \otimes b & \longmapsto & a^p b \end{array}$$

<u>Pf.</u> $(\Rightarrow)$

Suppose $\{\alpha_i\}$ is a basis of $E/_F$, and $\{c_j\}$ is a basis of $F/_{F^p}$. Then $\alpha_i^p \otimes c_j$ is an F^p-basis of $E^p \otimes_{F^p} F$. Now suppose for some $d_{ij} \in F$ we have

$$\sum d_{ij}^p\, \alpha_i^p c_j = 0 \iff \sum_i \big(\underbrace{\sum_j d_{ij}^p c_j}_{\text{in } F}\big)\, \alpha_i^p = 0.$$

Since α_i / F are indep. $\Rightarrow \alpha_i^p / _F$ are indep.

So, $\forall i$, $\sum_j d_{ij}^p c_j = 0 \Rightarrow d_{ij}^p = 0$.

Hence $\alpha_i^p c_j$'s are F^p linearly indep. $\Rightarrow E^p \otimes_{F^p} F \hookrightarrow E$.

($\Leftarrow$) a similar argument. ■

<u>Lemma</u>. For $E/_F$ finite extension,

$$E/_F \text{ is separable}' \iff E/_F \text{ is separable}.$$

<u>Pf.</u> ($\Rightarrow$) Exercise. ($\Leftarrow$) $E = F[\alpha]$. Let f be the minimal poly. of α / F and g be its min. poly. $/ F^{1/p}$. Then $g \mid f \mid g^p$ (?) by separablity, $f = g$. So $E \otimes_F F^{1/p}$ is a field. ■

<u>Theorem</u>. $E/_F$ separable$'$ $\iff$ $E/_F$ separably generated.

(we are assuming that $E/_F$ is of finite type.)

<u>Pf.</u> • If char $F = 0$, there is nothing to prove.

• Suppose char $F = p > 0$.

($\Rightarrow$) $E = F(\alpha_1, \dots, \alpha_m)$ and $\alpha_1, \dots, \alpha_d$ are algebraically independent and $d = \operatorname{tr.deg}_F E$.

* If $d = m$, then $E/_F$ is sep. gen.

* If $d < m-1$, then by the induction hypothesis

$\exists\, x_1, \dots, x_d \in F(\alpha_1, \dots, \alpha_{m-1})$ s.t.

$F(\alpha_1, \dots, \alpha_{m-1}) / F(x_1, \dots, x_d)$ is finite separable

and $x_1, \dots, x_d$ are alg. indep.

<u>And</u> $\exists\, y_1, \dots, y_d \in F(x_1, \dots, x_d, \alpha_m)$ s.t.

{here we}

$F(x_1, \dots, x_d, \alpha_m) / F(y_1, \dots, y_d)$ is finite separable here we are using $d<m-1$

$F(x_1,\dots,x_d,\alpha_m)/F(y_1,\dots,y_d)$ is finite separable and $y_1,\dots,y_d$ are alg. indep.

$$\Rightarrow \left.\begin{array}{l} F(\alpha_1,\dots,\alpha_m)/F(x_1,\dots,x_d,\alpha_m) \text{ sep. finite} \\ F(x_1,\dots,x_d,\alpha_m)/F(y_1,\dots,y_d) \quad \text{sep. finite} \end{array}\right\} \Rightarrow$$

$$E/F(y_1,\dots,y_d) \text{ sep. finite} \Rightarrow E/F \text{ sep. gen.}$$

$*$ Now we assume $d=m-1$.

$\exists$ a non-zero polynomial $f\in F[\underline{x}_1,\dots,\underline{x}_m]$ s.t.

$f(\alpha_1,\dots,\alpha_m)=0$.

- If $\partial_i f=0$, $\forall i$, then all the monomials of f are of the form $\left(\underline{x}_1^{i_1}\cdots\underline{x}_m^{i_m}\right)^p$ for some $i_j\in\mathbb{Z}^{\geq 0}$.

Since E/F is separable and $\alpha_I^p := \left(\alpha_1^{i_1}\cdots\cdots\alpha_m^{i_m}\right)^p$ are linearly depen. $/F$, so are α_I's. So

$\exists f\in F[\underline{x}_1,\dots,\underline{x}_m]$ s.t. ① $f(\alpha_1,\dots,\alpha_m)=0$.

② $\nabla f\neq 0$.

$$\Rightarrow \quad F[\underline{x}_1,\dots,\underline{x}_m]^m/\langle\nabla f\rangle \twoheadrightarrow \Omega_{F[\alpha_1,\dots,\alpha_m]/F}$$

$$\Rightarrow \quad F(\underline{x}_1,\dots,\underline{x}_m)^m/\langle\nabla f\rangle \twoheadrightarrow F(\alpha_1,\dots,\alpha_m)\otimes\Omega_{F[\alpha_1,\dots,\alpha_m]/F}$$

$$\simeq \Omega_{E/F}.$$

$\Rightarrow$ dim Ω $< m-1 = $ trdeg E

So by the previous theorem, $\dim_E \Omega_{E/F} = \mathrm{tr.deg}_F E$, which implies E/F is sep. gen.

$(\Leftarrow)$ E/F sep. gen $\Rightarrow \exists E'$ s.t. ① E/E' is sep. finite

② E'/F is purely trans.

By the above lemma, E/E' is separable$'$.

$F^p(\underline{x}_1^p, \dots \underline{x}_n^p) \otimes_{F^p} F \simeq F(\underline{x}_1, \dots, \underline{x}_n) \Rightarrow E'/F$ is separable$'$.

$$\left.\begin{array}{l} E^p \otimes_{E'^p} E' \hookrightarrow E \\ E'^p \otimes_{F^p} F \hookrightarrow E' \end{array}\right\} \Rightarrow E^p \otimes_{E'^p} (E'^p \otimes_{F^p} F) \hookrightarrow E^p \otimes_{E'^p} E' \hookrightarrow E$$

$$\Rightarrow E^p \otimes_{F^p} F \hookrightarrow E. \qquad \blacksquare$$

<u>Corollary</u>. Suppose F is perfect. Then for any E/E' finite type extensions of F we have

$$E/E' \text{ is separable gen.} \Leftrightarrow E \otimes_{E'} \Omega_{E'/F} \hookrightarrow \Omega_{E/F}$$

$$\Leftrightarrow \mathrm{Der}_F(E,E) \to \mathrm{Der}_F(E',E) \to 0$$

<u>Pf.</u> F: perfect $\Rightarrow E/F$ and E'/F are separable$'$

$\Rightarrow E/F$ and E'/F are sep. gen.

$\Rightarrow \dim_E \Omega_{E/F} = \mathrm{tr.deg}_F E$ and

$\dim_{E'} \Omega_{E'/F} = \mathrm{tr.deg}_F E'$

On the other hand, we have that

$$E \otimes_{E'} \Omega_{E'/F} \xrightarrow{\alpha} \Omega_{E/F} \longrightarrow \Omega_{E/E'} \longrightarrow 0$$

is an exact sequence. So

$$\dim_E \Omega_{E/F} = \dim_E \Omega_{E/E'} + \dim(\operatorname{Im}\alpha)$$

$$= \dim_E \Omega_{E/E'} + \dim_{E'} \Omega_{E'/F} - \dim \ker(\alpha)$$

$$\Rightarrow \operatorname{tr.deg}_F E - \operatorname{tr.deg}_F E' + \dim \ker(\alpha) = \dim_E \Omega_{E/E'}$$

$$\Rightarrow \operatorname{tr.deg}_{E'} E + \dim \ker(\alpha) = \dim_E \Omega_{E/E'}.$$

So $\ker(\alpha) = 0 \iff E/_{E'}$ is separably gen. ∎