Background on AG: simple points

Friday, May 12, 2017 1:34 PM

Theorem. Let X be an (affine) irreducible variety of dimension d.

Then (a) The simple points of X form a non-empty open subset of X.

(b) $\forall x \in X$, $\dim T_x X \geq d$.

(c) If $x_0 \in X$ is a simple point, $\exists$ an affine open nbhd U of x_0 s.t. Ω_U is a free $k[U]$-mod.

(a) Let $k[X] = k[\underline{x_1}, \dots, \underline{x_n}] / \langle f_1, \dots, f_m \rangle$. Recall that

$$\Omega_X := \Omega_{k[X]/k} \simeq k[X]^n / \langle ([\partial_1 f_i], \dots, [\partial_n f_i]) \rangle \quad , \text{ and}$$

$$T_x X = \mathrm{Der}_k(k[X], k_x) = \mathrm{Hom}_{k[X]}(\Omega_X, k_x)$$

$$= \mathrm{Hom}_{k_x}(\underbrace{\Omega_X \otimes_{k[X]} k_x}, k_x)$$

$$\simeq (\Omega_X / \mathfrak{m}_x \Omega_X)^* =: (\Omega_X(x))^*$$

So $\dim_{k_x} T_x X = \dim_{k_x} \Omega_X(x)$

Let $J = \begin{bmatrix} [\partial_1 f_1] & \cdots & [\partial_n f_1] \\ \vdots & & \vdots \\ [\partial_1 f_m] & \cdots & [\partial_n f_m] \end{bmatrix} \in M_{m \times n}(k[X]) \subseteq M_{m\times n}(k(X))$.

Let $r := \mathrm{rank}(J)$. Then

$$\dim_{k(X)} \Omega_{k(X)/k} = \dim_{k(X)} \Omega_X \otimes_{k[X]} k(X) = n - r.$$

$$\begin{array}{l} \quad \| \\ \mathrm{tr.deg}_k\, k(X) \\ \quad \| \\ \dim k[X] \\ \quad \| \\ \dim X. \end{array}$$

(k is alg. closed $\Rightarrow$ k is perfect
$\Rightarrow k(X)/_k$ is sep.'
$\Rightarrow k(X)/_k$ is sep. gen.)

On the other hand, $\exists f \in k[X]$ (prod. of non-zero minors of J) s.t.

$\Omega_X \otimes_{k[X]} k[X][1/f]$ is a free, rank $n-r$, $k[X][1/f]$-mod.

So, $\forall\, x \in X \setminus V(f)$, we have $\dim_{k_x} \Omega_X(x) = n-r$.

And so $\dim_k T_x X = \dim X \implies x$ is a simple point.

(b) $\mathrm{Rank}(J(x)) \leq \mathrm{Rank}(J) \quad \forall\, x \in X$

$\Rightarrow \dim T_x X \geq n-r = \dim X.$

(c) In $D_f := X \setminus V(f)$, Ω_{D_f} is a free, rank d, $k[D_f]$-mod.

■

Ex. X: irred. affine.

Ω_X: free $k[X]$-mod $\Rightarrow X$ is smooth.

Def. • $\phi: X \longrightarrow Y$ irred. varieties.

ϕ is called <u>dominant</u> if $\overline{\phi(X)} = Y$.

• $\phi : X \to Y$ irred. varieties ; ϕ : dominant;

ϕ is called <u>separable</u> if $k(X) / \phi^*(k(Y))$ is <u>separably generated</u>.

Theorem. Let $X \xrightarrow{\phi} Y$ be a morphism of irreducible varieties. Suppose ϕ is dominant.

(i) Suppose $\exists\, x_0 \in X$ s.t. ① x_0 is simple,

② $\phi(x_0)$ is simple,

③ $d\phi_{x_0} : T_{x_0} X \to T_{\phi(x_0)} Y$ is surjective

$\Rightarrow$ ϕ is separable.

(ii) Suppose ϕ is dominant and separable. Then

$\{ x \in X \mid ①\ x : \text{simple} \ \ ②\ \phi(x) : \text{simple} \ \ ③\ d\phi_x \text{ surjective} \}$

is a non-empty open subset of X.

Pf.. x_0 and $\phi(x_0)$ are simple $\Rightarrow$ going to open affine nbhds of x_0 and $\phi(x_0)$, we can assume

① Ω_X is a free $k[X]$-mod (and so X is smooth.)

② Ω_Y is a free $k[Y]$-mod (and so Y is smooth.)

$\phi^*: k[Y] \longrightarrow k[X]$ induces $(\phi^*)^\circ: \Omega_Y \longrightarrow \Omega_X$

and so we get $\psi: k[X] \otimes_{k[Y]} \Omega_Y \longrightarrow \Omega_X$

which is a $k[X]$-mod. homomorphism. And both of them are free $k[X]$-modules. Take $k[X]$-basis for Ω_X and $k[X] \otimes_{k[Y]} \Omega_Y$ and write the matrix $A_\psi \in M_{\dim X \times \dim Y}(k[X])$ of ψ with respect to this basis. Then, for $x_0 \in X$, we get $A_\psi(x_0) \in M_{\dim X \times \dim Y}(k_{x_0})$ by evaluating the entries of A_ψ at x_0.

This is the same as considering

$$\begin{tikzcd} \Omega_Y \otimes_{k[Y]} k[X] \otimes k_{x_0} \arrow[r, "\psi \otimes \mathrm{id}."] & \Omega_X \otimes k_{x_0} \\ \Omega_Y(\phi(x_0)) = \Omega_Y \otimes_{k[Y]} k_{\phi(x_0)} \arrow[u, "\cong"] \arrow[ur, "\psi(x_0)"'] & \end{tikzcd}$$

Recall that $T_{x_0} X \simeq \mathrm{Hom}_{k_{x_0}}(\Omega_X(x_0), k_{x_0})$ and

$T_{\phi(x_0)} Y \simeq \mathrm{Hom}_{k_{\phi(x_0)}}(\Omega_Y(\phi(x_0)), k_{\phi(x_0)})$, and

$$d\phi_{x_0}: T_{x_0} X \longrightarrow T_{\phi(x_0)} Y, \quad d\phi_{x_0} = \psi(x_0)^*.$$

(i) If $d\phi_{x_0}$ is surjective $\Rightarrow$ $\psi(x_0)$ is injective

$\Rightarrow$ rank $\psi(x_0) = \dim Y$ (number of columns)

$\Rightarrow$ rank $\Psi_\infty = \dim Y$ where

$$\Psi_\infty : k(X) \otimes_{k[Y]} \Omega_Y \longrightarrow k(X) \otimes_{k[X]} \Omega_X \simeq \Omega_{k(X)/k}.$$

Since ϕ is dominant, $\phi^* : k[Y] \to k[X]$ is injective.

$\Rightarrow$ Ψ_∞ induces an embedding

$$k(X) \otimes_{k(Y)} \Omega_{k(Y)/k} \longrightarrow \Omega_{k(X)/k}$$

$\Rightarrow$ $k(X)/_{\phi^* k(Y)}$ is separably generated.

(ii) If ϕ is separable, then

$k(X)/_{\phi^* k(Y)}$ is separably generated $\Rightarrow$

Ψ_∞ is injective $\Rightarrow$ its matrix is of rank $= \dim Y$

$\Rightarrow$ $\forall\, x \in X$ s.t. a non-zero minor of full dim.

does NOT vanish at x, we have

that $d\phi_x$ is surjective and $\phi(x)$ is simple.

(We can go to open affine smooth subvariaty of X.). ∎