Lie algebra

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Let G be an affine algebraic group. Let $\mathfrak{g} := T_e G$. Then, as G is smooth, we have $\dim_k \mathfrak{g} = \dim G$.

$\forall g \in G$, let $c_g : G \longrightarrow G$, $c_g(x) = gxg^{-1}$. Then c_g is an isomorphism of affine algebraic groups, and $c_g(e) = e$. So $dc_g|_e : T_e G \longrightarrow T_e G$. Let $Ad(g) := dc_g|_e : \mathfrak{g} \longrightarrow \mathfrak{g}$.

Using the 1^{st} definition of $T_e G$ we can identify it with

$$\{v \in k^n \mid e + \varepsilon v \in \ker(G(k[\varepsilon]) \longrightarrow G(k))\}.$$

And
$$c_g(e + \varepsilon v) = c_g(e) + \varepsilon\, dc_g|_e(v)$$
$$\Rightarrow \quad g(e + \varepsilon v)g^{-1} = e + \varepsilon\, Ad(g)(v)$$

So for any linear (algebraic) embedding $G \hookrightarrow GL_m(k)$ we have
$$g(I + \varepsilon v)g^{-1} = I + \varepsilon\, gvg^{-1} = I + \varepsilon\, Ad(g)(v)$$
$$\Longrightarrow \quad Ad(g)(v) = g v g^{-1}.$$

In particular all the entries of $Ad(g)(v)$ are regular functions on $G \times \mathfrak{g}$. So

$$Ad : G \longrightarrow GL(\mathfrak{g}) \text{ is an algebraic group homomorphism.}$$

$Ad : G \to GL(\mathfrak{g})$ is an algebraic group homomorphism.

It is called the <u>adjoint representation</u> of G.

So we get $d(Ad)|_e : \mathfrak{g} \longrightarrow T_I(GL(\mathfrak{g})) \underset{?}{=} End_k(\mathfrak{g})$

For $x \in \mathfrak{g}$, let $ad(x) := d(Ad)|_e(x) : \mathfrak{g} \longrightarrow \mathfrak{g}$.

Again using the 1^{st} def. of the tangent space, we have

$\forall x \in \mathfrak{g}, \qquad Ad(e + \varepsilon x) = I + \varepsilon\, ad(x)$

So for a linear algebraic embedding $G \hookrightarrow GL_m(k)$ we have,

$$\forall x, y \in \mathfrak{g}, \quad Ad(I + \varepsilon x)(y) = (I + \varepsilon x)\, y\, (I + \varepsilon x)^{-1}$$

$$= (I + \varepsilon x)\, y\, (I - \varepsilon x)$$

$$= y + \varepsilon(xy - yx)$$

$$\Rightarrow \boxed{ad(x)(y) = xy - yx}.$$

Hence $\mathfrak{g}$ is a sub-Lie algebra of $\mathfrak{gl}_m(k)$.

<u>Lemma</u>. Let $\phi : G_1 \longrightarrow G_2$ be an affine algebraic group homomorphism. Then $d\phi|_e : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is a Lie algebra homomorphism i.e. $d\phi|_e \circ ad = ad \circ d\phi|_e$.

<u>Pf.</u> $\phi \circ c_g = c_{\phi(g)} \Rightarrow d\phi|_e \circ Ad(g) = Ad \circ \phi(g)$.

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So $d\phi|_e \circ Ad : G \to End(\mathfrak{g})$ and

$Ad \circ \phi : G \to End(\mathfrak{g})$

are equal morphisms of affine varieties.

$$\Rightarrow d(d\phi|_e \circ Ad)|_e = d(Ad \circ \phi)|_e$$

$$\Rightarrow d(d\phi|_e) \circ d(Ad)|_e = d(Ad)|_e \circ d\phi|_e$$

$$\Rightarrow d\phi|_e \circ ad = ad \circ d\phi|_e . \quad \blacksquare$$

Embedding of $\mathfrak{g}$ into $End_k(k[G])$

$r_g : G \to G$, $r_g(g') = g'g \Rightarrow dr_g|_e : \mathfrak{g} \to T_g G$

$\forall\ \delta \in Der_k(k[G], k_e)$,

$dr_g|_e(\delta) : k[G] \to k_g$ is a k-derivation.

Let $D_\delta : k[G] \to Fun(G, k)$ be $D_\delta(f)(g) := dr_g|_e(\delta)(f)$.

Claim 1. $D_\delta(f) \in k[G]$.

Pf. $D_\delta(f)(g) = dr_g|_e(\delta)(f)$

$= \delta(r_g^*(f))$

$$\begin{tikzcd} k[G] \arrow[r, "r_g^*"] \arrow[d, dashed] & k[G] \arrow[d, "\delta"] \\ k_g \arrow[r, "\sim"] & k_e \end{tikzcd}$$

$r_g^*(f)(g') = f(g'g)$.

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Suppose $m^*(f) = \sum f_i^{(\ell)} \otimes f_i$

$$r_g^*(f)(g') = f(g'g) = \sum f_i^{(\ell)}(g')\, f_i(g)$$

$$\Rightarrow \quad r_g^*(f) = \sum f_i^{(r)}(g)\, f_i^{(\ell)}$$

$$\Rightarrow \quad \delta(r_g^*(f)) = \sum \delta(f_i^{(\ell)})\, f_i^{(r)}(g).$$

Hence $\quad D_\delta(f) = \sum \delta(f_i^{(\ell)})\, f_i^{(r)} \in k[G]$. ■

<u>Claim 2.</u> $D_\delta \in \mathrm{Der}_k(k[G], k[G])$.

<u>Pf.</u>
$$D_\delta(f_1f_2)(g) = \left(dr_g|_e(\delta)\right)(f_1f_2) = f_1(g)\left(dr_g|_e(\delta)\right)(f_2) + f_2(g)\left(dr_g|_e(\delta)\right)(f_1)$$
$$= f_1(g)\, D_\delta(f_2)(g) + f_2(g)\, D_\delta(f_1)(g)$$

$$\Rightarrow D_\delta(f_1f_2) = f_1\, D_\delta(f_2) + f_2\, D_\delta(f_1). \quad ■$$

<u>Claim 3.</u> $\forall g \in G$, $D_\delta \circ \rho(g) = \rho(g) \circ D_\delta$ where

$$\rho(g): k[G] \longrightarrow k[G], \quad (\rho(g)(f))(g') = f(g'g)$$

<u>Pf.</u> $\quad D_\delta(\rho(g)(f)) = ?$

$$\begin{tikzcd} k[G] \arrow[r, "m^*"] \arrow[d, "m^*"'] \arrow[dr, phantom, "\circlearrowright"] & k[G]\otimes k[G] \arrow[d, "m^*\otimes id"] \\ k[G] \arrow[r, "id\otimes m^*"'] & k[G]\otimes k[G]\otimes k[G] \end{tikzcd}$$

$\rho(g)(f) = \sum f_i^{(r)}(g)\, f_i^{(\ell)}$

if $\quad m^*(f) = \sum f_i^{(\ell)} \otimes f_i^{(r)}$.

So $\quad D_\delta(\rho(g)(f)) = \sum f_i^{(r)}(g)\, D_\delta(f_i^{(\ell)})$

if $\quad m^*(f_i^{(r)}) = \sum_j f_{ij}^{(\ell)} \otimes f_{ij}^{(r)}$, then if $m^*(f_i^{(r)}) = \sum_j f_{ij}^{(l)} \otimes f_{ij}^{(r)}$, then

$$D_\delta(f_i^{(l)}) = \sum_j \delta(f_{ij}^{(l)})\, f_{ij}^{(r)}.$$

Hence $$\boxed{D_\delta(\rho(g)(f)) = \sum_{i,j} f_i^{(r)}(g)\ \ \delta(f_{ij}^{(l)})\, f_{ij}^{(r)}.}$$

$$\rho(g)\big(D_\delta(f)\big) = \rho(g)\Big(\sum_i \delta(f_i^{(l)})\, f_i^{(r)}\Big)$$
$$= \sum_i \delta(f_i^{(l)})\, \rho(g)(f_i^{(r)})$$

if $m^*(f_i^{(r)}) = \sum h_{ij}^{(l)} \otimes h_{ij}^{(r)}$, then

$$\rho(g)(f_i^{(r)}) = \sum_j h_{ij}^{(r)}(g)\, h_{ij}^{(l)}$$

$$\Rightarrow \boxed{\rho(g)(D_\delta(f)) = \sum_{i,j} \delta(f_i^{(l)})\, h_{ij}^{(r)}(g)\, h_{ij}^{(l)}}$$

Hence we have to show

$$\forall g, g' \in G,\ \sum_{i,j} \delta(f_i^{(l)})\, h_{ij}^{(r)}(g)\, h_{ij}^{(l)}(g')$$
$$= \sum_{i,j} \delta(f_{ij}^{(l)})\, f_i^{(r)}(g)\, f_{ij}^{(r)}(g'). \quad \circledast$$

By co-associativity, we have

$$\sum f_i^{(l)} \otimes h_{ij}^{(l)} \otimes h_{ij}^{(r)} = \sum f_{ij}^{(l)} \otimes f_{ij}^{(r)} \otimes f_i^{(r)}$$

$$\Rightarrow \sum h_{ij}^{(r)}(g)\, h_{ij}^{(l)}(g')\, f_i^{(l)} = \sum f_i^{(r)}(g)\, f_{ij}^{(r)}(g')\, f_{ij}^{(l)}$$

which implies $\circledast$. ∎

Claim 4. $\delta \longmapsto D_\delta$ is 1-1.

Pf. $D_\delta(f)(e) = \left(dr_e\big|_e(\delta)\right)(f) = \delta(f)$. ∎

Claim 5.

$$\{D \in \mathrm{Der}_k(k[G], k[G]) \mid \forall g \in G,\ \rho(g)\circ D = D \circ \rho(g)\} \longrightarrow \mathrm{Der}_k(k[G], k_e)$$

$$D \longmapsto \delta_D$$

where $\delta_D(f) := D(f)(e)$.

is well-defined, and it is the inverse of $\delta \longmapsto D_\delta$.

Pf. • $\delta_D(f_1 f_2) = D(f_1 f_2)(e) = f_1(e)\,D(f_2)(e) + f_2(e)\,D(f_1)(e)$

$$= f_1 \cdot \delta_D(f_2) + f_2 \cdot \delta_D(f_1).$$

• By claim 4, $\delta_{(D_\delta)} = \delta$ for any $\delta \in \mathrm{Der}_k(k[G], k_e)$.

• We have to check $D_{(\delta_D)} = D$.

$D_{(\delta_D)}(f)(e) = \delta_D(f) = D(f)(e)$.

$\Rightarrow \quad D_{(\delta_D)}(\rho(g)(f))(e) = D(\rho(g)(f))(e)$

$\Rightarrow \quad \rho(g)\left(D_{(\delta_D)}(f)\right)(e) = \rho(g)\left(D(f)\right)(e)$

$\Rightarrow \quad D_{(\delta_D)}(f)(g) = D(f)(g)$

$\Rightarrow \quad D_{(\delta_D)}(f) = D(f) \quad \Rightarrow D_{(\delta_D)} = D$. ∎

Definition. For $f \in k[G]$, suppose $m^*(f) = \sum f_i^{(l)} \otimes f_j^{(r)}$.

For $\delta \in \mathrm{Der}_k(k[G], k_e)$, let

$$\delta * f := \sum \delta(f_i^{(l)})\, f_j^{(r)} \quad \text{and}$$

$$f * \delta := \sum \delta(f_j^{(r)})\, f_i^{(l)}. \qquad \text{(Convolution)}$$

<u>Remark.</u> We have already seen that $D_\delta(f) = \delta * f$.

The best way to find Lie algebra of an algebraic group is using dual numbers:

<u>Ex.</u> Find $\mathrm{Lie}(SL_n(k))$.

<u>Solution.</u> $x \in \mathrm{Lie}(SL_n(k)) \subseteq M_n(k) \Leftrightarrow I + \varepsilon x \in SL_n(k[\varepsilon])$

$$\Leftrightarrow \det(I + \varepsilon x) = 1 \Leftrightarrow \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} ([i = \sigma(i)] + \varepsilon x_{i\sigma(i)}) = 1$$

If $\sigma \neq id.$, then $\exists\, i_1 \neq i_2$ s.t. $\sigma(i_1) \neq i_1$ and $\sigma(i_2) \neq i_2$, which implies $\prod_{i=1}^{n} ([i = \sigma(i)] + \varepsilon x_{i\sigma(i)}) = 0$ as $\varepsilon^2 = 0$.

Hence
$$1 = \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} ([i = \sigma(i)] + \varepsilon x_{i\sigma(i)})$$
$$= \prod_{i=1}^{n} (1 + \varepsilon x_{ii}) = 1 + \varepsilon\, \mathrm{tr}(x).$$

Therefore $\mathrm{Lie}(SL_n(k)) = \mathfrak{sl}_n(k) := \{x \in \mathfrak{gl}_n(k) \mid \mathrm{tr}(x) = 0\}$. $\square$

<u>Ex.</u> Find $\mathrm{Lie}(Sp_{2n}(k))$ where $Sp_{2n}(k) = \{g \in SL_{2n}(k) \mid g \begin{bmatrix} & I \\ -I & \end{bmatrix} g^t = \begin{bmatrix} & I \\ -I & \end{bmatrix}\}$.