More on group actions and quotients

Wednesday, May 24, 2017 10:15 PM

Lemma. Suppose $V \subseteq k[G]$ is a finite-dimensiona subspace, and, $\forall f \in V$, $g \in G$, $\lambda(g)(f) \in V$. Let

$$\pi : G \longrightarrow GL(V), \quad \pi(g)(v) := \lambda(g)(v).$$

Then ① π is an algebraic group homomorphism.

② $\underbrace{\left(d\pi|_e(\delta)\right)(v)}_{\boxed{\text{we are identifying } Der(k[GL(V)], k_I) \text{ with } End(V).}} = -\delta * v$ for any $\delta \in Der_k(k[G], k_e)$

Pf. ① we have already proved: let $\{f_1, \dots, f_n\}$ be a basis of V, and extend it to a basis $\{f_i\}$ of $k[G]$. Suppose $m^*(f_i) = \sum_j a_{ij} \otimes f_j$ for same $a_{ij} \in k[G]$. Then

$$\lambda(g)(f_i)(g') = \sum_j a_{ij}(g^{-1}) f_j(g') \Rightarrow \lambda(g)(f_i) = \sum_j i^*(a_{ij})(g) f_j$$

$$\Rightarrow [\pi(g)]_{\{f_1, \dots, f_n\}} = [i^*(a_{ij})(g)] \in M_n(k),$$

and $i^*(a_{ij}) \in k[G]$.

②. $\delta * f_i = \sum_{j=1}^n \delta(a_{ij}) f_j$.

②. $\delta * f_i = \sum_{j=1}^{n} \delta(a_{ij})\, f_j$.

• To understand, $\big(d\pi|_e(\delta)\big)(f_i)$, we identify V with k^n and $k[V]=k[\underline{x}_1,\dots,\underline{x}_n]$; we have

$$k[V] \longrightarrow k[GL(V)]\otimes k[V] \longrightarrow k[G]\otimes k[V] \longrightarrow k_I \otimes k[V] \xrightarrow{\sim} k[V]$$

$$\underline{x}_j\big(d\pi|_e(\delta)(f_i)\big) = \ ?$$

$$\underline{x}_j \longmapsto \sum_s \underline{x}_{js}\otimes \underline{x}_s \longmapsto \sum \iota^*(a_{sj})\otimes \underline{x}_s \longmapsto \sum \delta(\iota^*(a_{sj}))\, \underline{x}_s .$$

So $\underline{x}_j\big(d\pi|_e(\delta)(f_i)\big) = \delta(\iota^*(a_{ij}))$, which implies

$$d\pi|_e(\delta)(f_i) = \sum_{j=1}^{n} \delta(\iota^*(a_{ij}))\, f_j \underset{(?)}{=} -\sum_{j=1}^{n} \delta(a_{ij})\, f_j$$

$$= -\delta * f_i . \qquad \blacksquare$$

<u>Lemma</u>. $\iota: G\to G$, $\iota(g)=g^{-1} \Rightarrow d\iota|_e(X) = -X$.

So $\forall\, \delta\in \mathrm{Der}_k(k[G],k_e)$, $\delta\circ\iota^* = -\delta$.

<u>Pf</u>. Using dual numbers: $(I+\varepsilon X)(I-\varepsilon X)=I$

$$\Rightarrow \iota(I+\varepsilon X) = I-\varepsilon X \Rightarrow d\iota|_e(X) = -X . \qquad \blacksquare$$

<u>Lemma</u>. Let H be a closed subgroup of G. Then

$$\mathfrak{h} = \{\delta\in\mathfrak{g} \mid \delta(I_H)=0\}$$

$$= \{\delta\in\mathfrak{g} \mid \delta * I_H \subseteq I_H\}$$

Pf. • $e+\varepsilon x \in H(k[\varepsilon]) \Rightarrow \forall f \in I_H,\ f(e+\varepsilon x)=0$

$$\Rightarrow f(e+\varepsilon x) = f(e)+\varepsilon\, \delta_X(f) = \varepsilon\, \delta_X(f)$$

$$\Rightarrow \delta_X(f)=0.$$

So $\forall \delta \in \mathfrak{h},\ \delta(I_H)=0$.

• $\delta(I_H)=0 \Rightarrow \forall f\in I_H,\ f(e+\varepsilon x_\delta) = f(e)+\varepsilon\,\delta(f)=0$

$$\Rightarrow e+\varepsilon x_\delta \in H(k[\varepsilon])$$

$$\Rightarrow x_\delta \in \mathfrak{h}.$$

• $f\in I_H \Rightarrow \forall h\in H,\ \rho(h)(f)\in I_H$

$$\Rightarrow 0=\delta(\rho(h)(f)) = D_\delta(\rho(h) f)(e) = \rho(h)\, D_\delta(f)(e)$$

$$= D_\delta(f)(h)$$

$$\Rightarrow D_\delta(f)\in I_H.$$

• $\forall f\in I_H,\quad \delta(f) = D_\delta(f)(e) \underset{D_\delta(f)\in I_H}{=} 0 \Rightarrow \delta\in\mathfrak{h}.$ ∎

Lemma. Suppose G is an affine algebraic group and H is a closed subgp. Then $\exists$ an algebraic finite-dimensional rep'n $\pi: G\longrightarrow GL(V)$ and a subspace W of V s.t.

① $\{g\in G \mid \pi(g)(W)=W\} = H$.

② $\{\delta\in\mathfrak{g} \mid d\pi|_e(\delta)(W)\subseteq W\} = \mathfrak{h}$.

Pf. Part ① has been proved; we will show the same choice of (V, W) works for ② as well:

Since $k[G]$ is a locally finite G-mod, $\exists$ a finite-dimensional G-invar. subspace V of $k[G]$ s.t. $I_H = \langle V \cap I_H \rangle$.

Let $W := V \cap I_H$.

We have seen that, $\forall \delta \in \mathfrak{g}$, $v \in V$,

$$(d\pi|_e(\delta))(v) = -\delta * v.$$

$\forall f \in I_H$, $\exists \omega_i \in W$, $f_i \in k[G]$ s.t. $f = \sum f_i \omega_i$.

$\Rightarrow \delta * f = \sum (\delta * f_i)\omega_i + f_i(\delta * \omega_i) \in \langle \omega_i, \delta * \omega_i \rangle \subseteq \langle W \rangle = I_H$.

$\Rightarrow \delta * I_H \subseteq I_H \Rightarrow \delta \in \mathfrak{h}$.

The other side is easy. ∎

Theorem. Let G be an affine algebraic group, and H be a closed subgroup of G. Then $\exists$ an algebraic group homomorphism $\pi: G \to GL(V)$ and a line $\ell \subseteq V$ s.t.

① $\{g \in G \mid \pi(g)\ell = \ell\} = H$

② $\{x \in \mathfrak{g} \mid d\pi|_e(x)(\ell) \subseteq \ell\} = \mathfrak{h}$.

We have already proved part ① using the exterior power of V. Using the same idea, it is enough to prove the following:

Lemma. Let V be a finite-dim. space and W a proper subspace.

Let $\pi_d : GL(V) \longrightarrow GL(\overset{d}{\wedge} V)$ be $\pi_d(g)(v_1 \wedge \cdots \wedge v_d) := (gv_1) \wedge \cdots \wedge (gv_d)$

where $d = \dim W$. Then

$$\{x \in \mathfrak{gl}(V) \mid d\pi_d\big|_I (x)(\overset{d}{\wedge} W) \subseteq \overset{d}{\wedge} W\} = \{x \in \mathfrak{gl}(V) \mid x(W) \subseteq W\}.$$

Pf. Let $\{e_1, \dots, e_d\}$ be a basis of W and $\{e_1, \dots, e_n\}$ be a basis of V.

$$\Rightarrow \pi_d(g)(e_I) = \sum_{\substack{J \subseteq \{1,\dots,n\} \\ |J| = d}} \det(g_{JI})\, e_J \quad \text{where } e_J = e_{j_1} \wedge \cdots \wedge e_{j_d}$$

for $J = \{j_1 < \cdots < j_d\}$.

So $$\pi_d(I + \varepsilon X)(e_{I_0}) = \sum_J \det((I + \varepsilon X)_{J I_0})\, e_J$$

where $I_0 = \{1, \dots, d\}$.

$$I + \varepsilon X = \left[\begin{array}{cccc:c} 1+\varepsilon x_{11} & \varepsilon x_{12} & \cdots & \varepsilon x_{1d} & \\ \varepsilon x_{21} & 1+\varepsilon x_{22} & \cdots & \varepsilon x_{2d} & * \\ \vdots & & \ddots & \vdots & \\ \varepsilon x_{d1} & \cdots & & 1+\varepsilon x_{dd} & \\ \hdashline & & & & \\ & \varepsilon\, x_{ij} & & & * \\ & & & & \end{array}\right]$$

- $\det((I + \varepsilon X)_{J I_0}) = 0$ if J has at least two rows from the lower part.
- $\det((I + \varepsilon X)_{I_0 I_0}) = (1 + \varepsilon x_{11}) \cdots (1 + \varepsilon x_{dd}) = 1 + \varepsilon (x_{11} + \cdots + x_{dd})$.
- $\det((I + \varepsilon X)_{\dots}) = \pm (1 + \varepsilon x_{11}) \cdots (\widehat{1 + \varepsilon x_{\dots}}) \cdots (1 + \varepsilon x_{\dots})(\varepsilon x_{\dots})$

$(\ldots)_{(I_0 \setminus \{i\} \cup \{i'\})\, I_0}$

$$= \pm \varepsilon\, x_{i'i} .$$

$$\Rightarrow d\pi_d\big|_I(x)(e_{I_0}) \in k\, e_{I_0} \Leftrightarrow \forall\ 1 \le i \le d < i' \le n, \quad x_{i'i} = 0 .$$

$$\Leftrightarrow X(W) \subseteq W . \qquad \blacksquare$$

Ex. $$d\pi_d\big|_I (X)(v_1 \wedge \cdots \wedge v_d) = (Xv_1) \wedge v_2 \wedge \cdots \wedge v_d + v_1 \wedge (Xv_2) \wedge \cdots \wedge v_d + \cdots + v_1 \wedge \cdots \wedge v_{d-1} \wedge (Xv_d) .$$

[Solution. $\pi_d(I + \varepsilon X)(v_1 \wedge \cdots \wedge v_d) = (v_1 + \varepsilon X v_1) \wedge \cdots \wedge (v_d + \varepsilon X v_d)$

$$= v_1 \wedge \cdots \wedge v_d + \varepsilon \sum_{i=1}^{d} v_1 \wedge \cdots \wedge (X \cdot v_i) \wedge \cdots \wedge v_d \ ]$$

Ex. Let $\pi : G \longrightarrow GL(V)$ be as above. Then

$$\{ g \in G(k[\varepsilon]) \mid \pi(g)(\ell \otimes k[\varepsilon]) = \ell \otimes k[\varepsilon] \} = H(k[\varepsilon]) .$$

And it is equiv. to the above properties.

[Pf. • RHS $\subseteq$ LHS ?

• $g = g_1 + \varepsilon g_2$ in LHS $\Rightarrow \pi(g_1)\ell = \ell \Rightarrow g_1 \in H(k)$. $\circledast$

$\Rightarrow g_1^{-1} g = I + \varepsilon g_1^{-1} g_2 \in$ LHS

$\Rightarrow \pi(I + \varepsilon g_1^{-1} g_2)\, \ell \subseteq \ell \otimes k[\varepsilon]$

$\Rightarrow \forall \omega \in \ell, \ d\pi\big|_e (g_1^{-1} g_2)(\omega) \in \ell$

$\Rightarrow g_1^{-1} g_2 \in \mathfrak{h} \Rightarrow \left.\begin{array}{l} I + \varepsilon g_1^{-1} g_2 \in H(k[\varepsilon]) \\ \circledast \end{array}\right\} \Rightarrow g \in H(k[\varepsilon])$.

• If the above equality holds, then $\forall g \in G = G(k)$

$$\pi(g)(\ell) = \ell \Rightarrow \pi(g)(\ell \otimes k[\varepsilon]) = \ell \otimes k$$
$$\Rightarrow g \in H(k[\varepsilon]) \cap G(k) = H(k).$$

$\forall x \in \mathfrak{g}$, $d\pi|_e(x)(\ell) \subseteq \ell \Rightarrow \pi(I + \varepsilon x)(\ell \otimes k[\varepsilon]) \subseteq \ell \otimes k[\varepsilon]$

$\Rightarrow I + \varepsilon x \in H(k[\varepsilon]) \Rightarrow x \in \mathfrak{h}$.]

We need the following theorems from A.G.:

Theorem 1. X, Y: irreducible varieties;

$\phi : X \to Y$ dominant ; $r := \dim X - \dim Y$.

$\Rightarrow \exists\, \emptyset \neq U \subseteq X$ open s.t.

① $\phi|_U$ is an open morphism.

② $Y' \subseteq Y$ irreducible
X': irred. comp of $\phi^{-1}Y'$ which inters. U $\Bigg\} \Rightarrow \dim X' = \dim Y' + r$

In particular, dim of any irred. comp. of $\phi^{-1}(y)$ which intersects U is r.

③ If $[k(Y) : \phi^*(k(X))] < \infty$, then, $\forall x \in X$,
$$|\phi^{-1}(\phi(x))| = [k(Y) : \phi^*(k(X))]_{sep}.$$

Theorem 2. X, Y: irredu.

$\phi : X \to Y$ bijective

$\phi^* : k(Y) \xrightarrow{\sim} k(X)$

Y: normal, i.e. $\mathcal{O}_{Y,y} \subseteq k(Y)$ is integrally closed.

Y: normal, i.e. $\mathcal{O}_{Y,y} \subseteq k(Y)$ is integrally closed.

$\Rightarrow \phi$ is an isomorphism

<u>Corollary</u>. Let $\phi: G_1 \to G_2$ be an algebraic group homomorphism.

Suppose ϕ is an abstract group isomorphism, and G_i's are connected.

Then ϕ is an isomorphism of algebraic groups $\Leftrightarrow$ ϕ is separable.

$\Leftrightarrow d\phi|_e$ is surjective.

<u>Pf.</u> We have already proved that ϕ is separable $\Leftrightarrow d\phi|_e$ is surj.

If ϕ is an isomorphism, then $d\phi|_e$ is an isomorphism.

Suppose ϕ is separable $\Rightarrow$ $k(G_2)/\phi^* k(G_1)$ is separably generated. (I)

On the other hand, $\forall g_1 \in G_1$, $|\phi^{-1}(\phi(g_1))| = 1 \Rightarrow$

$$\left.\begin{array}{l} \dim G_1 = \dim G_2 \Rightarrow [k(G_2) : \phi^* k(G_1)] < \infty \\ \qquad\qquad |\phi^{-1}(\phi(g_1))| = 1 \end{array}\right\} \Rightarrow [k(G_2):\phi^* k(G_1)]_{sep} = 1. \quad \text{(II)}$$

$$\left.\begin{array}{l} \text{(I)},\text{(II)} \Rightarrow k(G_2) = \phi^* k(G_1) \\ \quad G_2 : \text{smooth} \Rightarrow G_2 : \text{normal} \end{array}\right\} \Rightarrow \phi : \text{isomorphism (of varieties.)} \quad \blacksquare$$

<u>Corollary</u>. X, Y: irreducible G-homogen. spaces

$\phi: X \to Y$: G-equivar. morphism

ϕ: bijection.

<u>TFAE</u>: (1) $d\phi|_{x_0}$ is surjective for some x_0.

(2) ϕ is separable.

(3) ϕ is an isomorphism.

(The same proof as above.)

Theorem. Let G be an algebraic group and H be a closed subgp of
Then $\exists$ a quasi-projective G-homogen. space X and $x_0 \in$
s.t. ① $\{g \in G \mid g \cdot x_0 = x_0\} = H$.
② $\psi: G^\circ \to X^\circ$, $\psi(g) := g \cdot x_0$ is a separable morphism

Pf. We have already proved ①:

Let $\pi: G \to GL(V)$ be an algebraic group homomorphism and $l \subset V$ be a line as in the previous theorem.

Let $X := \pi(G)\, l \subseteq \mathbb{P}(V)$. Then $\exists\, \mathcal{O} \subseteq X$ which is open in the closure $\overline{X} \subseteq \mathbb{P}(V)$ of X. Since X is G-homogen., X is open in $\overline{X}$. And so it is quasi-projective G-homog. space. Let $x_0 := l$, and so the way (π, l) are chosen implies part 1.

To show ψ is separable, we will show $d\psi|_e$ is surjective.

- X is open in $\overline{X} \Rightarrow T_{x_0} X = T_{x_0} \overline{X}$

$\overline{X}$ is closed in $\mathbb{P}(V) \Rightarrow T_{x_0} \overline{X} \hookrightarrow T_{[l]}(\mathbb{P}(V))$.

So we need to study $T_l(\mathbb{P}(V))$. For this, we need an open affine neighborhood U of ℓ in $\mathbb{P}(V)$.

Let $v_1 \in V$ be s.t. $\ell = k\, v_1$; and let's extend it to a basis $\{v_1, \dots, v_n\}$ of V. Suppose $\{v_1^*, \dots, v_n^*\}$ is the dual basis.

Let $U := \{ [x] \in \mathbb{P}(V) \mid v_1^*(x) \neq 0 \}$. Then

$$U \longrightarrow k^{n-1}, \quad [x] \longmapsto \left(\frac{v_2^*(x)}{v_1^*(x)}, \dots, \frac{v_n^*(x)}{v_1^*(x)} \right) \quad \text{is an isomorphism.}$$

And so U is an open affine nbhd of ℓ, and $T_\ell(\mathbb{P}(V))$ can be identified with k^{n-1} as above. So, if $p: V\setminus\{0\} \to \mathbb{P}(V)$, $p(v) := [v]$

then

$$\begin{tikzcd}
\omega \arrow[d, mapsto] & T_{v_1}(V\setminus\{0\}) \arrow[r, "dp|_{v_1}"] \arrow[d] & T_{[v_1]}(\mathbb{P}(V)) \arrow[r, "\sim"] \arrow[d, phantom, "\circlearrowleft"] & T_0 \mathbb{A}^{n-1} \arrow[d] \\
(v_1^*(\omega), \dots, v_n^*(\omega)) & k^n \arrow[rr] & & k^{n-1}
\end{tikzcd}$$

$$(x_1, \dots, x_n) \longmapsto (x_2, \dots, x_n)$$

And so $\ker(dp|_{v_1}) = \ell$.

Now notice that

$$\begin{tikzcd}
G \arrow[r, "\pi"] \arrow[rrr, bend right, "\varphi"'] & GL(V) \arrow[r, "c"] & V\setminus\{0\} \arrow[r, "p"] & \mathbb{P}(V),
\end{tikzcd}$$

where $c(g) := g\, v_1$. Therefore

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "d\pi|_e"] \arrow[rrr, bend right, "d\varphi|_e"'] & \mathfrak{gl}(V) \arrow[r, "dc|_I"] & V \arrow[r, "dp|_{v_1}"] & T_\ell(\mathbb{P}(V))
\end{tikzcd}$$

$$\xrightarrow{\quad d\psi|_e \quad}$$

$$x \longmapsto d\pi|_e(x) \longmapsto d\pi|_e(x)(v_1) \longmapsto dp|_{v_1}\big(d\pi|_e(x)(v_1)\big).$$

Hence $x \in \ker\left(d\psi|_e\right) \iff d\pi|_e(x)(v_1) \in \ker dp|_{v_1} = \ell$

$$\iff d\pi|_e(x)(\ell) \subseteq \ell.$$

$$\iff x \in \mathfrak{h}.$$

Thus $\dim \operatorname{Im}\left(d\psi|_e\right) = \dim \mathfrak{g} - \dim \ker\left(d\psi|_e\right)$

$$= \dim \mathfrak{g} - \dim \mathfrak{h} = \dim G - \dim H. \qquad \mathrm{I}$$

- X is G-homogen $\Rightarrow$ X is smooth $\Rightarrow$ $\dim X = \dim T_\ell X$. II
- $G \xrightarrow{\psi} X$ is dominant and G°, X° are irreducible $\Rightarrow$

  $\exists$ an open subset U of G s.t.

  $\forall x \in X$ and an irreducible compon. Z of $\psi^{-1}(x)$, we have

  $\dim Z = \dim G - \dim X$.

Since $X^\circ = G^\circ \cdot x_0$, $\psi^{-1}(x) = g_x(H \cap G^\circ)$ for some $g_x \in G$.

$\Rightarrow$ $\dim \psi^{-1}(x) = \dim(H \cap G^\circ) = \dim H$

Therefore $\dim H = \dim G - \dim X$. III

$\mathrm{I}, \mathrm{II}, \mathrm{III} \Rightarrow d\psi|_e$ is surjective. $\blacksquare$

<u>Def.</u> A quotient of G by H is a pair (X, x) of a G-homoge. space X and a point $x \in X$ s.t. the following

universal property holds:

Y: G-homog. space ; $\mathrm{Stab}_G(y) \supseteq H$

$\Rightarrow \exists!$ G-equivar. morphism $X \xrightarrow{\phi} Y$ s.t.

$\phi(x) = y$.

$\boxed{\text{Theorem. } G\text{: affine algebraic; } H\text{: closed subgroup} \Rightarrow \text{A quotient of } G \text{ by } H \text{ exists; it is unique; it is quasi-projective; it is denoted by } G/H\text{, and } G^\circ \to (G/H)^\circ \text{ is separable.}}$

__Theorem__. G: affine algebraic; H: closed subgroup

$\Rightarrow$ A quotient of G by H exists ; it is unique;

it is quasi-projective; it is denoted by G/H , and

$G^\circ \longrightarrow (G/H)^\circ$ is separable.

__Pf.__ We have proved the existence of (X, x_0) s.t.

① X : G-homogen.; and so smooth (and therefore normal)

② $\mathrm{Stab}\, x_0 = H$

③ $G^\circ \longrightarrow G^\circ \cdot x_0$ is separable.

__Claim__ (X, x_0) satisfies the universal property of the quotient space.

(Notice that the uniqueness is a clear conseq. of the universal property.)

Let (Y, y) be a pair of a G-homog. space, and $y \in Y$ s.t.

$\mathrm{Stab}(y) \supseteq H$. So $\phi: X \longrightarrow Y$, $\phi(g \cdot x_0) := g \cdot y$

is a well-defined, G-equivariant function. (__Set theoretic__).

And ϕ is the unique G-equ. map which sends x_0 to y.

Topology. Why is ϕ continuous?

Suppose $U \subseteq Y$ is open $\Rightarrow$

$\tilde{U} := \{g \in G \mid g \cdot y \in U\}$ is open in G, and $\tilde{U}H = \tilde{U}$,

and $\phi^{-1}(U) = \psi(\tilde{U})$, where $\psi(g) := g \cdot x_0$.

So it is enough to show ψ is an open function.

$\psi: G^\circ \to X^\circ$ is a dominant morphism of irredu. varieties

and X° is smooth (, and so normal).

$\Rightarrow \exists\, \mathcal{O} \subseteq G^\circ$ non-empty open s.t. $\psi|_{\mathcal{O}}$ is open

$\Rightarrow \psi|_{g\mathcal{O}}$ is open $\forall g \in G \Rightarrow \psi$ is open as G has a (finite) open cover $\bigcup g\mathcal{O}$.

Ringed space. Why is ϕ a ringed space homomorphism?

Suppose $G = G^\circ$.

We will get an intrinsic understanding of $\mathcal{O}_X(U)$ for any open subset U of X.

Proposition. $\psi^*: \mathcal{O}_X(U) \xrightarrow{\sim} \mathcal{O}_G(\psi^{-1}(U))^H$,

where $\mathcal{O}_G(\psi^{-1}(U))^H = \left\{ f : \psi^{-1}(U) \to k \;\middle|\; \begin{array}{l} \text{① } f \text{ is regular} \\ \text{② } f(gh) = f(g) \\ \quad \forall g \in \psi^{-1}(U), h \in H \end{array} \right\}$.

Pf of proposition. $\psi^*(f)(gh) = f(\psi(gh)) = f(\psi(g))$

$$= \psi^*(f)(g).$$

$$\Longrightarrow \ \mathrm{Im}\, \psi^* \subseteq \mathcal{O}_G(\psi^{-1}(U))^H .$$

- Suppose $f \in \mathcal{O}_G(\psi^{-1}(U))^H$. Let $\Gamma_f \subseteq \psi^{-1}(U) \times \mathbb{A}^1$ be the graph of f; and

$$\begin{tikzcd}
G \arrow[r] & \Gamma_f \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & \Gamma_f' \arrow[dr] \arrow[d, phantom, "\cap" rotate=90] & \\
& \psi^{-1}(U) \times \mathbb{A}^1 \arrow[r, "(\psi, id.)"'] & U \times \mathbb{A}^1 \arrow[r, "pr"'] & U ,
\end{tikzcd}$$

where $\Gamma_f' = \{ (\psi(g), f(g)) \mid g \in \psi^{-1}(U) \}$.

<u>Claim 1.</u> Γ_f' is close in $U \times \mathbb{A}^1$.

<u>Pf of claim 1.</u> We have to show $U \times \mathbb{A}^1 \setminus \Gamma_f'$ is open in $X \times \mathbb{A}^1$. Since Γ_f is close in $\psi^{-1}(U) \times \mathbb{A}^1$ and $U \times \mathbb{A}^1 \setminus \Gamma_f' = (\psi, id.)(\psi^{-1}(U) \times \mathbb{A}^1 \setminus \Gamma_f)$, it is enough to prove $(\psi, id.)$ is an open function, which can be proved as above using the homgen. of G and X. $\blacksquare$

<u>Claim 2.</u> $\Gamma_f' \xrightarrow[pr]{} U$ is a bijection.

<u>Pf of Claim 2.</u> $\psi(g_1) = \psi(g_2) \Rightarrow g_1 = g_2 h$ for some $h \in H$

$$\Longrightarrow f(g_1) = f(g_2 h) = f(g_2) \Longrightarrow (\psi(g_1), f(g_1)) = (\psi(g_2), f(g_2)).$$

$\blacksquare$

<u>Claim 3.</u> $\Gamma_f' \xrightarrow[pr]{} U$ is separable.

Pf of claim 3. $\psi^{-1}(U) \longrightarrow \Gamma_0 \longrightarrow \Gamma_0' \xrightarrow{pr} U$, and ψ is separable

<u>Pf of claim 3.</u> $\psi^{-1}(U) \longrightarrow \Gamma_f \longrightarrow \Gamma_f' \xrightarrow{pr} U$, and ψ is separable

$$\begin{array}{ccc} \psi^{-1}(U) & & U \\ \text{open } \cap\!\!\mid & & \text{open } \cap\!\!\mid \\ G & \xrightarrow[\psi]{\hspace{3em}} & X \end{array}$$

$\Rightarrow \Gamma_f' \xrightarrow{pr} U$ is separable. ■

<u>Claim 4.</u> $\Gamma_f' \xrightarrow[pr]{} U$ is birational.

<u>Pf of claim 4.</u> In an open set, dimension of any irred. compon. of the fibers are $\dim \Gamma_f' - \dim U$. So $\dim \Gamma_f' = \dim U \Rightarrow k(\Gamma_f') / _{pr^*(k(U))}$ is a finite extension. $\Rightarrow$ in an open subset of Γ_f' we have

$$\left.\begin{array}{r} 1 = |pr^{-1}(pr(x))| = [k(\Gamma_f') : pr^*(k(U))]_{sep} \\ pr : \Gamma_f' \longrightarrow U \text{ separable} \end{array}\right\} \Rightarrow \Gamma_f' \xrightarrow[pr]{} U \text{ is birational. } ■$$

<u>Claim 5.</u> $\Gamma_f' \xrightarrow[pr]{} U$ is an isomorphism.

<u>Pf of claim 5.</u> $\left.\begin{array}{l} U \text{ is open in } X \\ X \text{ smooth} \end{array}\right\} \Rightarrow U$ is normal

$\Gamma_f' \xrightarrow[pr]{} U$: bijection, birational $\Rightarrow \Gamma_f' \xrightarrow[pr]{} U$ is an isomorphism.

. Let $F : = \Gamma_f' \longrightarrow k$ be the projection to the 2^{nd} compon.

$$\begin{array}{c} \Gamma_f' \\ \cap\!\!\mid \\ U \times \mathbb{A}^1 \end{array}$$

And $G \longrightarrow \Gamma_f \longrightarrow \Gamma_f'$, with $G \xrightarrow[\eta]{} \Gamma_f'$. Then $\eta^*(F)(g) = F(\eta(g))$
$= f(g)$.

$g \longmapsto (\psi(g), f(g))$

$$g \longmapsto^{\eta} (\psi(g), f(g))$$

$$= f(g).$$

$$\Rightarrow f \in \eta^*(\mathcal{O}_{\Gamma_f'}(\eta(\psi^{-1}(U))))$$

$$\Rightarrow f \in \psi^*(\mathcal{O}_U(U)) = \psi^*(\mathcal{O}_X(U)).$$ ∎

Ex. complete the proof. ∎

<u>Corollary</u>. G : affine algebraic group, X : variety, $G \curvearrowright X$ $\Rightarrow$ $\forall x \in X$, $\exists!$ G-equivariant morphism $\phi : G/G_x \to G\cdot x$

And ϕ is an isomorphism if and only if $\begin{array}{l} G \to G\cdot x \\ g \mapsto g\cdot x \end{array}$ is separable.

In particular, $G/G_x \xrightarrow{\sim} G\cdot x$ if $\mathrm{char}(k) = 0$.

<u>Corollary</u>. If G/H is affine, then $k[G/H] \simeq k[G]^H$.

<u>Corollary</u>. If G is connected, then $k(G/H) \simeq k(G)^H$.

<u>Ex.</u> G/H is affine $\Leftrightarrow$ ① $k[G]^H$ is finitely generated.

② $\forall g_1 H \neq g_2 H$, $\exists f \in k[G]^H$ s.t. $f(g_1 H) \neq f(g_2 H)$.

<u>Theorem</u> ⓐ G : affine algebraic group, $N \triangleleft G$: a closed normal subgp $\Rightarrow$ $\exists$ $\pi : G \to GL(V)$ an algebraic group homomorphism s.t. ① $\ker(\pi) = N$.

$N \trianglelefteq G$: a closed normal subgp s.t. ① $\ker(\bar{\pi}) = N$.

② $\ker(d\bar{\pi}|_e) = \mathrm{Lie}(N)$.

ⓑ $G/_N$ is an affine algebraic group.

Pf. ⓐ We have proved $\exists\, \pi: G \to GL(V)$ an algebraic group homomorphism, and a line $\ell \subseteq V$ s.t. ① $\{g \in G \mid g\,\ell = \ell\} = N$,

② $\{x \in \mathfrak{g} \mid d\pi|_e(x)(\ell) \subseteq \ell\} = \mathrm{Lie}\ N$.

Let v_0 be a non-zero vector in ℓ. Then, for any $n \in N$,

$\pi(n) = \chi_0(n)\, v_0$ for some $\chi_0(n) \in k^\times$. Then

① Since π is a morphism, $\chi_0: N \to k$ is a morphism.

② Since π is a homomorphism, $\chi_0: N \to k^\times$ is an abstract group homomorphism.

One can use ①, ② to show that $\chi_0: N \to GL_1(k)$ is an algebraic group homomorphism. ($X^*(N) := \mathrm{Hom}(N, GL_1(k))$ is called the set of characters of N.)

For any $\chi \in X^*(N)$, let $V_\chi := \{v \in V \mid \pi(n)(v) = \chi(n)\, v\}$; let $\overline{V} := \sum_{\chi \in X^*(N)} V_\chi$.

Lemma. If $\chi_1, \dots, \chi_m$ are distinct characters of a group H and $V_{\chi_i} \neq 0$, then $V_{\chi_1}, \dots, V_{\chi_m}$ are linearly independent, i.e.

and $V_{\chi_i} \neq 0$, then $V_{\chi_1}, \ldots, V_{\chi_m}$ are linearly independent, i.e.
$\left(v_i \in V_{\chi_i} \text{ and } v_1 + \cdots + v_m = 0 \right) \Rightarrow v_i = 0$.

Pf of Lemma You have seen this lemma, when you were studying Galois theory:

Suppose m is the smallest positive integer s.t.

$$\exists \; v_i \in V_{\chi_i} \setminus \{0\} \text{ where } \sum_{i=1}^{m} v_i = 0. \quad \circledast$$

Let $h \in H$ be s.t. $\chi_1(h) \neq \chi_2(h)$. Then

$$0 = \left(h \cdot \sum_{i=1}^{m} v_i \right) - \chi_1(h) \left(\sum_{i=1}^{m} v_i \right) = \sum_{i=1}^{m} \left(\chi_i(h) - \chi_1(h) \right) v_i$$

$$= \sum_{i=2}^{m} \left(\chi_i(h) - \chi_1(h) \right) v_i \quad ? \Bigg] , \text{ which contracts } \circledast .$$

$$\left(\chi_2(h) - \chi_1(h) \right) v_2 \in V_{\chi_2} \setminus \{0\}$$

$\blacksquare$

Corollary. For any group H acting linearly on a vector space V,

$$\left| \{ \chi \in \mathrm{Hom}(H, k^\times) \mid V_\chi \neq 0 \} \right| \leq \dim V. \quad \blacksquare$$

- $\forall g \in G$, $\chi \in X^*(N)$, let ${}^g\chi : N \to k^\times$, ${}^g\chi(n) := \chi(g^{-1} n g)$
  $\Rightarrow {}^g\chi \in X^*(N)$.

- $\forall g \in G$, $\chi \in X^*(N)$, $v \in V_\chi$, $n \in N$,

$$\pi(n)(\pi(g) v) = \pi(g) \left(\pi(g^{-1} n g) v \right)$$

$$= \pi(g) \left(\chi(g^{-1} n g) v \right)$$

$$= {}^{g}\chi(n)\ \pi(g)v$$

$$\Rightarrow \pi(g)v \in V_{{}^{g}\chi}.$$

$\Rightarrow \overline{V}$ is G-invariant, and $\ell \subseteq V_{\chi_0} \subseteq \overline{V}$;

So, by restricting $\pi(g)$ to its action on $\overline{V}$, w.l.o.g. we can and will assume $\overline{V} = V$.

Let $\{e_{i,\chi}\}_i$ be a basis of V_χ. Then, in this basis, $\pi(N)$ is of the form:

$$\left[\begin{array}{c:c:c} \chi_1(n)\, I_{V_{\chi_1}} & \cdots & 0 \\ \hdashline \vdots & \ddots & \vdots \\ \hdashline 0 & \cdots & \chi_m(n)\, I_{V_{\chi_m}} \end{array}\right] \subseteq \begin{bmatrix} GL(V_{\chi_1}) & & \\ & \ddots & \\ & & GL(V_{\chi_m}) \end{bmatrix}.$$

Let $W \subseteq \mathrm{End}(\bigoplus V_{\chi_i})$ be the subspace consisting of linear maps L s.t. $L(V_{\chi_i}) \subseteq V_{\chi_i}$ for any i. In the above basis, W cosists of matrices of the form

$$\begin{array}{c} \left[\begin{array}{c:c:c:c} * & 0 & \cdots & 0 \\ \hdashline 0 & * & \cdots & 0 \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline 0 & 0 & \cdots & * \end{array}\right] \\ \begin{array}{cccc} \underleftrightarrow{V_{\chi_1}} & \underleftrightarrow{V_{\chi_2}} & & \underleftrightarrow{V_{\chi_m}} \end{array} \end{array}.$$

So $W \subseteq C_{\mathrm{End}(V)}(\pi(N))$; and

$\forall g \in G,\ L \in W,\ \chi \in X^*(N),\ \pi(g)^{-1} L \pi(g) V_\chi = \pi(g)^{-1} L V_{g\chi}$

$$\subseteq \pi(g)^{-1} V_{g\chi} = V_{g^{-1}g\chi} = V_\chi .$$

$$\Longrightarrow \pi(g)^{-1} L \pi(g) \in W.$$

Let $\bar{\pi} : G \longrightarrow GL(W)$, $\bar{\pi}(g)(L) := \pi(g) L \pi(g)^{-1}$.

Then, by the above discussion and since π is an algebraic group homomorphism, $\bar{\pi}$ is an algebraic group homomorphism.

- $g \in \ker \bar{\pi} \Longleftrightarrow \pi(g)$ commutes with $\bigoplus \mathrm{End}(V_{\chi_i})$

$$\Longleftrightarrow \pi(g) \text{ acts via scalar multi: on } V_{\chi_i}$$

$$\begin{cases} \Longleftarrow g \in N \\ \Longrightarrow \pi(g)(\ell) = \ell \Rightarrow g \in N. \end{cases}$$

So $\ker \bar{\pi} = N \Longrightarrow \mathrm{Lie}\, N \subseteq \ker(d\bar{\pi}|_e)$

- $x \in \ker(d\bar{\pi}|_e) \Longleftrightarrow d\bar{\pi}|_e(x) = 0$

Consider $\quad G \xrightarrow{\pi} GL(V) \xrightarrow{Ad} GL(\mathrm{End}\, V)$.

$$\forall L \in W,\ \bar{\pi}(g)(L) = Ad(\pi(g))(L).$$

$$\Longrightarrow d\bar{\pi}|_e(x) = \left(d\,Ad|_I \circ d\pi|_e(x) \right)\Big|_W .$$

$$\Longrightarrow d\bar{\pi}|_e(x) = 0 \Longleftrightarrow ad\,(d\pi|_e(x))\Big|_W = 0.$$

$$\Longleftrightarrow [d\pi|_e(x), W] = 0$$

$$\Longleftrightarrow d\pi|_e(x) \subset C \quad (W).$$

$$\Leftrightarrow \quad d\pi|_e(X) \subseteq C_{\mathrm{End}(V)}(W).$$

$$\Leftrightarrow \quad d\pi|_e(X) \text{ acts by scalars on } V_\chi$$

$$\Rightarrow \quad d\pi|_e(X)(\ell) \subseteq \ell$$

$$\Leftrightarrow \quad X \in \mathrm{Lie}(N).$$

So $\ker(d\bar{\pi}|_e) = \mathrm{Lie}\, N$.

(b) Let $\bar{G}$ be the image of $\bar{\pi}$. Then as we have proved earlier, $\bar{G}$ is a closed subgroup of $GL(W)$.

So $\bar{\pi}: G \to \bar{G}$ is a surjective algebraic group homomorphism and $\ker \bar{\pi} = N$ and $\ker d\bar{\pi}|_e = \mathrm{Lie}\, N$.

$\Rightarrow$ (1) $\dim \bar{G} = \dim G - \dim N$

(2) $\bar{\pi}$ is separable.

$\Rightarrow$ $\exists$ an isomorphism of varieties $\psi: G/N \to \bar{G}$ which is G-equivariant (why? within the proof of the existence of a quotient we have proved:

X: G-homog. and $g \mapsto g \cdot x_0$ separable

implies $X \simeq G/G_{x_0}$.)

$\Rightarrow$ (1) G/N is an affine algebraic group.

(2) $\psi: G \to G/N$, $\psi(g) := gN$ is an algebraic group homomor.

③ $\ker \psi = N$.

④ ψ is separable ; $d\psi|_e$ is surjective ; $\ker d\psi|_e = \mathrm{Lie}\, N$; ∎

<u>Theorem</u> Let $\pi : G_1 \to G_2$ be an algebraic group homomorphism. Then the following is a bijective algebraic group homomorphism $\overline{\pi} : G_1/_{\ker \pi} \longrightarrow \mathrm{Im}\, \pi$,

$$\overline{\pi}(g \ker \pi) := \pi(g);$$

$\dim G_1 = \dim(\mathrm{Im}\, \pi) + \dim(\ker \pi)$;

Moreover TFAE :

① $\overline{\pi}$ is an isomorphism of algebraic groups.

② $\overline{\pi}$ is separable.

③ $\ker(d\overline{\pi}|_e) = \mathrm{Lie}(\ker \pi)$

④ $\ker\left(G_1(k[\varepsilon]) \xrightarrow{\pi_{k[\varepsilon]}} G_2(k[\varepsilon])\right) = (\ker \pi)(k[\varepsilon])$

⑤ $d\overline{\pi}|_e$ is surjective.

(Pf. <u>Exercise</u>)