Jordan decomposition

Tuesday, May 30, 2017 11:57 AM

<u>Recall</u>. (Jordan normal form) [Additive]

Let $a \in \mathrm{End}(V)$, where V is a finite-dimensional k-vector space.

Let's view V as a $k[\underline{x}]$-mod where $\underline{x} \cdot v := a(v)$.

Since $\dim_k V < \infty$, it is a f.g. $k[\underline{x}]$-mod. By the classification of f.g. modules of a PID, we have

$$V \simeq \bigoplus k[\underline{x}]/\langle p_i^{n_i} \rangle \quad \text{where } p_i \in k[\underline{x}] \text{ is irreducible.}$$

Since k is algebraically closed, $p_i = \underline{x} - \lambda_i$.

[In the basis $1, [\underline{x}-\lambda], \ldots, [\underline{x}-\lambda]^{n-1}$ of $k[\underline{x}]/\langle (\underline{x}-\lambda)^n \rangle$, matrix of multip. by $\underline{x}-\lambda$ is $\begin{bmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}$. So the matrix of multiplication by $\underline{x}$ is $J_n(\lambda) := \begin{bmatrix} \lambda & & 0 \\ 1 & \ddots & \\ 0 & 1 & \lambda \end{bmatrix}$; this gives the Jordan form of a.]

So $V \simeq \bigoplus_i \bigoplus_j k[\underline{x}]/\langle (\underline{x}-\lambda_i)^{n_{ij}} \rangle$, where λ_i's are distinct elements of k.

Let $E_i : V \to V$ be the projection

$$E_i(v) = \begin{cases} 0 & v \in k[\underline{x}]/\langle (\underline{x}-\lambda_{i'})^{n_{i'j}} \rangle \text{ and } i' \neq i \\ v & v \in k[\underline{x}]/\langle (\underline{x}-\lambda_i)^{n_{ij}} \rangle . \end{cases}$$

We'd like to show $E_i(v) = f_i(\underline{x}) \cdot v$ for some polynomial f_i.

This happens if and only if $f_i(\underline{x}) \in \langle \underline{x} - \lambda_{i'} \rangle^{n_{i'j}}$ for $i' \neq i$

and $(f_i(\underline{x}) - 1) \in \langle \underline{x} - \lambda_i \rangle^{n_{ij}}$;

By Chinese Remainder Theorem,

$\exists f_i(\underline{x})$ s.t. $f_i(\underline{x}) \equiv 0 \pmod{(\underline{x} - \lambda_{i'})^{\max_j n_{i'j}}}$

and $f_i(\underline{x}) \equiv 1 \pmod{(\underline{x} - \lambda_i)^{\max_j n_{ij}}}$.

Let $f_{sem}(\underline{x}) := \sum_i \lambda_i f_i(\underline{x})$, and $f_{nil}(\underline{x}) := \underline{x} - f_{sem}(\underline{x})$.

Then, for $v \in k[x]/\langle \underline{x} - \lambda_i \rangle^{n_{ij}}$, we have $f_{sem}(\underline{x}) \cdot v = \lambda_i v$

and so $f_{nil}(\underline{x}) \cdot v = (\underline{x} - \lambda_i) \cdot v$. Therefore

- $a = f_{sem}(a) + f_{nil}(a)$
- $f_{sem}(a)$ is diagonalizable (semisimple element) $\leftarrow a_s$
- $f_{nil}(a)$ is nilpotent $\leftarrow a_n$

$\Rightarrow$
- $a_s a_n = a_n a_s$
- $C_{End(V)}(a) = C_{End(V)}(a_s) \cap C_{End(V)}(a_n)$

Ex. $S \subseteq M_n(k)$: a set of pairwise commuting matrices

① $\exists x \in GL_n(k)$ s.t. $x S x^{-1} \subseteq$ lower triangular matrices

② If S consists of semisimple elements, then

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② If S consists of semisimple elements, then $\exists x \in GL_n(k)$ s.t. $xSx^{-1} \subseteq$ diagonal matrices.

(Use induction on $\underline{n}$.)

<u>Ex.</u> Prove that there is a unique pair (s, n) of a semisimple element s and a nilpotent element n s.t.
① $a = s + n$ ② $sn = ns$.

<u>Corollary</u>. $a \in \mathrm{End}(V)$; $W \subseteq V$; $a(W) \subseteq W$

$$\Rightarrow a_s|_W \text{ and } a_n|_W \text{ give us the Jordan decomposition of } a|_W.$$

(Use the above mentioned uniqueness.)

<u>Corollary</u>.

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "b"] \\ V \arrow[r, "\phi"] & W \end{tikzcd} \Rightarrow \begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "a_s"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "b_s"] \\ V \arrow[r, "\phi"] & W \end{tikzcd} \text{ and } \begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "a_n"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "b_n"] \\ V \arrow[r, "\phi"] & W \end{tikzcd}$$

<u>Pf</u>. Using the above corollary, we can assume ϕ is surjective.

$$\Rightarrow V \simeq \bigoplus_{i,j} k[x]/\langle (x-\lambda_i)^{n_{ij}} \rangle \text{ and } W \simeq \bigoplus_{i,j} k[x]/\langle (x-\lambda_i)^{m_{ij}} \rangle$$

and $n_{ij} \geq m_{ij}$. So $f^{(V)}_{sem}$ works as well for $f^{(W)}_{sem}$.

$(\max_j n_{ij} \geq \max_j m_{ij})$. ∎

<u>Corollary</u>. $a \oplus b \in GL(V \oplus W) \Rightarrow \begin{cases} a \otimes b = (a_s \otimes b_s)(a_u \otimes b_u) \\ a \oplus b = (a_s \oplus b_s)(a_u \oplus b_u) \end{cases}$

Corollary: $a \oplus b \in GL(V \oplus W) \Rightarrow \begin{cases} a \otimes b = (a_s \otimes b_s)(a_u \otimes b_u) \\ a \oplus b = (a_s \oplus b_s)(a_u \oplus b_u) \end{cases}$

are Jordan decompositions

Theorem (Jordan decomposition: multiplicative version)

$\forall g \in GL(V)$, $\exists!\ g_s, g_u$ s.t. ① g_s: semisimple; g_u: unipo.

② $g_u \cdot g_s = g_s \cdot g_u$.

For such g_s and g_u we have:

$$g_s = f_{sem}(g) \quad \text{and} \quad g_u = f_{uni}(g)$$

And a similar commuting diagram holds for this decomposition.

Pf. Let $g_u := (I + g_s^{-1} g_n)$. $\square$

Theorem (Jordan decomp.: locally-finite case)

Suppose $x \in End(V)$ and $g \in GL(V)$; $\dim V = \infty$;

$\forall v \in V$, $\exists\ W \subseteq V$ s.t. ① $v \in W$ ② $\dim W < \infty$

③ $xW, gW \subseteq W$.

$\Rightarrow \exists!\ x_s, x_n \in End(V)$ s.t. ① $x_s x_n = x_n x_s$, $x = x_s + x_n$

② $x_s|_W$ is s.s., $x_n|_W$ is nilpot.

for any finite-dim. x-invari. subspace W

$\exists!\ g_s, g_u \in GL(V)$ s.t. ① $g_s g_u = g_u g_s$, $g = g_s g_u$

$\exists!\ g_s, g_u \in GL(V)$ s.t. ① $g_u g_s = g_s g_u$, $g = g_s g_u$

② $g_s|_W$: s.s. ; $g_u|_W$: unipo.

for any finite-dim. g-invar. subsp. W.

Pf. Using finite-dim. filter. and compati. of Jordan decompositions, prove this. □

Theorem. (Jordan decomposition : affine algebraic group case)

Let G be an affine algebraic group. Then for any $g \in G$

① $\exists!\ g_u, g_s \in G$ s.t. $\rho(g_s) = \rho(g)_s$ and $\rho(g_u) = \rho(g)_u$

where $\rho(g) : k[G] \to k[G]$, $(\rho(g) f)(g') = f(g'g)$

(in particular, $g_u g_s = g$.)

② If $\phi : G \to G'$ is an algebraic group homomorphism, then

$$\phi(g_s) = \phi(g)_s \quad \text{and} \quad \phi(g_u) = \phi(g)_u .$$

③ If $G \subseteq GL_n$, then $g_s \cdot g_u$ is the (matrix) Jordan decomposition of g.

Pf. ① Let $\rho(g)_s \cdot \rho(g)_u$ be the Jordan decomposition of $\rho(g)$.

We'd like to find $g_s, g_u \in G(k) = \mathrm{Hom}(k[G], k)$ such that

$$\rho(g)_s = \rho(g_s) \iff \forall f \in k[G],\ \rho(g)_s(f) = \rho(g_s)(f)$$

What is the relation between an element $k[G] \xrightarrow{g} k$ and $k[G] \xrightarrow{\rho(g)} k[G]$?

$g(f)$ is the evaluation map $f \mapsto f(g)$. So g can be identified with $f \mapsto \big(\rho(g)(f)\big)(e)$.

So we should consider $f \mapsto (\rho(g)_s f)(e)$ and show it is a k-algebra homomorphism $k[G] \to k$. Let m be the multip. homomorphism. So

$$\begin{tikzcd} k[G]\otimes k[G] \arrow[r, "m"] \arrow[d, "\rho(g)\otimes\rho(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & k[G] \arrow[d, "\rho(g)"] \\ k[G]\otimes k[G] \arrow[r, "m"] & k[G] \end{tikzcd} \Rightarrow \begin{tikzcd} k[G]\otimes k[G] \arrow[r, "m"] \arrow[d, "\rho(g)_s\otimes\rho(g)_s"'] \arrow[dr, phantom, "\circlearrowleft"] & k[G] \arrow[d, "\rho(g)_s"] \\ k[G]\otimes k[G] \arrow[r] & k[G] \end{tikzcd}$$

as $\rho(g) \in Aut(k[G])$

$\Rightarrow \rho(g)_s \in Aut(k[G]) \Rightarrow$ $f \mapsto \big(\rho(g)_s f\big)(e)$ is in $Hom(k[G], k)$;

so we get $g_s \in G$ s.t.

$$(\rho(g)_s f)(e) = (\rho(g_s) f)(e).$$

Since $\lambda(x)$ commutes with $\rho(g)$ we get

$$(\rho(g)_s f)(x) = \big(\lambda(x^{-1})(\rho(g)_s f)\big)(e)$$

$$= \left(\rho(g)_s \left(\lambda(x^{-1}) f\right)\right)(e)$$

$$= \rho(g_s) \left(\lambda(x^{-1}) f\right)(e)$$

$$= \left(\lambda(x^{-1}) f\right)(g_s)$$

$$= f(x g_s) = \left(\rho(g_s) f\right)(x)$$

$$\Rightarrow \quad \rho(g)_s f = \rho(g_s) f .$$

Similarly $\rho(g)_u = \rho(g_u)$.

② $G \longrightarrow \mathrm{Im}\,\phi \longrightarrow G'$. So it is enough to consider the following cases:

ⓐ G is a closed subgroup of G'

ⓑ ϕ is surjective.

$\boxed{\text{Case ⓐ}}$ $\quad G = \{ g \in G' \mid \rho(g) I_G = I_G \}$.

$\Rightarrow$ Let $g_s g_u$ be the Jordan decomp. of $g \in G$ in G'.

$$\Rightarrow \left\{ \begin{array}{l} \rho(g_s) I_G = \rho(g)_s I_G = I_G \\ \rho(g_u) I_G = \rho(g)_u I_G = I_G \end{array} \right\} \Rightarrow g_s, g_u \in G .$$

$\Rightarrow$ $g_s g_u$ is the Jordan decomp of g in G as well.

$\boxed{\text{Case ⓑ}}$ $\quad k[G'] \xhookrightarrow{\phi^*} k[G]$

and $\phi^* k[G']$ is $\rho(g)$-invariant

$$\Longrightarrow \quad \phi^* k[G'] \text{ is } \rho(g)_s\text{-invariant}$$

and

$$\begin{tikzcd} k[G'] \arrow[r, "\phi^*", "\sim"'] \arrow[d, "\rho(g)_s = \rho(g_s)"'] \arrow[dr, phantom, "\circlearrowleft"] & \phi^* k[G'] \arrow[d, "\rho(\phi(g_s))"] \\ k[G'] \arrow[r, "\phi^*", "\sim"'] & \phi^* k[G'] \end{tikzcd}.$$

$$\left.\begin{array}{l} \Longrightarrow \quad \rho(\phi(g_s)) = \rho(\phi(g))_s . \\ \text{Similarly we have } \rho(\phi(g_u)) = \rho(\phi(g))_u \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \phi(g_s) = \phi(g)_s , \\ \phi(g_u) = \phi(g)_u . \end{array}\right.$$

3 Using part 2 it is enough to prove it for $G = GL(V)$.

Let $c^*: k[V] \longrightarrow k[V] \otimes k[G]$ be the co-map of $V \curvearrowleft G$.

Notice that $V^* := \mathrm{Hom}(V, k)$ can be realized as (degree 1) elements of $k[V]$. So we have

$$\begin{tikzcd} V^* \arrow[r, hook] \arrow[d, "g^*"] & k[V] \arrow[r] \arrow[dr, phantom, "\circlearrowleft"] & k[G] \otimes k[V] \arrow[d, "\rho(g) \otimes \mathrm{id}."] \\ V^* \arrow[r, hook] & k[V] \arrow[r] & k[G] \otimes k[V] \end{tikzcd}$$

<u>why</u> is it a commuting diagram?

For $f \in V^*$, suppose $c^*(f) = \sum q_i \otimes f_i$. Then

for any $g \in G$ and $v \in V$ we have $f(vg) = \sum q_i(v) f_i(g)$.

So $\quad g^*(f)(v) = f(vg) = \sum q_i(v) f_i(g)$

$$\Longrightarrow \quad g^*(f) = \sum f_i(g)\, q_i .$$

Suppose $c^*(q_i) = \sum q_{ij} \otimes f_{ij}$. Then

$$c^*(g^*(f)) = \sum_{i,j} f_i(g)\, q_{ij} \otimes f_{ij}$$

$$\Rightarrow \quad c^*(g^*(f))(v, g') = \sum_{i,j} f_i(g)\, f_{ij}(g')\, q_{ij}(v)$$

On the other hand, suppose $m^*(f_i) = \sum f_{ij}^{(l)} \otimes f_{ij}^{(r)}$. Then

$$\rho(g) f_i = \sum f_{ij}^{(r)}(g)\, f_{ij}^{(l)} \Rightarrow$$

$$(id. \otimes \rho(g))\,(c^*(f)) = (id. \otimes \rho(g))\left(\sum q_i \otimes f_i\right)$$

$$= \sum_{i,j} f_{ij}^{(r)}(g)\, q_i \otimes f_{ij}^{(l)}$$

We have

$$\begin{tikzcd}
f \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \sum q_i \otimes f_i \arrow[ddd, mapsto] \\
& k[V] \arrow[r, "c^*"] \arrow[d, "c^*"'] \arrow[dr, phantom, "\circlearrowleft"] & k[V] \otimes k[G] \arrow[d, "c^* \otimes id."] & \\
& k[V] \otimes k[G] \arrow[r, "id. \otimes m^*"'] & k[V] \otimes k[G] \otimes k[G] & \\
\sum q_i \otimes f_i \arrow[rr, mapsto] & & \sum q_i \otimes f_{ij}^{(l)} \otimes f_{ij}^{(r)} \arrow[r, phantom, "="] & \sum q_{ij} \otimes f_{ij} \otimes f_i
\end{tikzcd}$$

So $\forall\, v, g, g'$:

$$\underset{\substack{\parallel \\ \left((id.\otimes \rho(g))\,(c^*(f))\right)(v, g')}}{\sum q_i(v)\, f_{ij}^{(l)}(g')\, f_{ij}^{(r)}(g)} = \underset{\substack{\parallel \\ c^*(g^*(f))(v, g'),}}{\sum q_{ij}(v)\, f_{ij}(g')\, f_i(g)}$$

which shows the above diagram is commutative.

which shows the above diagram is commutative.

$\Rightarrow$ the linear Jordan decomposition is compatible with the algebraic group Jordan decomposition:

- $\rho(g_s) \otimes id = (\rho(g) \otimes id)_s$ and $(\rho(g) \otimes id)_s \big|_{V^*}$ is the semisimple part of g^*. And we have a similar statement for the unipotent part.

[• Let V be the space of $1 \times n$ matrices; And identify V^* with the space of $n \times 1$ matrices:

$$f(v) := \langle v || f \rangle \quad \text{(matrix multiplication)}.$$

$\forall g \in GL_n(k)$, $\langle v \bullet g | := \langle v | g$ matrix multiplication.

$$\Rightarrow g^*(f)(v) := f(v \bullet g) = \langle v \bullet g || f \rangle = \langle v | g | f \rangle$$

$$= \langle v | (g | f \rangle) \Rightarrow | g^*(f) \rangle = g | f \rangle.$$

So in this basis g^* can be represented by the matrix g.] ■

<u>Theorem.</u> $G \subseteq GL_n(k)$ consists of unipotent elements

$\Rightarrow \exists\, x \in GL_n(k)$ s.t. $x G x^{-1} \subseteq$ upper-triangular matrices.

<u>Pf.</u>. We proceed by induction on n.

. If $V = k^n$ is NOT an irreducible G-module, then $\exists\ 0 \subsetneq W \subsetneq V$ which is G-invariant $\Rightarrow$ by induction

$\exists\ 0 \subsetneq W \subsetneq V$ which is G-invariant $\Rightarrow$ by induction hypoth. used for the actions of G on W and V/W, we get the result.

Suppose $V = k^n$ is irreducible; then let A be the k-span of G. $\Rightarrow V$ is a simple faithful A-module.

$\Rightarrow A = M_n(k)$ (Here one should use a theorem from non-commutative algebra, e.g. density theorem; or Schur + double centralizer; or ...)

$\Rightarrow \exists\ g_1, \dots, g_{n^2} \in G$ which form a basis of $M_n(k)$.

On the other hand, $\forall g \in G$, $tr(g) = tr(I)$ as g is unipotent.

$\Rightarrow tr((g-I)g_i) = tr(gg_i) - tr(g_i) = 0 \quad \forall g \in G, i.$

$\Rightarrow \forall x \in M_n(k)$, $tr((g-I)x) = 0 \Rightarrow g - I = 0$

$\Rightarrow g = I \Rightarrow n = 1$, we are done. ∎

<u>Corollary</u>. A unipotent algebraic group is a nilpotent group.

<u>Corollary</u>. $\left.\begin{array}{l} U \curvearrowright X \\ U: \text{unipotent algebraic gp} \\ X: \text{affine} \end{array}\right\} \Rightarrow$ all the orbits are closed.

<u>Pf.</u> $\forall x \in X$, $O_x := U \cdot x$ is open in its closure $\overline{O_x} \subseteq X$.

$\Rightarrow Y := \overline{O_x} \setminus O_x$ is a closed subset of the affine variety $\overline{O_x}$

$\Rightarrow Y := \overline{\mathcal{O}_x} \setminus \mathcal{O}_x$ is a closed subset of the affine variety $\overline{\mathcal{O}_x}$.

Let $I(Y) \lhd k[\overline{\mathcal{O}_x}] \Rightarrow I(Y)$ is a locally-finite U-mod.

$\Rightarrow \exists\ f \in I(Y)$ which is U-invariant

$\Rightarrow f|_{\mathcal{O}_x}$ is constant $\Rightarrow f|_{\overline{\mathcal{O}_x}}$ is constant $\Rightarrow f|_{\overline{\mathcal{O}_x}} = 0$

$\mapsto Y = \emptyset \Rightarrow \mathcal{O}_x = \overline{\mathcal{O}_x}$. ∎