Midterm MAT 214: Theorems and Problem Sets.

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1-(15 points) Let a and b be two non-zero integers. Show that there are integers r and s such that g.c.d.$(a, b) = ar + bs$.

2-(15 points) State a version of Wolstenholme's theorem and prove it.

3-(20 points) Define *primitive root* and show that there is a primitive root modulo a prime number.

4- Let $f(n) = \sum_{m \leq n,\ (m,n)=1} m$. Show that

a) (5 points) $f(n) = n\phi(n)/2$.

b) (10 points) If $f(n) = f(m)$, then $n = m$.

5-(5 points) Find all positive integers n such that, for any k, the binomial coefficient $\binom{n}{k}$ is odd.

6-(5 points) Show that g.c.d.$(2^{2^n} + 1, 2^{2^m} + 1) = 1$.

7-a) (10 points) Let p be a prime number and k a positive integer. Show that

$$\sum_{i=0}^{p-1} i^k \equiv \begin{cases} 0 \pmod{p} & \text{if } p-1 \nmid k, \\ -1 \pmod{p} & \text{if } p-1 | k. \end{cases}$$

b) (15 points) Let $f \in (\mathbb{Z}/p\mathbb{Z})[x_1, \ldots, x_n]$ be a polynomial in n variables with coefficients in $\mathbb{Z}/p\mathbb{Z}$. Show that if degree of f is less than n, then p divides $\# V_f(\mathbb{Z}/p\mathbb{Z})$, where

$$V_f(\mathbb{Z}/p\mathbb{Z}) = \{\mathbf{a} \in (\mathbb{Z}/p\mathbb{Z})^n |\ f(\mathbf{a}) \equiv 0 \pmod{p}\}.$$