Discrete subgroups acting transitively on vertices of a Bruhat-Tits building

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Abstract

We describe all the discrete subgroups of $\mathrm{Ad}(\mathbb{G}_0)(F) \rtimes \mathrm{Aut}(F)$ which act transitively on the set of vertices of $\mathfrak{B} = \mathfrak{B}(F, \mathbb{G}_0)$ the Bruhat-Tits building of a pair $(F, \mathbb{G}_0)$ of a characteristic zero non-archimedean local field and a simply-connected absolutely almost simple F-group if $\mathfrak{B}$ is of dimension at least 4. In fact, we classify all the maximal such subgroups. We show that there are exactly eleven families of such subgroups and explicitly construct them. Moreover, we show that four of these families act simply transitively on the vertices. In particular, we show that there is no such actions if either the dimension of the building is larger than 7, F is not isomorphic to $\mathbb{Q}_p$ for some prime p, or the building is associated to $\mathbb{SL}_{n,D_0}$ where D_0 is a non-commutative division algebra. Along the way we also give a new proof of Siegel-Klingen theorem on the rationality of certain Dedekind zeta functions and L-functions.[1]

1 Introduction and statement of the results.

1.1

In the 80s, several mathematicians constructed discrete subgroups acting transitively on the chambers of a Bruhat-Tits building [Ka85, KMW90, Me86, We85]. Among other things what makes these rare groups interesting is that they produce finite building-like geometries. In 1987, a classification of such groups was announced in [KLT87]. In this work, authors show that in the "generic case" there is no discrete chamber-transitive automorphism group of a Bruhat-Tits building. In comparison to our work, their approach is more geometric and makes use of similar results in the finite setting by Seitz [Se73].

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In [BP89], Borel and Prasad show that there are only finitely many $(F, \mathbb{G}, \Gamma)$ consisting of a non-archimedean local field F of characteristic zero, an absolutely almost simple F-group $\mathbb{G}$ of absolute rank at least 2, and a discrete subgroup Γ of $\mathbb{G}(F)$ which acts transitively on the set of facets of a given type of the associated Bruhat-Tits building as long as the given type has at least two elements. In this article, we deal with the missing case; namely we look for $(F, \mathbb{G}, \Gamma)$ such that Γ acts transitively on the set of vertices of the associated Bruhat-Tits building. In contrast to the previous cases, we construct infinitely many such triples. It is worth mentioning that despite this contrast, we use a similar method as in [BP89] to get the main results. The major difference between our work and [BP89] is that we are looking for a classification result and they asked for a finiteness result. In fact, recently there have been lots of interest to get various quantitative versions of the results in [BP89], e.g. [Be, PY08, PY07, Sa09, BGLS10, Sa].

In this article, the action of a discrete subgroup of $\mathrm{Aut}(\mathfrak{B})$ which acts transitively on the vertices of $\mathfrak{B}$ is called *a discrete vertex transitive action* and, for simplicity, it will be abbreviated to a DVT action. In [CMSZ93(I), CMSZ93(II)], the problem of finding and classifying DVT actions was studied. But these works only dealt with the case of a two-dimensional affine building where any vertex has at most as many neighbors as the order of $\mathrm{PGL}_3(\mathbb{Z}/3\mathbb{Z})$. In [CS98], Cartwright and Steger constructed a family of $\widetilde{A}_n$-groups, i.e. subgroups of $\mathrm{Aut}(\mathfrak{B})$ which act simply transitively on the vertices, for any $n \geq 2$ over a local field of positive characteristic. These groups have been also used in the construction of explicit Ramanujan complexes in [LSV05].

1.2

In the current work, we would like to classify all the DVT actions on a Bruhat-Tits building of dimension at least 4 over a local field of characteristic zero F. We show that in contrast to the positive characteristic case [CS98], in this situation, there are only finitely many possible dimensions. It is worth mentioning that by Tits's classification of buildings [T74, T86], any irreducible affine building of dimension at least 4 is a Bruhat-Tits building associated to a pair $(F, \mathbb{G}_0)$ of a non-archimedean local field and a simply connected, almost simple F-group. In particular, the group of automorphisms of the building coincides with the group of automorphisms of $\mathbb{G}_0(F)$, which is isomorphic to $\mathrm{Aut}(\mathbb{G}_0)(F) \rtimes \mathrm{Aut}(F)$. It is well-known that $\mathrm{Aut}(\mathbb{G}_0)(F)/\mathrm{Ad}(\mathbb{G}_0)(F)$ is isomorphic to a subgroup of the group of symmetries of the absolute Dynkin diagram. Though most of our arguments can be carried on in the full generality, but for technical reasons we restrict ourselves to the discrete subgroups of $\mathrm{Ad}(\mathbb{G}_0)(F) \rtimes \mathrm{Aut}(F)$. It is worth mentioning that by the above remark $\mathrm{Ad}(\mathbb{G}_0)(F) \rtimes \mathrm{Aut}(F)$ is of index at most two in $\mathrm{Aut}(\mathfrak{B})$ as the existence of a DVT action implies that $\mathbb{G}_0$ is of type A. In fact, our results hold for the full group of automorphisms of $\mathfrak{B}$ if the dimension of $\mathfrak{B}$ is large enough. But in lower dimensions we cannot handle the extra factor of two and studying the full group of automorphisms needs a more delicate analysis. To avoid more complications we will restrict ourselves to subgroups of $\mathrm{Ad}(\mathbb{G}_0)(F) \rtimes \mathrm{Aut}(F)$.

Theorem A [Construction]. *All the following lattices act transitively on the vertices of the corresponding Bruhat-Tits buildings.*

1.
$$\overline{\Gamma} = N_{\mathrm{PGL}_m(\mathbb{Q}_{p_0})}(\mathrm{Ad}(\{g \in \mathrm{SL}_m(\mathcal{O}_l[1/\mathfrak{p}_0]) \mid \rho(g)(h') = h'\})),$$
where $l = \mathbb{Q}[\sqrt{-a_l}]$ is a complex quadratic extension of $\mathbb{Q}$, $p_0 = \mathfrak{p}_0 \cdot \overline{\mathfrak{p}}_0$ splits over l, $\mathcal{O}_l$ is the ring of integers of l, g^* is the conjugate transpose of g, and $\rho(g)(h') = gh'g^*$, $h' = q \cdot q^*$, and these parameters are given as follows.

m	a_l	p_0	q
8	3	$1 \pmod 3$	$\begin{bmatrix} I_2 & & & \\ & I_2 & & \\ \mathcal{Y}_1 & & I_2 & \\ & \mathcal{Y}_1 & & I_2 \end{bmatrix} \mathrm{diag}(\frac{1}{2\sqrt{-3}} I_4, 2I_4)$
7	3	$1 \pmod 3$	$\begin{bmatrix} 1 & & \\ v & I_3 & \\ w & \mathcal{Y}_2 & I_3 \end{bmatrix} \mathrm{diag}(\frac{1}{\sqrt{-3}}, \frac{1}{2}, \frac{1}{2\sqrt{-3}} I_2, 2I_3)$
6	1	$1 \pmod 4$	$\begin{bmatrix} I_2 & & \\ & \frac{1}{2} I_2 & \\ & \frac{1}{2}\mathcal{Y}_3 & (1-i)I_2 \end{bmatrix}$
5	3	$1 \pmod 3$	$\begin{bmatrix} 1 & & \\ & \frac{1}{2} I_2 & \\ & \frac{\beta}{2} I_2 & 2I_2 \end{bmatrix}$
5	3	$1 \pmod 3$	$\begin{bmatrix} 1 & & \\ & I_2 & \\ & \mathcal{Y}_1 & I_2 \end{bmatrix} \mathrm{diag}(1, \frac{1}{2\sqrt{-3}} I_2, 2I_2)$
5	3	$1 \pmod 3$	$\begin{bmatrix} 1 & & \\ & I_2 & \\ & \mathcal{Y}_4 & I_2 \end{bmatrix} \mathrm{diag}(1, \frac{1}{2\sqrt{-3}}, \frac{1}{2}, 2I_2)$
5	1	$1 \pmod 4$	$\begin{bmatrix} 1 & & \\ & \frac{1}{2} I_2 & \\ & \frac{1}{2}\mathcal{Y}_3 & (1-i)I_2 \end{bmatrix}$

$$\alpha = 10 - 3\sqrt{-3},\ \beta = -2 + \sqrt{-3},\ v = -4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ w = \alpha v,\ \mathcal{Y}_1 = \begin{bmatrix} \alpha & -4 \\ 4 & \alpha \end{bmatrix},$$

$$\mathcal{Y}_2 = \begin{bmatrix} \alpha & & \\ & -\alpha & -4 \\ & 4 & -\alpha \end{bmatrix},\ \mathcal{Y}_3 = \begin{bmatrix} -2+i & 1+i \\ -1+i & -2-i \end{bmatrix},\ \mathcal{Y}_4 = \begin{bmatrix} \beta & \\ 4 & \beta \end{bmatrix}.$$

2. *Let $l = \mathbb{Q}[\sqrt{-7}]$, $\mathfrak{p}_0 = \frac{1+\sqrt{-7}}{2}$, D an l-central division algebra such that*

i) $\mathrm{inv}_{\mathfrak{p}_0}(D) = -\mathrm{inv}_{\overline{\mathfrak{p}}_0}(D) = \frac{1}{5}$ or $\frac{2}{5}$,

ii) $\mathrm{inv}_{\mathfrak{p}}(D) = 0$ for $\mathfrak{p} \notin \{\mathfrak{p}_0, \overline{\mathfrak{p}}_0\}$,

where $\mathrm{inv}_{\mathfrak{p}}(D)$ *is the local Hasse invariant of* D*,* τ *is an involution of second kind on* D*, and* $\mathbb{G}$ *is the unique simply connected, absolutely almost simple, unitary group associated to* $(D, 1, \tau)$*,* $\{\mathbf{P}_p\}$ *is a coherent family of parahoric subgroups of* $\mathbb{G}(\mathbb{Q}_p)$ *such that* $\mathbf{P}_p$ *is hyper-special for all* p *except for* $p = 2, 7$*,* $\mathbf{P}_2 = \mathbb{G}(\mathbb{Q}_2)$*,* $\mathbf{P}_7$ *is a special parahoric,* p_0 *is an odd prime which is congruent to either 1, 2, or 4 modulo 7,*

$$\Lambda = \mathbb{G}(\mathbb{Q}) \cap \prod_{p \neq p_0} \mathbf{P}_p, \text{and } \overline{\Gamma} = N_{\mathrm{PGL}_5(\mathbb{Q}_{p_0})}(\Lambda).$$

Furthermore in the second case, $\overline{\Gamma}$ *acts simply transitively on the vertices of the associated Bruhat-Tits building.*

we give 11 new families of DVT actions on Bruhat-Tits buildings of dimension at least 4. Four of these families act simply transitively on the set of vertices.

Theorem B [Classification]. *Let* $\mathfrak{B}$ *be an irreducible affine building of dimension at least 4. Assume that there is a discrete subgroup* $\widehat{\Gamma}_0$ *of the group of automorphisms of* $\mathfrak{B}$ *which acts transitively on the set of vertices of* $\mathfrak{B}$*. Then* $\mathfrak{B}$ *is a Bruhat-Tits building associated with a pair* $(F, \mathbb{G}_0)$ *of a non-archimedean local field and a simply connected, absolutely almost simple* F*-group of type A. If* F *is of characteristic zero and* $\widehat{\Gamma}_0$ *is a subgroup of* $\mathrm{Ad}(\mathbb{G})(F) \rtimes \mathrm{Aut}(F)$*, then*

i) F *is isomorphic to* $\mathbb{Q}_{p_0}$ *for an odd prime* p_0 *which is congruent to either 1 modulo 4, 1 modulo 3, or 1,2, or 4 modulo 7.*

ii) $\mathbb{G}_0$ *is* F*-isomorphic to* $\mathbb{SL}_m$*, where* m *is either 5, 6, 7 or 8.*

Furthermore any such $\widehat{\Gamma}_0$ *is isomorphic to a subgroup of one of the lattices described in Theorem A.*

1.3 Structure of the paper and outline of the argument.

In this article, we heavily use Bruhat-Tits theory and Tits's indices. We refer the reader to [T79] for the definitions and the exact formulation of the needed results. However for the ease of the reader, we shall review Tits's indices of groups of absolute type A in Section 2.

In order to classify DVT actions, we start with a lattice in $\mathrm{Ad}(\mathbb{G}_0)(F)$ and describe its structure by Margulis's Arithmeticity Theorem. This theorem essentially says that in order to understand the commensurability group of a lattice, it is enough to understand the possible global forms of $\mathbb{G}_0$, i.e. a pair $(k, \mathbb{G})$ of a number field and a k-group (see Section 3.2 for a detailed description). To be able to describe the structure of the lattice itself rather than its commensurability group, we shall work with maximal lattices and use Rohlfs's maximality criterion. All this will be done in Section 3.

Section 4 is devoted to giving a good upper-bound on the index of a "principle congruence subgroup" in a desired maximal lattice, and describing the action of $H^1(k,\mu)$ the first Galois cohomology of the center of $\mathbb{G}$ on the local Dynkin diagrams along the way. The point is that we can more or less compute the co-volume of a "principle congruence subgroup". So the above bound can help us to get a lower bound on the co-volume of a maximal lattice.

In Section 5, we evaluate co-volume of the "principle congruence subgroup" of the desired maximal lattices, volume of maximal parahorics, and conclude a series of inequalities which will be used in the rest of the article.

Now having all the needed tools in hand, we start to get the following information one by one: possible number fields k, possible k-groups $\mathbb{G}$, possible coherent families of parahoric subgroups and finally check if our candidates really arise to a DVT action.

In Section 6, we prove $k = \mathbb{Q}$, and as a consequence there is no Galois group in the group of automorphisms of the building. It is more delicate to describe possible $\mathbb{G}$. To do so we start with finding the quadratic complex extensions l of $\mathbb{Q}$ over which the quasi-split inner form of $\mathbb{G}$ splits (Section 7). And, by the end of Section 8, using classification of hermitian forms over global number fields and over division algebras, Witt ring of $\mathbb{Q}$, and Brauer-Hasse-Noether theorem, we give a full description of all the possible $\mathbb{G}$. We get a complete list of all candidates for a DVT action at the end of Section 9, by getting the possible types of coherent families of parahoric subgroups.

Checking if our candidates truly arise to a DVT action is a delicate process. First we prove strong approximation for the adjoint form of $\mathbb{G}$ with respect to the possible parahoric subgroups and it enables us to "glue" the local information. As a result we get new families of lattices which act simply transitively on the set of vertices of the building of $\mathrm{SL}_5(\mathbb{Q}_{p_0})$. Furthermore using the "gluing" argument we describe certain lattices in $\mathbb{C}^m$ and compute the order of their group of linear symmetries, which enables us to complete the checking process.

Appendix C is devoted to providing a new proof of Siegel-Klingen theorem on rationality of certain Dedekind zeta and L-function values, using co-volume of lattices in $\mathrm{SL}_m(F)$, which in part gives an upper bound on the denominator of the product of values of certain Dedekind zeta and L-functions. Such a bound is needed to get the exact value of this product using a software.

1.4

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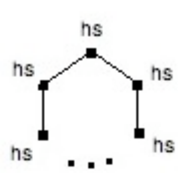
2 Notation, conventions and preliminaries.

2.1

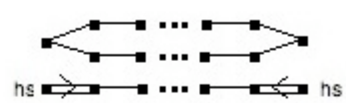
For a given number field k, let $V(k)$ (resp. $V_\infty(k)$, $V_f(k)$) be the set of places (resp. archimedean places, non-archimedean places) of k. For a given place $\mathfrak{p}$, let $k_\mathfrak{p}$ denote the completion of k with respect to $\mathfrak{p}$. If $\mathfrak{p}$ is a non-archimedean place, let $\mathcal{O}_\mathfrak{p}$ be the valuation ring of $k_\mathfrak{p}$, $\pi_\mathfrak{p}$ a uniformizer, and $\mathfrak{f}_\mathfrak{p}$ the residue field. Moreover $|\cdot|_\mathfrak{p}$ is normalized such that $|\pi_\mathfrak{p}|_\mathfrak{p} = (\#\mathfrak{f}_\mathfrak{p})^{-1}$. Let $\mathbb{A}_k$ be the adele ring of k. For S a finite set of places of k, let $\mathbb{A}_{k,S}$ be the projection of $\mathbb{A}_k$ onto $\prod_{\mathfrak{p}\notin S} k_\mathfrak{p}$. $\mathcal{O}_k$ denotes the ring of integers of k. h_k is the class number of k, and D_k denotes the absolute value of the discriminant of k. For a given local field F, let $\widehat{F}$ be the maximal unramified extension of F, $\mathcal{O}_F$ the valuation ring of F, π_F a uniformizer, and $\mathfrak{f}_F$ the residue field.

2.2

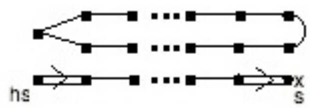
We assume that reader is fairly familiar with the Bruhat-Tits theory. All the notations and results of the Bruhat-Tits theory which we use can be found in [T79]. Let $\mathbb{H}/F$ be an absolutely almost simple group of type A. If $\mathbb{H}$ is quasi-split, its local Dynkin diagram is one of the following:



(I) The split case

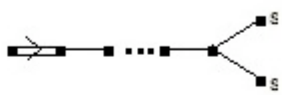


(II) quasi – split, Even dim,
Split/unramified



(III) quasi − split, Odd dim,
Split/unramified

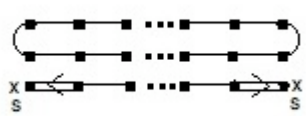
(IV) quasi − split, Odd dim,
Nonsplit/unramified



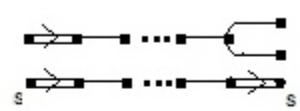
(V) quasi − split, Even dim,
Nonsplit/unramified

If $\mathbb{H}$ is an inner form and non-split, then $\mathbb{H}$ is isomorphic to $\mathbb{SL}_{d,R}$, where R is an F-central division algebra [PR94, Proposition 2.17]. In this case, we call $\mathbb{H}$ of kind (VI), and the absolute local Dynkin diagram is a cycle of length $\mathrm{ind}(R) \cdot d$ on which $Gal(\widehat{F}/F)$ acts through a cyclic group of order d generated by a rotation of the cycle. The relative diagram is a cycle of length d all vertices of which are special but not hyper-special.

If $\mathbb{H}$ is an outer form and non-quasi-split, then its local Dynkin diagram is one of the following:



(VII) Non − quasi − split,
Split/unramified



(VIII) Non − quasi − split,
Nonsplit/unramified

3 Arithmetic structure and maximality.

3.1

Let $\mathbb{G}_0$ be an absolutely almost simple F-group, where F is a finite extension of $\mathbb{Q}_p$, and $\mathcal{B} = \mathcal{B}(\mathbb{G}_0, F)$ the associated Bruhat-Tits building. From this point on, we assume that there is a DVT action on $\mathfrak{B}$. So the group of symmetries of the local Dynkin diagram of $\mathbb{G}_0$ over F acts transitively on the vertices.

Hence by looking at the possible local Dynkin diagrams given in [T79, Section 4], we have that $\mathbb{G}_0$ is an inner form of type A. Thus $\mathbb{G}_0 = \mathbb{SL}_{n,D_0}$ where D_0 is an F-central division algebra [PR94, Proposition 2.17]. Tits [T74] proved that for $n > 4$, the group of combinatorial automorphisms of $\mathcal{B}$ is isomorphic to $\mathrm{Aut}(\mathbb{G}_0)(F) \rtimes \mathrm{Aut}(F)$. It is well known that we have the following exact sequence

$$1 \to \mathrm{Ad}(\mathbb{G}_0)(F) \to \mathrm{Aut}(\mathbb{G}_0)(F) \to \mathbb{Z}/2\mathbb{Z},$$

where $\mathbb{Z}/2\mathbb{Z}$ comes from the group of symmetries of the absolute Dynkin diagram of $\mathbb{G}_0$. As we said in the introduction, in this article for simplicity, we only consider a DVT action which arises from a subgroup $\widehat{\Gamma}_0$ of $\mathrm{PGL}_n(D_0) \rtimes \mathrm{Aut}(F)$. By a linear action on $\mathcal{B}$, we mean an action which arises from a subgroup of $\mathrm{PGL}_n(D_0)$. It is worth mentioning that if the dimension of the building is large enough our argument is sufficient to conclude that there are no DVT actions. But for small dimensions one needs more careful computations and possibly new ideas. Since $\mathrm{Aut}(F)$ is a finite group, $\overline{\Gamma}_0 = \widehat{\Gamma}_0 \cap \mathrm{PGL}_n(D_0)$ is a discrete subgroup of $\mathrm{PGL}_n(D_0)$ which is of finite index in $\widehat{\Gamma}_0$. In particular, if $\widehat{\Gamma}_0$ is a lattice in $\mathrm{Aut}(\mathcal{B})$, so is $\overline{\Gamma}_0$ in $\mathrm{PGL}_n(D_0)$. Consequently any considered DVT action on $\mathcal{B}$ gives us co-compact lattices $\widehat{\Gamma}_0$ and $\overline{\Gamma}_0$ in $\mathrm{PGL}_n(D_0) \rtimes \mathrm{Aut}(F)$ and $\mathrm{PGL}_n(D_0)$, respectively. Let $\mathrm{Ad} : \mathbb{SL}_{n,D_0} \to \mathbb{PGL}_{n,D_0}$ be the adjoint map and Γ_0 the pre-image of $\overline{\Gamma}_0$ under the adjoint map from $\mathrm{SL}_n(D_0)$ to $\mathrm{PGL}_n(D_0)$.

3.2

In this paper, we will restrict ourselves to the case of $n > 4$ (Some of the statements are also valid for $2 \leq n \leq 4$.) Thus, by Margulis's arithmeticity [Ma91], Γ_0 is an arithmetic subgroup. This means that there is a number field k, a non-archimedean place $\mathfrak{p}_0$, a simply connected absolutely almost simple k-group $\mathbb{G}$ with the following properties:

i) $k_{\mathfrak{p}_0}$ is isomorphic to F.

ii) $\mathbb{G}_0$ is $k_{\mathfrak{p}_0}$-isomorphic to $\mathbb{G}$, where F is identified with $k_{\mathfrak{p}_0}$ by means of the above isomorphism.

iii) There is K a compact subgroup of $\mathbb{G}(\mathbb{A}_{k,S})$, where $S = V_\infty(k) \cup \{\mathfrak{p}_0\}$, such that Λ the projection of $\mathbb{G}(k) \cap K$ to the $\mathfrak{p}_0$ factor is commensurable to Γ_0.

Since $\widehat{\Gamma}_0$ acts transitively on the vertices of $\mathfrak{B}$, it is clear that Γ_0 is a co-compact lattice. Hence because of (iii), we further know:

iv) $\mathbb{G}$ is k-anisotropic.

v) $\mathbb{G}$ is $k_{\mathfrak{p}}$-anisotropic, for any $\mathfrak{p} \in V_\infty(k)$. In particular, k is totally real.

By the classification of k-forms of absolutely almost simple groups of type A [PR94, Chapter 2], we know that if $\mathbb{G}$ is an inner form of type A and k-anisotropic, then there is a k-central division algebra D of index m such that

$\mathbb{G} = \mathbb{SL}_{1,D}$, where $\mathrm{M}_d(D) \otimes_k k_{\mathfrak{p}_0} \simeq \mathrm{M}_n(D_0)$ as $k_{\mathfrak{p}_0}$-algebras. Hence for $m > 2$ there is no inner form of type A which satisfies both (iv) and (v). If $\mathbb{G}$ is an outer form of type A, then there are

i) l a quadratic extension of k such that $\mathfrak{p}_0$ splits over l, i.e. $l \otimes_k k_{\mathfrak{p}_0} \simeq l_{\mathfrak{P}} \oplus l_{\bar{\mathfrak{P}}}$, and because of the above property (v), l is totally complex,

ii) D an l-central division algebra such that

$$\mathrm{M}_d(D) \otimes_k k_{\mathfrak{p}_0} \simeq \mathrm{M}_n(D_0) \oplus \mathrm{M}_n(D_0),$$

iii) τ an involution of second kind on D, whose restriction to l is the generator of the Galois group of l/k,

iv) h a non-degenerate Hermitian form on D^d with respect to τ,

such that $\mathbb{G} = \mathbb{SU}_h$. (For the definition of undefined terms, we refer the reader to [PR94, Chapter 2].) Let us briefly say that any outer form of type A is a special unitary group over a finite dimensional module over a division algebra as we described in (iii) and (iv). Let us also add that (i) and (ii) hold as $\mathbb{G} \simeq \mathbb{SL}_{n,D_0}$ over $k_{\mathfrak{p}_0}$. Following [BP89], let $\mathcal{G}/k$ be the unique inner form of $\mathbb{G}$ which is k-quasi-split, i.e. $\mathcal{G} \simeq \mathbb{SU}_{h_0}$ where h_0 is a hermitian form on $l^{\mathrm{ind}(D)\cdot d}$, which is either

$$\begin{bmatrix} 0 & I_{\frac{r}{2}} \\ I_{\frac{r}{2}} & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_{\frac{r-1}{2}} \\ 0 & I_{\frac{r-1}{2}} & 0 \end{bmatrix},$$

where $r = \mathrm{ind}(D) \cdot d$.

3.3

When $\mathfrak{p}$ is a finite place, $\mathrm{Ad}(\mathbb{G})(k_{\mathfrak{p}})$ acts on the local Dynkin diagram $\mathcal{D}_{\mathfrak{p}}$. Let $\xi_{\mathfrak{p}} : \mathrm{Ad}(\mathbb{G})(k_{\mathfrak{p}}) \to \mathrm{Aut}(\mathcal{D}_{\mathfrak{p}})$ be the corresponding homomorphism. The simply connected group acts trivially on $\mathcal{D}_{\mathfrak{p}}$, i.e. $\mathrm{Ad}(\mathbb{G}(k_{\mathfrak{p}}))$ is a subgroup of $\ker(\xi_{\mathfrak{p}})$. On the other hand, the short exact sequence $1 \to \mu \to \mathbb{G} \to \mathrm{Ad}(\mathbb{G}) \to 1$ gives

$$\begin{array}{ccccc} 1 \to \mu(k) & \to \mathbb{G}(k) & \to \mathrm{Ad}(\mathbb{G})(k) & \xrightarrow{\delta} H^1(k,\mu) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 \to \mu(k_{\mathfrak{p}}) & \to \mathbb{G}(k_{\mathfrak{p}}) & \to \mathrm{Ad}(\mathbb{G})(k_{\mathfrak{p}}) & \xrightarrow{\delta_{\mathfrak{p}}} H^1(k_{\mathfrak{p}},\mu) \end{array}$$

Moreover, when $\mathfrak{p}$ is a finite place, $H^1(k_{\mathfrak{p}}, \mathbb{G}) = \{1\}$. Therefore $H^1(k_{\mathfrak{p}}, \mu)$ can be identified with $\mathrm{Ad}(\mathbb{G})(k_{\mathfrak{p}})/\mathrm{Ad}(\mathbb{G}(k_{\mathfrak{p}}))$. In particular, $\xi_{\mathfrak{p}}$ induces a homomorphism from $H^1(k_{\mathfrak{p}}, \mu)$ to $\mathrm{Aut}(\mathcal{D}_{\mathfrak{p}})$. Let us denote it also by $\xi_{\mathfrak{p}}$, and $\Xi_{\mathfrak{p}} = \mathrm{Im}(\xi_{\mathfrak{p}})$.

There is a correspondence between parahoric subgroups of $\mathbb{G}(k_{\mathfrak{p}})$ up to conjugacy and subsets of $\mathcal{D}_{\mathfrak{p}}$, called their type. $\{\mathbf{P}_{\mathfrak{p}}\}_{\mathfrak{p} \in V_f(k)}$ a collection of parahoric subgroups $\mathbf{P}_{\mathfrak{p}}$ of $\mathbb{G}(k_{\mathfrak{p}})$ is said to be *coherent* if $\prod_{\mathfrak{p} \in V_\infty(k)} \mathbb{G}(k_{\mathfrak{p}}) \cdot \prod_{\mathfrak{p} \in V_f(k)} \mathbf{P}_{\mathfrak{p}}$ is an open subgroup of $\mathbb{G}(\mathbb{A}_k)$. Let $V_f(k)^\circ = V_f(k) \setminus \{\mathfrak{p}_0\}$, and $\mathbf{P}_{\mathfrak{p}_0}$ a fixed standard parahoric subgroup of $\mathbb{G}(k_{\mathfrak{p}_0})$ with maximal volume. $\Theta = \{\Theta_\mathfrak{p}\}_{\mathfrak{p}\in V_f(k)^\circ}$ collection of types is called a $\mathfrak{p}_0$-*global-type* if the standard parahoric subgroups of type $\Theta_\mathfrak{p}$ form a coherent collection. A type $\Theta_\mathfrak{p}$ is called *maximal* if $\mathcal{D}_\mathfrak{p} \setminus \Theta_\mathfrak{p}$ the complement is an orbit of a subgroup of $\Xi_\mathfrak{p}$. A $\mathfrak{p}_0$-global-type $\{\Theta_\mathfrak{p}\}_{\mathfrak{p}\in V_f(k)^\circ}$ is called maximal if all $\Theta_\mathfrak{p}$'s are. For a given type $\Theta_\mathfrak{p} \subseteq \mathcal{D}_\mathfrak{p}$, let $\Xi_{\Theta_\mathfrak{p}}$ be the stabilizer of $\Theta_\mathfrak{p}$ in $\Xi_\mathfrak{p}$, and $H^1(k_\mathfrak{p},\mu)_{\Theta_\mathfrak{p}}$ the stabilizer of $\Theta_\mathfrak{p}$ in $H^1(k_\mathfrak{p},\mu)$. For $\Theta = \{\Theta_\mathfrak{p}\}_{\mathfrak{p}\in V_f(k)^\circ}$ a given $\mathfrak{p}_0$-global-type, let

$$\delta(\mathrm{Ad}(\mathbb{G})(k))^\circ_\Theta = \delta(\mathrm{Ad}(\mathbb{G})(k)) \cap \prod_{\mathfrak{p}\in V_f(k)^\circ} H^1(k_\mathfrak{p},\mu)_{\Theta_\mathfrak{p}}, \text{ and}$$

$$\delta(\mathrm{Ad}(\mathbb{G})(k))_\Theta = \delta(\mathrm{Ad}(\mathbb{G})(k)) \cap \{1\} \cdot \prod_{\mathfrak{p}\in V_f(k)^\circ} H^1(k_\mathfrak{p},\mu)_{\Theta_\mathfrak{p}},$$

where $\{1\}$ is the identity element in $H^1(k_{\mathfrak{p}_0},\mu)$. Using Rohlfs's maximality criterion, one proves the following.

Theorem 1. *There is $\{\mathbf{P}_\mathfrak{p}\}_{\mathfrak{p}\in V_f(k)^\circ}$ a coherent collection of parahoric subgroups $\mathbf{P}_\mathfrak{p}$ of maximal type $\Theta_\mathfrak{p}$ such that*

i) If we set $\Lambda = \mathbb{G}(k) \cap \prod_{\mathfrak{p}\in V_f(k)^\circ} \mathbf{P}_\mathfrak{p}$, $\Gamma = N_{\mathbb{G}(k_{\mathfrak{p}_0})}(\Lambda)$, $\overline{\Lambda} = \mathrm{Ad}(\Lambda)$, and $\overline{\Gamma} = N_{\mathrm{Ad}(\mathbb{G})(k_{\mathfrak{p}_0})}(\overline{\Lambda})$. Then $\Gamma_0 \subseteq \Gamma$ & $\overline{\Gamma}_0 \subseteq \overline{\Gamma}$.

ii) Γ and $\overline{\Gamma}$ are lattices in $\mathbb{G}(k_{\mathfrak{p}_0})$ and $\mathrm{Ad}(\mathbb{G})(k_{\mathfrak{p}_0})$, respectively.

iii) $\Lambda = \Gamma \cap \mathbb{G}(k)$.

iv) The following sequence is exact

$$1 \to \mu(k_{\mathfrak{p}_0})/\mu(k) \to \Gamma/\Lambda \to \delta(\mathrm{Ad}(\mathbb{G})(k))_\Theta \to 1.$$

where $\Theta = \{\Theta_\mathfrak{p}\}_{\mathfrak{p}\in V_f(k)^\circ}$. Moreover

$$\overline{\Gamma}/\overline{\Lambda} \simeq \delta(\mathrm{Ad}(\mathbb{G})(k))^\circ_\Theta.$$

Proof. See [CR97, R79], or [BP89, Propositions 1.4, 2.9]. □

4 An upper bound for $\#\overline{\Gamma}/\overline{\Lambda}$.

4.1

Since $\mathcal{G}$ is an inner form of $\mathbb{G}$, their centers are k-isomorphic. Hence the following sequence is exact

$$1 \to \mu \to R_{l/k}(\mu_m) \xrightarrow{N} \mu_m \to 1,$$

where μ_m is the k-group of m^{th}-roots of unity and N is induced by the norm map. It gives rise to the following long exact sequence

$$\mu_m(l) \to \mu_m(k) \to H^1(k,\mu) \to H^1(k, R_{l/k}(\mu_m)) \to H^1(k,\mu_m).$$

Consequently

$$1 \to \mu_m(k)/N(\mu_m(l)) \to H^1(k,\mu) \to \ker(l^\times/(l^\times)^m \to k^\times/(k^\times)^m) \to 1.$$

Let $l_0/(l^\times)^m = \ker(l^\times/(l^\times)^m \to k^\times/(k^\times)^m)$. If m is odd, then $\mu_m(k) = N(\mu_m(l))$, and so $H^1(k,\mu) \simeq l_0/(l^\times)^m$. Similarly $H^1(k_\mathfrak{p},\mu) = \{1\}$, for any $\mathfrak{p} \in V_\infty(k)$. On the other hand, the following diagram is commutative and the horizontal sequences are "exact".

$$\begin{array}{ccccc} \mathrm{Ad}(\mathbb{G})(k) & \xrightarrow{\delta} & H^1(k,\mu) & \to & H^1(k,\mathbb{G}) \\ \downarrow & & \downarrow & & \downarrow\simeq \\ \prod_{\mathfrak{p}\in V_\infty(k)} \mathrm{Ad}(\mathbb{G})(k_\mathfrak{p}) & \xrightarrow{(\delta_\mathfrak{p})} & \prod_{\mathfrak{p}\in V_\infty(k)} H^1(k_\mathfrak{p},\mu) & \to & \prod_{\mathfrak{p}\in V_\infty(k)} H^1(k_\mathfrak{p},\mathbb{G}), \end{array} \tag{1}$$

where the vertical correspondence is because of Hasse local-global theorem [PR94]. Thus $\delta(\mathrm{Ad}(\mathbb{G})(k)) = l_0/(l^\times)^m$.

Now assume m is even. Since l is totally complex, and k is totally real, we have

$$\begin{array}{ccccc} 1 \to & \{\pm 1\} & \to H^1(k,\mu) \to & l_0/(l^\times)^m & \to 1 \\ & \downarrow & \downarrow & \downarrow & \\ 1 \to & \{\pm 1\} & \to H^1(k_\mathfrak{p},\mu) \to & 1 & \end{array} \tag{2}$$

for any $\mathfrak{p} \in V_\infty(k)$. In particular, $H^1(k,\mu) \simeq \{\pm 1\} \rtimes l_0/(l^\times)^m$ as the first row splits. On the other hand, $\mathbb{G}(k_\mathfrak{p}) \xrightarrow{\mathrm{Ad}} \mathrm{Ad}(\mathbb{G})(k_\mathfrak{p})$ is surjective, for any $\mathfrak{p} \in V_\infty(k)$. So the fiber over the trivial element in $H^1(k_\mathfrak{p},\mathbb{G})$ is trivial in $H^1(k_\mathfrak{p},\mu)$. Thus combining with (1) and (2), we again have $\delta(\mathrm{Ad}(\mathbb{G})(k)) \simeq l_0/(l^\times)^m$.

4.2

Looking at the local Dynkin diagrams given in Section 2.2, one can see that $\Xi_\mathfrak{p} = \{1\}$ except possibly when the local Dynkin diagram of $\mathbb{G}$ over $k_\mathfrak{p}$ is either of type I, II, V, VI or VII. Types I and VI occur if and only if l is a subfield of $k_\mathfrak{p}$. In all other possible cases, m is even. Type II and VII occur if and only if $\mathfrak{p}$ is prime over l. Type V occurs if and only if $\mathfrak{p}$ is a ramified prime over l and $\mathbb{G}$ is quasi-split over $k_\mathfrak{p}$. Altogether we have the following possibilities:

i) $\mathfrak{p}$ splits over l, i.e. $l \otimes_k k_\mathfrak{p} = l_\mathfrak{P} \oplus l_{\bar{\mathfrak{P}}}$ where $l_\mathfrak{P}$ and $l_{\bar{\mathfrak{P}}}$ are isomorphic to $k_\mathfrak{p}$.

ii) m is even, and $\mathfrak{p}$ is a prime over l, i.e. $l \otimes_k k_\mathfrak{p} = l_\mathfrak{p}$ is an unramified field extension of $k_\mathfrak{p}$.

iii) m is even, $\mathfrak{p}$ is a ramified prime over l, and $\mathbb{G}$ is quasi-split over $k_\mathfrak{p}$.

If $\mathbb{G}$ splits over $\hat{k}_\mathfrak{p}$, then since $\Xi_\mathfrak{p}$ is a subquotient of $\widehat{\Xi}_\mathfrak{p}$ (e.g. see [BP89, Lemma 2.3]) and the later is generated by $\pi_\mathfrak{p}(\hat{k}_\mathfrak{p}^\times)^m$, after identifying $H^1(\hat{k}_\mathfrak{p}, \mu)$ with $\hat{k}_\mathfrak{p}^\times/(\hat{k}_\mathfrak{p}^\times)^m$, one can describe the action of $l_0/(l^\times)^m$ on the local Dynkin diagram in the first and the second cases:

i) Let $\mathfrak{p}$ be a finite place on k which splits over l. Then $\mathbb{G} \simeq \mathbb{SL}_{n_\mathfrak{p}, D_\mathfrak{p}}$ where $D_\mathfrak{p}$ is a $k_\mathfrak{p}$-central division algebra and $d_\mathfrak{p} \cdot n_\mathfrak{p} = m$ for $d_\mathfrak{p} = \mathrm{ind}(D_\mathfrak{p})$. If we identify $l_0/(l^\times)^m$ with $\delta(\mathrm{Ad}(\mathbb{G})(k))$ via the isomorphism in Section 4.1, then it acts on $\mathcal{D}_\mathfrak{p}$ via the following homomorphism to $\mathbb{Z}/m\mathbb{Z}$

$$x(l^\times)^m \mapsto d_\mathfrak{p} v_\mathfrak{P}(x) \mod m.$$

ii) Let m be even and $\mathfrak{p}$ a finite place on k which is also a prime over l. Then $\Xi_\mathfrak{p}$ is a group with two elements generated by $\xi_\mathfrak{p}(\pi_\mathfrak{p}{}^{\frac{m}{2}}(l_\mathfrak{p}^\times)^m)$.

To understand the third case, one can identify $\Xi_\mathfrak{p}$ with $\overline{\mathbb{T}}(k_\mathfrak{p})/\overline{T}_0 \cdot \mathrm{Ad}(\mathbb{T}(k_\mathfrak{p}))$, where $\mathbb{T}$ is a maximal $k_\mathfrak{p}$-torus, $\overline{\mathbb{T}}$ its image under the adjoint map, and $\overline{T}_0 = \{t \in \overline{\mathbb{T}}(k_\mathfrak{p}) | \, \forall \alpha \in X_{k_\mathfrak{p}}(\overline{\mathbb{T}}) \; |\alpha(t)| = 1\}$, and deduce that $\pi_\mathfrak{P}\overline{\pi_\mathfrak{P}}^{-1}(l_\mathfrak{P}^\times)^m$ acts non-trivially on $\mathcal{D}_\mathfrak{p}$ and so gives rise to the generator of $\Xi_\mathfrak{p}$. If $\mathfrak{P}$ does not divide 2, then one can choose a traceless uniformizer $\pi_\mathfrak{P}$, and so $(-1)(l^\times)^m$ gives us the generator.

Indeed in the third case, $l_\mathfrak{P}$ is a ramified extension of $k_\mathfrak{p}$ and $\mathbb{G}$ is isomorphic to $\mathbb{SU}_h$ where h is the following hermitian form over $l_\mathfrak{P}^m$

$$\begin{bmatrix} 0 & I_{\frac{m}{2}} \\ I_{\frac{m}{2}} & 0 \end{bmatrix}.$$

Hence we can choose $\mathbb{T}$ such that

$$\mathbb{T}(k_\mathfrak{p}) = \{\mathrm{diag}(d_1, \ldots, d_{m/2}, \bar{d}_1^{-1}, \ldots, \bar{d}_{m/2}^{-1}) | \, d_i \in l_\mathfrak{P}^\times, N_{l_\mathfrak{P}/k_\mathfrak{p}}(\prod_i d_i) = 1\},$$

and

$$\overline{\mathbb{T}}(k_\mathfrak{p}) = \{\mathrm{diag}(d_1, \ldots, d_{m/2}, k\bar{d}_1^{-1}, \ldots, k\bar{d}_{m/2}^{-1}) | \, d_i \in l_\mathfrak{P}^\times, k \in k_\mathfrak{p}^\times\} \cdot \Delta(l_\mathfrak{P}^\times)/\Delta(l_\mathfrak{P}^\times),$$

where $\Delta(l_\mathfrak{P}^\times) = \{\mathrm{diag}(d, \ldots, d) | \, d \in l_\mathfrak{P}^\times\}$ and $\bar{d}_i$ is the Galois conjugate of d_i. Therefore it is clear that

$$\mathrm{diag}(\pi_\mathfrak{P}, 1, \ldots, 1, \pi_\mathfrak{P}^{-1}, 1, \ldots, 1)\Delta(l_\mathfrak{P}^\times)$$

is non-trivial in $\overline{\mathbb{T}}(k_\mathfrak{p})/\overline{T}_0 \cdot \mathrm{Ad}(\mathbb{T}(k_\mathfrak{p}))$, which in turn gives us the non-trivial element of the group of symmetries of the local Dynkin diagram. Now we send it to $H^1(k_\mathfrak{p}, \mu)$ via the boundary map. Then it is sent to $H^1(l_\mathfrak{P}, \mu_m)$ and the latter is identified with $l_\mathfrak{P}^\times/(l_\mathfrak{P}^\times)^m$. Following these homomorphisms and looking at the way they are defined, we see that $\pi_\mathfrak{P}\overline{\pi_\mathfrak{P}}^{-1}(l_\mathfrak{P}^\times)^m$ acts non-trivially on $\mathcal{D}_\mathfrak{p}$, which is exactly what we claimed above.

4.3

For any $\mathfrak{p} \in V_f(k)$, $H^1(k,\mu)$ acts on $\mathcal{D}_\mathfrak{p}$ via $\xi_\mathfrak{p}$. Let $\xi : H^1(k,\mu) \to \prod_{\mathfrak{p}\in V_f(k)} \Xi_\mathfrak{p}$ be the corresponding homomorphism, and $\xi^\circ : H^1(k,\mu) \to \prod_{\mathfrak{p}\in V_f(k)^\circ} \Xi_\mathfrak{p}$. Here after identifying $\delta(\mathrm{Ad}(\mathbb{G})(k))$ with $l_0/(l^\times)^m$ via isomorphism given in Section 4.1, we get

$$l_\xi/(l^\times)^m = \ker(\xi) \cap \delta(\mathrm{Ad}(\mathbb{G})(k)) \text{ and } l_{\xi^\circ}/(l^\times)^m = \ker(\xi^\circ) \cap \delta(\mathrm{Ad}(\mathbb{G})(k)).$$

Let $x \in l$. The fractional ideal generated by x can be written in a unique way as product of prime ideals

$$\prod_{\mathfrak{P}} \mathfrak{P}^{i_1}\overline{\mathfrak{P}}^{i_2} \cdot \prod_{\mathfrak{p}} \mathfrak{p}'^{i'} \cdot \prod_{\mathfrak{P}''} \mathfrak{P}''^{i''},$$

where intersection of $\mathfrak{p} = \mathfrak{P}\overline{\mathfrak{P}}$, $\mathfrak{p}'$, and $\mathfrak{p}'' = \mathfrak{P}''\overline{\mathfrak{P}''}$ with $\mathcal{O}_k$ are prime ideals, $\mathfrak{P} \neq \overline{\mathfrak{P}}$, i.e. $\mathfrak{p}$ splits over l, and $\mathfrak{P}'' = \overline{\mathfrak{P}''}$, i.e. $\mathfrak{p}''$ is ramified over l. If x is in l_0, then, by definition, $N_{l/k}(x)$ is in $(k^\times)^m$, and so m divides $i_1 + i_2$, $2i'$, and i''. Moreover by discussions in Section 4.2, having this decomposition, we can understand action of x on the local Dynkin diagrams for primes which are unramified. Indeed x induces a trivial action on local Dynkin diagrams $\mathcal{D}_\mathfrak{p}$ (resp. $\mathcal{D}_{\mathfrak{p}'}$) if and only if m divides $d_\mathfrak{p} i_1$ (resp. i'). Let

$$T_1 = \{\mathfrak{p} \in V_f(k) |\ \mathfrak{p} \text{ splits}/l \ \&\ \mathbb{G} \text{ not split}/k_\mathfrak{p}\},$$

and $T_1^\circ = T_1 \setminus \{\mathfrak{p}_0\}$. Let T_1^l be a subset of $V(l)$ such that

$$\{\mathfrak{P} \in V(l) |\ \exists \mathfrak{p} \in T_1 : \mathfrak{P}|\mathfrak{p}\} = T_1^l \sqcup \overline{T_1^l}.$$

Then by the above discussion

$$0 \to (l_m \cap l_0)/(l^\times)^m \to l_\xi/(l^\times)^m \xrightarrow{(v_\mathfrak{P})_{\mathfrak{P}\in T_1^l}} (\mathbb{Z}/m\mathbb{Z})^{\#T_1}, \tag{3}$$

where $l_m = \{x \in l \mid \forall \mathfrak{P} \in V(l) : m | v_\mathfrak{P}(x)\}$. On the other hand, as it is discussed in [BP89, Proposition 0.12],

$$1 \to U_l/U_l^m \to l_m/(l^\times)^m \to \mathcal{P} \cap \mathcal{I}^m/\mathcal{P}^m \to 1,$$

where $\mathcal{I}$ is the group of all fractional ideals, $\mathcal{P}$ is the group of principle fractional ideals, and U_l is the group of units of $\mathcal{O}_l$. By virtue of Dirichlet's unit theorem, $N_{l/k}$ induces an isomorphism between the non-torsion factors of U_l and U_k. Therefore

$$\#l_m \cap l_0/(l^\times)^m \leq \#((l_m \cap l_0 \cap U_l) \cdot U_l^m)/U_l^m \cdot \#\mathcal{C}_{l,m} \leq \#\mu_m(l) \cdot h_{l,m},$$

where $\mathcal{C}_{l,m}$ is the subgroup of all elements of the class group of l whose order is a divisor of m, and $h_{l,m} = \#\mathcal{C}_{l,m}$. Combining with (3), we get

$$\#l_\xi/(l^\times)^m \leq m^{\#T_1} \cdot h_{l,m} \cdot \#\mu_m(l),$$

and similarly

$$\#l_{\xi^\circ}/(l^\times)^m \leq m^{\#T_1^\circ+1} \cdot h_{l,m} \cdot \#\mu_m(l). \tag{4}$$

Let Θ be as in Theorem 1. Then clearly

$$\#\delta(\mathrm{Ad}(\mathbb{G})(k))_\Theta^\circ \leq \#l_{\xi^\circ}/(l^\times)^m \cdot \prod_{\mathfrak{p}\in V^\circ(k)} \#\Xi_{\Theta_\mathfrak{p}}.$$

Hence by Theorem 1, (4), and the above inequality,

$$\#\overline{\Gamma}/\overline{\Lambda} \leq m^{\#T_1^\circ+1} \cdot h_{l,m} \cdot \#\mu_m(l) \cdot \prod_{\mathfrak{p}\in V^\circ(k)} \#\Xi_{\Theta_\mathfrak{p}}. \tag{5}$$

5 Volume formula and estimates.

5.1

Throughout this paper, we will use notations and results of [P89] and [BP89]. Here we will recall Prasad's result and adapt it to our setting. Let $\{\mathbf{P}_\mathfrak{p}\}_{\mathfrak{p}\in V^\circ(k)}$ be as in Theorem 1. Let $\mathcal{G}$ be the unique k-inner form of $\mathbb{G}$ which is k-quasi-split. Let $\{\mathbf{P}_\mathfrak{p}^m\}_{\mathfrak{p}\in V_f(k)}$ and $\{\mathcal{P}_\mathfrak{p}\}_{\mathfrak{p}\in V_f(k)}$ be coherent families of parahoric subgroups $\mathbf{P}_\mathfrak{p}^m$ (resp. $\mathcal{P}_\mathfrak{p}$) of $\mathbb{G}(k_\mathfrak{p})$ (resp. $\mathcal{G}(k_\mathfrak{p})$), such that, for any $\mathfrak{p}$, volume of $\mathbf{P}_\mathfrak{p}^m$ (resp. $\mathcal{P}_\mathfrak{p}$) is maximum among all parahoric subgroups of $\mathbb{G}(k_\mathfrak{p})$ (resp. $\mathcal{G}(k_\mathfrak{p})$). It describes a unique parahoric subgroup up to an element of $\mathrm{Ad}(\mathbb{G})(k_\mathfrak{p})$ (resp. $\mathrm{Ad}(\mathcal{G})(k_\mathfrak{p})$), unless $\mathbb{G}$ (resp. $\mathcal{G}$) is of kind (IV). In that case, $\mathbf{P}_\mathfrak{p}^m$ (resp. $\mathcal{P}_\mathfrak{p}$) corresponds to the right special vertex in the diagram given in Section 2.2. (Since hyper-special parahoric subgroups, if exists, are of maximum volume among the parahoric subgroups [BP89], it is clear that such a coherent collection exits.) Bruhat-Tits theory associates $\mathbf{G}_\mathfrak{p}$ (resp. $\mathbf{G}_\mathfrak{p}^m, \mathcal{G}_\mathfrak{p}$) an $\mathcal{O}_\mathfrak{p}$-smooth scheme to each parahoric $\mathbf{P}_\mathfrak{p}$ (resp. $\mathbf{P}_\mathfrak{p}^m, \mathcal{P}_\mathfrak{p}$). Let $\overline{\mathbf{G}}_\mathfrak{p}$ (resp. $\overline{\mathbf{G}}_\mathfrak{p}^m, \overline{\mathcal{G}}_\mathfrak{p}$) be its (resp. their) special fiber(s). Let $\overline{\mathbf{M}}_\mathfrak{p}$ (resp. $\overline{\mathbf{M}}_\mathfrak{p}^m, \overline{\mathcal{M}}_\mathfrak{p}$) be the reductive part of $\overline{\mathbf{G}}_\mathfrak{p}$ (resp. $\overline{\mathbf{G}}_\mathfrak{p}^m, \overline{\mathcal{G}}_\mathfrak{p}$). Type of the semisimple part of $\overline{\mathbf{M}}_\mathfrak{p}$ can be determined by dropping the vertices in $\Theta_\mathfrak{p}$, which is the type of $\mathbf{P}_\mathfrak{p}$, from the local Dynkin diagram.

5.2

Let vol be the unique Haar measure on $\mathbb{G}(k_{\mathfrak{p}_0})$ such that $\mathrm{vol}(\mathrm{P}_{\mathfrak{p}_0}^{\mathrm{m}}) = 1$. Then the main result of [P89] implies that

$$\mathrm{vol}(\mathrm{G}/\Lambda) = \mathrm{D}_{\mathrm{k}}^{\frac{1}{2}(\mathrm{m}^2-1)} \cdot \left(\frac{\mathrm{D}_{\mathrm{l}}}{\mathrm{D}_{\mathrm{k}}^2}\right)^{\frac{1}{2}\mathfrak{s}(\mathcal{G})} \cdot \left(\prod_{\mathrm{i}=1}^{\mathrm{m}-1} \frac{\mathrm{i}!}{(2\pi)^{\mathrm{i}+1}}\right)^{\mathrm{d}} \cdot \mathcal{E}, \tag{6}$$

where $G = \mathbb{G}(k_{\mathfrak{p}_0})$, $d = \dim_\mathbb{Q} k$, $\mathfrak{s}(\mathcal{G}) = \frac{1}{2}(m-2)(m+1)$ for m even, $\mathfrak{s}(\mathcal{G}) = \frac{1}{2}(m-1)(m+2)$ for m odd, $\mathcal{E} = \prod_{\mathfrak{p}\in V_f(k)} e(\mathbf{P}_\mathfrak{p})$, and

$$e(\mathbf{P}_\mathfrak{p}) = \frac{q_\mathfrak{p}^{(\dim \overline{\mathbf{M}}_\mathfrak{p} + \dim \overline{\mathcal{M}}_\mathfrak{p})/2}}{\#\overline{\mathbf{M}}_\mathfrak{p}(\mathfrak{f}_\mathfrak{p})}. \tag{7}$$

For almost all $\mathfrak{p}$, $\mathbf{P}_\mathfrak{p}$ is a hyper-special parahoric subgroup, in which case, $e(\mathbf{P}_\mathfrak{p})$ equals to the local factor of

$$Z(l/k, m) = \zeta_k(2) \cdot L_{l/k}(3) \cdot \ldots \cdot *(m),$$

where the last term is either $\zeta_k(m)$ if m is even, or $L_{l/k}(m)$ if m is odd. This claim is a direct consequence of Equations (6), (7) and the formulas for the order of $\mathrm{SL}_m(\mathfrak{f}_\mathfrak{p})$ and the finite special unitary groups.

Thus

$$\mathrm{vol(G/\Lambda) = D_k^{\frac{1}{2}(m^2-1)} \cdot \left(\frac{D_l}{D_k^2}\right)^{\frac{1}{2}\mathfrak{s}(\mathcal{G})} \cdot \left(\prod_{i=1}^{m-1} \frac{i!}{(2\pi)^{i+1}}\right)^{d} \cdot Z(l/k, m) \cdot \prod e'(\mathbf{P}_\mathfrak{p})}, \quad (8)$$

where $e'(\mathbf{P}_\mathfrak{p})$ is one for almost all $\mathfrak{p}$.

Lemma 2. *For any $\mathfrak{p}$, $e'(\mathbf{P}_\mathfrak{p})$ is a rational integer.*

Proof. It is clear that if $\mathbf{P}'_\mathfrak{p}$ contains $\mathbf{P}_\mathfrak{p}$, then $e'(\mathbf{P}_\mathfrak{p})$ is an integral multiple of $e'(\mathbf{P}'_\mathfrak{p})$. So without loss of generality we can assume that $\mathbf{P}_\mathfrak{p}$ is a maximal parahoric.

i) $\mathfrak{p}$ *splits over l:* The local Dynkin diagram of $\mathbb{G}$ (resp. $\mathcal{G}$) over $k_\mathfrak{p}$ is of type (VI) (resp. I) in Section 2.2. Hence there is no difference between maximal parahorics and

$$e'(\mathbf{P}_\mathfrak{p}) = \prod_{d_\mathfrak{p} \nmid i, i=1}^{m-1} (q_\mathfrak{p}^i - 1).$$

ii) $\mathfrak{p}$ *a prime over l, and $\mathbb{G}$ quasi-split over $k_\mathfrak{p}$:* Either $\mathbb{G}$'s local Dynkin diagram is either of type (II) or (III). In the either case, for some i between 1 and $\lceil m/2 \rceil$,

$$e'(\mathbf{P}_\mathfrak{p}) = \prod_{j=1}^{i'} \frac{q_\mathfrak{p}^{j+m-i'} - (-1)^{j+m-i'}}{q_\mathfrak{p}^j - (-1)^j},$$

where $i' = 2i - 2$.

iii) $\mathfrak{p}$ *a prime over l, and $\mathbb{G}$ not quasi-split over $k_\mathfrak{p}$:* In this case, the local Dynkin diagram of $\mathbb{G}$ (resp. $\mathcal{G}$) over $k_\mathfrak{p}$ is of type (VII) (resp. (II)). Hence for some i between 1 and $m/2$,

$$e'(\mathbf{P}_\mathfrak{p}) = \prod_{j=1}^{i'} \frac{q_\mathfrak{p}^{j+m-i'} - (-1)^{j+m-i'}}{q_\mathfrak{p}^j - (-1)^j},$$

where $i' = 2i - 1$.

iv) $\mathfrak{p}$ *ramified over* l*, and* $\mathbb{G}$ *quasi-split over* $k_\mathfrak{p}$*:* If m is odd, then $\mathbb{G}/k_\mathfrak{p}$ is of type (IV) and for some i

$$e'(\mathbf{P}_\mathfrak{p}) = \frac{\prod_{j=i}^{\frac{m-1}{2}}(q_\mathfrak{p}^{2j}-1)}{\prod_{j=1}^{\frac{m+1}{2}-i}(q_\mathfrak{p}^{2j}-1)}.$$

If m is even, then $\mathbb{G}/k_\mathfrak{p}$ is of type (V) and for some i

$$e'(\mathbf{P}_\mathfrak{p}) = \frac{\prod_{j=i}^{\frac{m}{2}}(q_\mathfrak{p}^{2j}-1)}{\prod_{j=1}^{\frac{m}{2}-i+1}(q_\mathfrak{p}^{2j}-1)} \cdot (q_\mathfrak{p}^{\frac{m}{2}-i+1}+1).$$

v) $\mathfrak{p}$ *ramified over* l*, and* $\mathbb{G}$ *not quasi-split over* $k_\mathfrak{p}$*:* In this case, m is even and $\mathbb{G}$ (resp. $\mathcal{G}$) over $k_\mathfrak{p}$ is of type (VIII) (resp. (V)). Hence for some i

$$e'(\mathbf{P}_\mathfrak{p}) = \frac{\prod_{j=i}^{\frac{m}{2}}(q_\mathfrak{p}^{2j}-1)}{\prod_{j=1}^{\frac{m}{2}-i+1}(q_\mathfrak{p}^{2j}-1)} \cdot (q_\mathfrak{p}^{\frac{m}{2}-i+1}-1).$$

To finish proof of the lemma, it is enough to note that for any non-negative integers i and i'

$$Q_{i,i'}(x,y) = \prod_{j=1}^{i'} \frac{x^{j+i}-y^{j+i}}{x^j-y^j}$$

is an integral polynomial in two variables x and y (since $\prod_{j=1}^{i'} \frac{x^{j+i}-1}{x^j-1} \in \mathbb{Z}[x]$.) □

Lemma 3. *In the above setting,*

$$\mathcal{R}(l/k,m) = D_k^{\frac{1}{2}(m^2-1)} \cdot \left(\frac{D_l}{D_k^2}\right)^{\frac{1}{2}\mathfrak{s}(\mathcal{G})} \cdot \left(\prod_{i=1}^{m-1} \frac{i!}{(2\pi)^{i+1}}\right)^d \cdot Z(l/k,m) =$$

$$\frac{c_m}{2^{d(m-1)}}\zeta_k(-1)\cdot L_{l/k}(-2)\cdot \ldots \cdot *(1-m),$$

where c_m *is equal to* 1 *(resp.* $(-1)^{m/2}$*) if* m *is odd (resp. even), and the last term is either* $\zeta_k(1-m)$ *if* m *is even, or* $L_{l/k}(1-m)$ *if* m *is odd.*

Proof. When m is odd, this is observed in [PY08]. One can verify it by induction on m, using functional equations of the Dedekind zeta functions and L-functions. □

Corollary 4. vol(G/Λ) *is an integral multiple of* $\mathcal{R}(l/k,m)$*.*

Proof. This is a consequence of equation (8), Lemma 2, and Lemma 3. □

5.3

Let $\widehat{K}$ (resp. $\overline{K}$, K) be the stabilizer of a vertex in $\widehat{G} = \mathrm{PGL}_n(D_0) \rtimes \mathrm{Aut}(F)$ (resp. $\mathrm{PGL}_n(D_0)$, $\mathrm{SL}_n(D_0)$). Then $\widehat{K} = \overline{K} \rtimes \mathrm{Aut}(F)$, $\overline{K} \simeq \mathrm{PGL}_n(\mathcal{O}_{D_0})$, and $K \simeq \mathrm{SL}_n(\mathcal{O}_{D_0})$. Let $\widehat{\mathrm{vol}}$ (resp. $\overline{\mathrm{vol}}$, vol) be the Haar measure on $\mathrm{PGL}_n(D_0) \rtimes \mathrm{Aut}(F)$ (resp. $\mathrm{PGL}_n(D_0)$, $\mathrm{SL}_n(D_0)$) such that $\widehat{\mathrm{vol}}(\widehat{K}) = 1$ (resp. $\overline{\mathrm{vol}}(\overline{K}) = 1$, $\mathrm{vol(K)} = 1$). Since $\widehat{\Gamma}_0$ acts transitively on the vertices of $\mathcal{B}$

$$\widehat{\mathrm{vol}}(\widehat{G}/\widehat{\Gamma}_0) = \frac{1}{\#\widehat{\Gamma}_0 \cap \widehat{K}}.$$

Hence,

$$\overline{\mathrm{vol}}(\mathrm{PGL}_n(D_0)/\overline{\Gamma}_0) = \frac{\#\widehat{\Gamma}_0/\overline{\Gamma}_0}{\#\widehat{\Gamma}_0 \cap \widehat{K}} \ \&$$

$$\begin{aligned}\mathrm{vol(G/\Lambda)} &= \frac{\#\widehat{\Gamma}_0/\overline{\Gamma}_0}{\#\widehat{\Gamma}_0 \cap \widehat{\mathrm{K}}} \cdot \frac{\#\overline{\mathrm{K}}/\mathrm{Ad(K)}}{\#\mathrm{PGL_n(D_0)/PSL_n(D_0)}} \cdot \frac{1}{\#\mu(\mathrm{k})} \cdot \frac{\#\overline{\Gamma}/\overline{\Lambda}}{\#\overline{\Gamma}/\overline{\Gamma}_0} \\ &= \frac{\#\widehat{\Gamma}_0/\overline{\Gamma}_0}{\#\widehat{\Gamma}_0 \cap \widehat{K}} \cdot \frac{1}{n} \cdot \frac{1}{\#\mu(k)} \cdot \frac{\#\overline{\Gamma}/\overline{\Lambda}}{\#\overline{\Gamma}/\overline{\Gamma}_0}. \end{aligned} \tag{9}$$

Corollary 5. *If (l,k) is an admissible pair of number fields, then any prime factor of the numerator of $\mathcal{R}(l/k,m)$ is either a prime factor of $\#\widehat{\Gamma}_0/\overline{\Gamma}_0$ (and consequently $\#\mathrm{Aut}(F)$), or m.*

Proof. This is a direct consequence of Theorem 1, Corollary 4, and Equation (9). □

From Equations (8) and (9), we further get $\frac{\#\widehat{\Gamma}_0/\overline{\Gamma}_0}{\#\mu(k)\cdot\#\overline{\Gamma}/\overline{\Gamma}_0\cdot\#\widehat{\Gamma}_0\cap\widehat{K}}$ is equal to

$$\frac{n}{\#\overline{\Gamma}_0/\overline{\Lambda}} \cdot D_k^{\frac{1}{2}(m^2-1)} \cdot \left(\frac{D_l}{D_k^2}\right)^{\frac{1}{2}\mathfrak{s}(\mathcal{G})} \cdot \left(\prod_{i=1}^{m-1} \frac{i!}{(2\pi)^{i+1}}\right)^d \cdot Z(l/k,m) \cdot \mathcal{E}',$$

where $\mathcal{E}' = \prod e'(\mathbf{P}_\mathfrak{p})$.

Corollary 6 (Main Inequality)**.** *In the above setting*

$$\#\widehat{\Gamma}_0/\overline{\Gamma}_0 \geq \frac{n \cdot e'(\mathbf{P}_{\mathfrak{p}_0})}{m} \cdot \mathcal{E}''^{\circ} \cdot \frac{B(\mathcal{G}) \cdot V_m^d}{h_{l,m} \cdot \#\mu_m(l)} \cdot Z(l/k,m),$$

where $B(\mathcal{G}) = D_k^{\frac{1}{2}(m^2-1)} \cdot \left(\frac{D_l}{D_k^2}\right)^{\frac{1}{2}\mathfrak{s}(\mathcal{G})}$, $V_m = \prod_{i=1}^{m-1} \frac{i!}{(2\pi)^{i+1}}$, *and*

$$\mathcal{E}''^{\circ} = \frac{\prod_{\mathfrak{p}\neq\mathfrak{p}_0} e'(\mathbf{P}_\mathfrak{p})}{m^{\# T_1^\circ} \cdot \prod_{\mathfrak{p}\in V^\circ(k)} \#\Xi_{\Theta_\mathfrak{p}}}.$$

Moreover, if $\dim_{\mathbb{Q}} k = d$*, we can take d as an upper bound in the above inequality.*

Proof. From Equations (8) and (9), we get

$$\frac{\#\widehat{\Gamma}_0/\overline{\Gamma}_0}{\#\mu(k)\cdot\#\overline{\Gamma}/\overline{\Gamma}_0\cdot\#\widehat{\Gamma}_0\cap\widehat{K}}=\frac{n}{\#\overline{\Gamma}_0/\overline{\Lambda}}\cdot B(\mathcal{G})\cdot V_m^d\cdot Z(l/k,m)\cdot\mathcal{E}',$$

Hence, by inequality (5), we get the claimed. To complete the proof, it is enough to note that $\widehat{\Gamma}_0/\overline{\Gamma}_0$ can be embedded into $\mathrm{Aut}(F)=\mathrm{Aut}(k_{\mathfrak{p}_0})$, and so it has at most d elements. □

Lemma 7. *Both* $\frac{n\cdot e'(\mathbf{P}_{\mathfrak{p}_0})}{m}$ *and* $\mathcal{E}''^{\circ}$ *are at least 1, when* $m>4$.

Proof. Let us start with the first factor. If D_0 is commutative, i.e. $D_0=F$, then clearly the first factor is 1 and there is nothing to discuss. If not, then as discussed in the proof of Lemma 2, $e'(\mathbf{P}_{\mathfrak{p}_0})=\prod_{d_{\mathfrak{p}_0}\nmid i,i=1}^{m-1}(q_{\mathfrak{p}}^i-1)$. To see why both of the factors are at least 1, it is enough to note that $2^{m-1}-1>m^2/2$, when $m>4$, and use the formula for $e'(\mathbf{P}_{\mathfrak{p}})$ given in Lemma 2. □

Lemma 8. $Z(l/k,m)>Z(m)>1$, *where*

$$Z(m)=\begin{cases}\prod_{i=1}^{\frac{m-1}{2}}\zeta(2di)^{\frac{1}{2}} & 2\nmid m\\ \prod_{i=1}^{\frac{m}{2}-1}\zeta(2di)^{\frac{1}{2}}\cdot\zeta(dm) & 2|m\end{cases}$$

Proof. This is a direct consequence of [PY08, lemma 1]. □

Corollary 9 (The First Estimate)**.** *In the above setting, we have*

$$f(m,d,\widehat{d},s)=\left[(\widehat{d}\cdot 50(s-1)s\cdot m)^{1/d}\cdot\frac{h(s)}{V_m}\right]^{\frac{2}{m^2-1-2s}}\geq D_k^{1/d},$$

where $\widehat{d}=\#\widehat{\Gamma}_0/\overline{\Gamma}_0$, $h(s)=\frac{\Gamma(s)\zeta(s)^2}{e^{0.1}(2\pi)^s}$ *and* $s>1$.

Proof. Using Brauer-Siegel theorem and a result of Zimmert [Z81], Prasad and Yeung [PY08] get the following inequality for the class number of l

$$\frac{1}{h_l}\geq\frac{1}{50s(s-1)}\cdot\frac{1}{h(s)^d}\cdot\frac{1}{D_l^{s/2}}.\tag{10}$$

Let $\widehat{d}=\#\widehat{\Gamma}_0/\overline{\Gamma}_0$. By Lemmas 7 and 8, and inequality (10), we have

$$\widehat{d}\geq\frac{B(\mathcal{G})\cdot V_m^d}{h_{l,m}\cdot\#\mu_m(l)}\geq\frac{D_k^{\frac{1}{2}(m^2-1)}\cdot\left(\frac{D_l}{D_k^2}\right)^{\frac{1}{2}\mathfrak{s}(\mathcal{G})}\cdot V_m^d}{50s(s-1)\cdot m\cdot h(s)^d\cdot D_l^{s/2}}\geq\frac{D_k^{\frac{1}{2}(m^2-1-2s)}\cdot V_m^d}{50s(s-1)\cdot m\cdot h(s)^d},$$

which finishes the proof. (Here we used the fact that $D_l\geq D_k^2$.) □

Corollary 10 (The Second Estimate)**.** *With the same notations as before, we have*

$$\left[\frac{\widehat{d}\cdot 50s(s-1)\cdot m}{Z(m)}\cdot\left(\frac{h(s)}{V_m}\right)^d\cdot D_k^{\mathfrak{s}(\mathcal{G})-\frac{1}{2}(m^2-1)}\right]^{\frac{2}{\mathfrak{s}(\mathcal{G})-s}}\geq D_l$$

Proof. It is a corollary of Main Inequality, Lemma 8, and inequality (10). □

When we have a candidate for an admissible pair (k,l), we check the following inequality, which follows from the Main Inequality, Lemma 7, and Lemma 8.

Corollary 11 ((k,l)-Checker)**.** $\widehat{d}\geq\frac{B(\mathcal{G})\cdot V_m^d}{h_{l,m}\cdot\#\mu_m(l)}$.

6 $k=\mathbb{Q}$.

6.1

Prasad and Yeung [PY08, Proposition 2] showed that if k is a totally real number field of degree $d\neq 1$. Then we get the following bounds on $D_k^{1/d}$.

$d\geq$	2	3	4	5
$D_k^{1/d}\geq$	2.23	3.65	5.18	6.8

(11)

At this point, we apply The First Estimate to get an upper bound on $D_k^{1/d}$ and apply the above mentioned result.

Lemma 12. *If k is an admissible number field of degree d, then we have the following.*

$m\geq$	13	9	6	5
$d\leq$	1	2	3	4

In particular, for $m\geq 13$, $k=\mathbb{Q}$.

Proof. $f(m,d,\widehat{d},s)$ is clearly increasing in $\widehat{d}$ for s less than $(m^2-1)/2$, and so it is at most $f(m,d,d,s)$. On the other hand, $f(m,d,d,s)$ is at most $f(m,e,e,s)$, for any m,d, and $s<(m^2-1)/2$, and it is decreasing in d for $d>e$. It is also decreasing in m, for $m>19$, since $(i-1)!>(2\pi)^i$ for $i\geq 19$.

By the above discussion and calculating $f(n,e,e,3)$ for n between 19 and 13, we get an upper bound 2.17 for $D_k^{1/d}$. Therefore by table 11, we get that $d=1$, and thus $k=\mathbb{Q}$.

m	19	18	17	16	15	14	13
$f(m,e,e,3)<$	1.48	1.56	1.65	1.76	1.88	2.01	2.17

The rest of the claim also follows by similar argument and calculating appropriate $f(m,d,d,s)$. Here are the needed $f(m,d,d,s)$ values.

m	12	11	10	9
$f(m,3,3,3)<$	2.35	2.58	2.86	3.22

,

m	8	7	6
$f(m,4,4,2)<$	3.63	4.25	5.18

.

The last needed value is $f(5,5,5,2)<6.59$. □

6.2

Let $d = 2$ and m between 9 and 13. We can calculate $f(m, 2, 2, 2.5)$ to get an upper bound on D_k.

m	12	11	10	9
$D_k \leq$	5	6	8	10
D_k	5	5	$5, 8$	$5, 8$

For each of the above possible discriminant, we will apply The Second Estimate, and we get the following upper bounds for D_l, which are impossible as the smallest D_l for l a totally complex quartic is 117. (We set s=2.5)

$D_k \setminus m$	12	11	10	9
5	34	45	85	103
8			75	113

Corollary 13. *For $m \geq 9$, $k = \mathbb{Q}$.*

6.3

Let $d = 3$. Again, we calculate $f(m, 3, 3, 2.5)$ to get an upper bound for D_k, and then use the list of cubic fields [1, degree3/t33.001]to write the possible discriminants.

m	8	7	6	5
$D_k \leq$	49	81	154	389
D_k	49	$49, 81$	$49, 81, 148$	$49, 81, 148, 169, 229, 257, 316, 321, 361$

As before, for each of the above possible D_k, we apply The Second Estimate (for s=2), and get the following upper bounds for D_l and $\delta_{l/k} = D_l/D_k^2$.

$D_k \setminus m$	8	7	6	5
49	$(2589, 1)$	$(6009, 2)$	$(46276, 19)$	$(70236, 29)$
81		$(6779, 1)$	$(34516, 5)$	$(83047, 12)$
148			$(24284, 1)$	$(101528, 4)$
169				$(106119, 3)$
229				$(117429, 2)$
257				$(122033, 1)$
316				$(130736, 1)$
321				$(131422, 1)$
361				$(136668, 1)$

To get more information, we appeal to the table of totally complex number fields of degree 6 [1, degree6/t60.001], and observe that because of the above restrictions,

$$(m, D_k, \delta_{l/k}) \in \{(5, 49, 7), (5, 81, 3), (5, 148, 4), (6, 49, 7), (6, 81, 3)\}.$$

Furthermore there is a unique number field with $D_l = -49^2 \times 7$, which is $l = \mathbb{Q}(\zeta_7)$ where ζ_7 is a primitive 7^{th} root of unity, $h_l = 1$, and the group of roots of unity in l has 14 elements. Using this data, we can use (k,l)-Checker, to see that this pair is not possible for $m = 6$, and for $m = 5$, as it has been computed in [PY08, Proof of theorem 1], the numerator of $\mathcal{R}(l/k, 5)$ has a prime factor other than 3 and 5. Hence it is not an admissible pair, by Corollary 5.

Similarly there is a unique number field with $D_l = -81^2 \times 3$, which is $l = \mathbb{Q}(\zeta_9)$ where ζ_9 is a primitive 9^{th} root of unity, $h_l = 1$, and the group of roots of unity in l has 18 elements. Again by (k,l)-Checker, we can see that this pair is not acceptable for $m = 6$. For $m = 5$, we once more refer to [PY08] for the computation of value of $\mathcal{R}(l/k, 5)$, and notice that its numerator is not a product of a power of 3 and a power of 5. Therefore, by Corollary 5, it is not an admissible pair.

Once more, looking at the table, we see that there is a unique totally complex number field with $D_l = -148^2 \times 4$. Moreover its class number is 1, and it has 4 roots of unity. (k,l)-Checker says that such a pair is not admissible for $m = 5$.

6.4

Let $d = 4$. Then by the discussion in Section 6.1, the only possible value for m is 5 (let us recall that we assumed m is at least 5). To get an upper bound for D_k, we again use The First Estimate. Calculating $f(5, 4, 4, 2.5)$ gives us that $D_k \leq 2222$. Looking at the table of totally real quartic number fields [1, degree4/t44.001], we see that there are exactly 6 fields with this property. Here are their discriminants.

$$D_k \in \{725, 1125, 1600, 1957, 2000, 2048\}.$$

Again by The Second Estimate (for $s = 2.5$), we get the following bounds for $\delta_{l/k}$.

D_k	725	1125	1600	2000	2048
$\delta_{l/k} \leq$	6	2	1	1	1

Prasad and Yeung [PY08] used the database in [2] and found out that the class number of any totally complex octic number field with discriminant less than 5000000 is 1. Now, applying (k,l)-Checker with $h_{l,m} = 1$ and m instead of $\#\mu_m(l)$, we get that the only possibility is $(D_k, \delta_{l/k}) = (725, 1)$. However the minimum D_l for l a totally complex octic number field is 1257728, which is larger than 725^2. Hence d is not 4.

6.5

Let $d = 2$ and m between 5 and 8. As always, using The First Estimate, we get an upper bound for D_k (for $s = 2$).

m	8	7	6	5
$D_k \leq$	14	20	32	63

Hence we have that

$$D_k \in \{5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 40, 41, 44, 52, 53, 56, 57, 60, 61\}.$$

By The Second Estimate, for a given D_k, we get an upper bound for $\delta_{l/k}$ (Let $s = 2$).

$m \setminus D_k$	≥ 44	$41 \geq \cdot \geq 33$	29	28	24	21	17	13	12	8	5
5	1	2	3	3	5	6	8	13	15	31	68
6			1	1	2	3	5	10	13	38	129
7							1	2	2	5	12
8								1	1	4	12

Further looking at the table of totally complex quartic number fields, we can see what the possible (D_l, h_l, r_l) are, where h_l is the class number of l, and r_l is the number of roots of unity in l. In particular, we observe that for all such number fields $h_l \leq 2$. Then we apply a variation of (k, l)-Checker to get an upper bound for $\delta_{k/l}$. Namely we apply

$$\left(\frac{\widehat{d} \cdot h_{l,m} \cdot \#\mu_m(l)}{D_k^{\frac{1}{2}(m^2-1)} \cdot V_m^d}\right)^{\frac{2}{s(\mathcal{G})}} \geq \delta_{l/k}, \tag{12}$$

for 2 (resp. m, d) instead of $h_{l,m}$ (resp. $\#\mu_m(l)$, $\widehat{d}$), and we will get the following bounds for $\delta_{l/k}$.

$m \setminus D_k$	≥ 37	33	29	28	24	21	17	13	12	8	5
5	0.9	1	1	1	2	2	3	5	6	13	30
6			0.64	0.7	1	1	2	4	5	15	51
7							1	1	1	3	9
8								1	1	3	9

Further looking at the mentioned table, we will see that the class number of all of the remaining fields is one. Applying inequality (12) for the second time, we will have the following modification of the upper bounds for $\delta_{l/k}$.

$m \setminus D_k$	33	29	28	24	21	17	13	12	8	5
5	1	1	1	1	2	3	5	6	12	27
6				0.93	1	2	4	5	14	46
7						0.99	1	1	3	8
8							0.97	1	2	8

Now looking at the table, we have a relatively small list of possibilities. Thus we can use (k,l)-Checker with $h_l = 1$ and the right value of $\#\mu_m(l)$, and altogether we get the following possibilities for $\delta_{l/k}$.

$m \setminus D_k$	5	8	12	21	24	28
5	5, 9, 16	4, 5, 8, 9	1, 3, 4	1	1	1
6	5, 9, 16	4, 5, 8, 9	1, 3, 4	1		
7	5		1			
8	5		1			

By the discussion in [PY08, Proof of theorem 1] and the above table, we get the following possibilities for (l,k). (These are the only possible pairs with the above prescribed discriminants and l containing k.)

$$\begin{array}{lll}
\mathcal{C}_1: & k = \mathbb{Q}(\sqrt{7}), & l = \mathbb{Q}(\sqrt{-1}, \sqrt{7}) \\
\mathcal{C}_2: & k = \mathbb{Q}(\sqrt{6}), & l = \mathbb{Q}(\sqrt{-3}, \sqrt{6}) \\
\mathcal{C}_3: & k = \mathbb{Q}(\sqrt{21}), & l = \mathbb{Q}(\sqrt{-3}, \sqrt{-7}) \\
\mathcal{C}_4: & k = \mathbb{Q}(\sqrt{3}), & l = \mathbb{Q}(\sqrt{-1}, \sqrt{3}) \\
\mathcal{C}_5, \mathcal{C}_6: & k = \mathbb{Q}(\sqrt{2}), & l = \mathbb{Q}(\sqrt{-1}, \sqrt{2}),\ \mathbb{Q}(\sqrt{2}, \sqrt{-3}) \\
\mathcal{C}_7, \mathcal{C}_8, \mathcal{C}_9: & k = \mathbb{Q}(\sqrt{5}), & l = \mathbb{Q}(\zeta_5),\ \mathbb{Q}(\sqrt{-3}, \sqrt{5}),\ \mathbb{Q}(\sqrt{-1}, \sqrt{5})
\end{array}$$

At this point calculating $\mathcal{R}(l/k, m)$ and using Corollary 5, we show that none of the above pairs are admissible. Most of the needed zeta function or L-function values are borrowed from [PY08], and the rest are computed using PARI/GP and functional equations. Having the values, we see that numerator of $\mathcal{R}(l/k, m)$ has the following prime factor, which is neither $d = 2$ nor a prime factor of m, and therefore (k,l) is not admissible and subsequently $d = 1$, i.e. $k = \mathbb{Q}$.

$m \setminus (k,l)$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$
5	113	19	11	23	11	11	293	31	587
6			11	23	11	11	293	31	587
7				23			293		
8				23			293		

Theorem 14. *In our setting $k = \mathbb{Q}$. In particular, if F is a local field of characteristic zero, D_0 an F-central division algebra, and there is a discrete vertex transitive action on the Bruhat-Tits building of* $\mathrm{SL}_n(D_0)$*, then $F = \mathbb{Q}_p$ for some prime number p, and* $\mathrm{Aut}(F) = 1$.

Remark 15. Since there is no Galois action, $\widehat{\Gamma}_0 = \overline{\Gamma}_0$ is contained in $\overline{\Gamma}$. Hence $\overline{\Gamma}$ also acts transitively on the vertices. From this point on, without loss of generality, we assume that $\overline{\Gamma}_0 = \overline{\Gamma}$.

7 Determining possible l's.

7.1

In the previous section, we have established that $k = \mathbb{Q}$ and $\mathrm{Aut}(F) = 1$ (so $\widehat{d} = 1$). Here first we use a variation of The Second Estimate to get an upper bound for the possible D_l's. In fact, we notice that $\#\mu_m(l)$ is at most 6 since l is a quadratic number field. Therefore

$$\left[300s(s-1)\cdot\frac{h(s)}{V_m}\right]^{\frac{2}{s(\mathcal{G})-s}} \geq D_l.$$

By the above inequality, we get the following ($s = 1.5$).

m	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$D_l \leq$	49	135	18	29	10	13	6	7	4	5	3	3	2	2	2

Moreover since the the left hand side of the above inequality decreases for $m \geq 19$, we get $D_l \leq 2$ for $m \geq 17$. On the other hand, there is no quadratic field with absolute value of discriminant less than 3.

Proposition 16 (First Bound)**.** *Let F be a local field of characteristic zero, and $\mathbb{G}_0$ an absolutely almost simple F-group, with absolute rank $r_{\mathbb{G}_0}$. If $r_{\mathbb{G}_0} > 15$, then no discrete subgroup of $\mathrm{Ad}(\mathbb{G}_0)(F) \rtimes \mathrm{Aut}(F)$ acts transitively on the set of vertices of the Bruhat-Tits building of $\mathbb{G}_0/F$.*

7.2

Since l is a complex quadratic field, there is a_l a square free positive integer such that $l = \mathbb{Q}(\sqrt{-a_l})$. It is well-known that $D_l = a_l$ (resp. $4a_l$) if $a_l = 3$ mod 4 (resp. otherwise). We define χ on prime numbers and then extend it multiplicatively to all the positive integers.

$$\chi(p) = \begin{cases} 1 & p \text{ splits over } l, \\ -1 & p \text{ is a prime over } l, \\ 0 & p \text{ ramifies over } l. \end{cases}$$

If $p \neq 2$, then clearly $\chi(p) = \left(\frac{-a_l}{p}\right)$, where $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol. It is also well-known that $p = 2$ is a ramified prime over l unless $a_l = 3 \pmod 4$, on that case $\chi(2) = 1$ (resp. -1) if $a_l = 7 \pmod 8$ (resp. $a_l = 3 \pmod 8$). If a_l is odd, then by the reciprocity law, for any odd prime p,

$$\chi(p) = (-1)^{\frac{p-1}{2}} \cdot (-1)^{\frac{p-1}{2}\cdot\frac{a_l-1}{2}} \cdot \left(\frac{p}{a_l}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{a_l+1}{2}} \cdot \left(\frac{p}{a_l}\right).$$

Coupling with the description of $\chi(2)$, we have that when $a_l = 3 \pmod 4$, for any natural number n,

$$\chi(n) = \left(\frac{n}{a_l}\right),$$

and consequently it is a primitive Dirichlet character whose conductor is $D_l = a_l$. When $a_l \neq 3 \pmod 4$, by the above discussion, χ is again a primitive Dirichlet character whose conductor is $D_l = 4a_l$. So altogether $L_{l/k}(s) = L(\chi, s)$, where the later is a Dirichlet L-function.

On the other hand, one can compute value of Dirichlet L-functions (resp. zeta function) at negative integers using generalized Bernoulli (resp. Bernoulli) numbers. Indeed, if

$$F_\chi(z) = \sum_{j=1}^{D_l} \frac{\chi(j) z e^{jz}}{e^{D_l z} - 1} = \sum_{j=0}^{\infty} B_{j,\chi} \frac{z^j}{j!},$$

then $B_{j,\chi}$'s are called generalized Bernoulli numbers, and $L(\chi, 1-j) = -\frac{B_{j,\chi}}{j}$. Similarly, if

$$F(z) = \frac{z e^z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!},$$

then B_j's are called Bernoulli numbers, and $\zeta(1-j) = -\frac{B_j}{j}$.

The von Staudt-Clausen theorem states that $B_{2j} + \sum_{(p-1)|2j} \frac{1}{p} \in \mathbb{Z}$. In particular, the denominator of B_{2j} is a square free number, and any of its prime factors is at most $2j+1$. Analog of this theorem for the generalized Bernoulli numbers is proved by Leopoldt [Le58] and Carlitz [Ca59]. In particular, prime factors of the denominator of $B_{j,\chi}/j$ are at most D_l. Indeed they prove that if D_l is not power of a prime number, then $B_{j,\chi}/j$ is an integer.

7.3

Computing the first 12 terms of $F(z)$, we calculate the Bernoulli numbers, and consequently $\zeta(-2j+1)$ for j between 1 and 6. In particular, $\zeta(-11) = \frac{691}{2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13}$. In Section 7.1, we saw that for $m > 11$, D_l is at most 7. Hence by the discussion in Section 7.2, 691 does not appear as a prime factor of any of the zeta or L-function values in $\mathcal{R}(l/\mathbb{Q}, m)$ for any possible l and $m \leq 16$. Hence $\mathcal{R}(l/\mathbb{Q}, m)$ has 691 as a prime factor of its numerator if $11 < m < 17$ and $(\mathbb{Q}, l)$ is an admissible pair, which contradicts Corollary 5. Hence we have the following improvement of Proposition 16.

Proposition 17 (Second Bound)**.** *Let $\mathbb{G}_0$, F, and $r_{\mathbb{G}_0}$ be as in Proposition 16. If $r_{\mathbb{G}_0} > 10$, then no discrete subgroup of $\mathrm{Ad}(\mathbb{G}_0)(F) \rtimes \mathrm{Aut}(F)$ acts transitively on the set of vertices of the Bruhat-Tits building of $\mathbb{G}_0/F$.*

7.4

To get a list of admissible $l = \mathbb{Q}(\sqrt{-a_l})$, we apply Corollary 5. To this end, we have to be able to effectively compute $\mathcal{R}(l/\mathbb{Q}, m)$ for a large list of l and small range of m, which is an easy task applying the generalized Bernoulli numbers and the discussion in Section 7.2. We use Mathematica to compute value of L-functions for the needed complex quadratic fields (see the Appendix B). We look for the cases where we do not have "bad" prime factors in the numerator of $\mathcal{R}(l/k,m)$. By Corollary 5, we know that there is no such prime factors. As a consequence, we get the following possible (m,a_l).

(m,a_l)	$(8,3)$	$(7,3)$
$\mathcal{R}(\mathbb{Q}(\sqrt{a})/\mathbb{Q},m)^{-1}$	$2^{15}\cdot 3^9\cdot 5^2$	$2^{10}\cdot 3^8\cdot 5$

(13)

(m,a_l)	$(6,3)$	$(6,1)$	$(6,7)$	$(6,31)$
$\mathcal{R}(\mathbb{Q}(\sqrt{a})/\mathbb{Q},m)^{-1}$	$2^{10}\cdot 3^7\cdot 5\cdot 7$	$2^{14}\cdot 3^4\cdot 7$	$2^3\cdot 3^4\cdot 5\cdot 7^2$	$2\cdot 3^{-1}\cdot 7$

(m,a_l)	$(5,3)$	$(5,1)$	$(5,7)$	$(5,31)$
$\mathcal{R}(\mathbb{Q}(\sqrt{a})/\mathbb{Q},m)^{-1}$	$2^7\cdot 3^5\cdot 5$	$2^{11}\cdot 3^2$	$3^2\cdot 5\cdot 7$	$2^{-2}\cdot 3^{-3}$

Proposition 18. *With the previous setting, the only possible pairs of (m,a_l)'s for $m>4$ are*

$$(8,3),(7,3),(6,3),(6,1),(6,7),(6,31),(5,3),(5,1),\text{and }(5,7).$$

Proof. By the above discussion, it is enough to exclude $(m,a_l)=(5,31)$. By the Main Inequality, Lemma 7, and Theorem 14, we have that

$$1\geq\frac{\mathcal{R}(l/\mathbb{Q},m)}{h_{l,m}\cdot\#\mu_m(l)}$$

for any possible m and l. On the other hand, $h_{\mathbb{Q}(\sqrt{-31})}=3$ and clearly no quadratic field has a primitive 5^{th} root of unity. Hence the right hand side of the above inequality, for $(m,a_l)=(5,31)$, is equal to $2^2\cdot 3^3$, which is a contradiction. $\square$

Corollary 19 (Third Bound)**.** *Let $\mathbb{G}_0$, F, and $r_{\mathbb{G}_0}$ be as in Proposition 16. If $r_{\mathbb{G}_0}>7$, then no discrete subgroup of $\mathrm{Ad}(\mathbb{G}_0)(F)\rtimes\mathrm{Aut}(F)$ acts transitively on the set of vertices of the Bruhat-Tits building of $\mathbb{G}_0/F$.*

8 Determining $\mathbb{G}$.

In this section, we will describe the possible global forms, via the description of the local forms of $\mathbb{G}$. Namely, we will find the possible set of primes over which $\mathbb{G}$ is not quasi-split.

8.1

Definition 20. For a given natural number $b>1$, a prime factor of b^c-1 is called a *primitive prime factor* if it does not divide $b^{c'}-1$ for any natural number c' less than c.

Lemma 21. *i) Let q be a positive integer. A primitive prime factor of $q^c - 1$ exists and any such factor is larger than 7 if $c \in \{5, 7, 10, 14, 8\}$.*

ii) If q is an integer larger than 2, then $(q^2 - q + 1)(q^2 + q + 1)$ has a prime factor larger than 7.

Proof. i) By Bang's theorem, any $(q, c) \neq (2, 6)$ has a primitive factor. Let p be a primitive prime factor of $q^c - 1$. Clearly p and q are co-prime. Hence $q^{p-1} - 1$ is also divisible by p. In particular, $p - 1$ is divisible by c, which finishes proof of the first part.

ii) If not, then p a primitive prime factor of $q^6 - 1$ should be equal to 7 (by Bang's theorem such a prime exists.) Hence 7 does not divide $q - 1$, $q + 1$, and $q^2 + q + 1$. Since $q^2 - q + 1$ and $q^2 + q + 1$ are co-prime, $7 | q^2 - q + 1$, and $q^2 + q + 1$ is an odd number, the only possible prime factors of $q^2 + q + 1$ are 3 and 5. On the other hand, if 5 divides $q^2 + q + 1$, then it also divides $q^3 - 1$, and consequently $q - 1$, which is a contradiction. Hence $q^2 + q + 1 = 3^\alpha$ for some positive integer α. Thus we have $(2q + 1)^2 = 4 \cdot 3^\alpha - 3$, and as a result 3 divides $4 \cdot 3^{\alpha-1} - 1$, which happens only when $\alpha = 1$. Therefore $q^2 + q + 1 = 3$ and so $q = 1$, which is a contradiction. □

8.2

Proposition 22. *As in the setting of Section 3.2, if D is non-commutative, then $(m, a_l) = (5, 7)$. Moreover, D does not split only over primes which divide 2. In particular, $D_0 = F$ and $n = m$.*

Proof. Let D be a non-commutative l-central division algebra, with an involution of second kind. Since there is no non-commutative division algebra over a local field with an involution of second kind, if D does not split over $\mathfrak{p}$, $(p) = N_{l/\mathbb{Q}}(\mathfrak{p})$ splits over l. In this case, as in the proof of Lemma 2, $e'(\mathbf{P}_p) = \prod_{d_p \nmid i, i=1}^{m-1} (q_p^i - 1)$.

If $m \neq 5$, then by Lemma 8.1, $e'(\mathbf{P}_p)$ has a prime factor larger than 7. On the other hand, in all of the possible cases, $\mathcal{R}(l/\mathbb{Q}, m)$ has no prime factor larger than 7 in its denominator. Hence we get a contradiction from

$$\frac{1}{\#\mu(\mathbb{Q}) \cdot \#\overline{\Gamma}_0 \cap \overline{K}} = \frac{n \cdot \mathcal{R}(l/\mathbb{Q}, m)}{\#\overline{\Gamma}_0 / \overline{\Lambda}} \cdot \prod e'(\mathbf{P}_p)$$

as the denominator of the right hand side of the above equality has no prime factor larger than 7. Hence D should split over all the finite places, and therefore D is commutative.

If $m = 5$, then by Proposition 18, $h_l = 1$. On the other hand, no quadratic field has a 5^{th} root of unity, and so $\#\mu_5(l) = 1$. Since 5 is a prime number, $D_0 = F$

and D either splits over a prime, or it remains a division algebra. Now by the Main Inequality, we have

$$1 \geq \mathcal{R}(l/\mathbb{Q}, 5) \prod_{p \in T_1^\circ} \frac{e'(\mathbf{P}_p)}{5} = \mathcal{R}(l/\mathbb{Q}, 5) \prod_{p \in T_1^\circ} \frac{(p-1)(p^2-1)(p^3-1)(p^4-1)}{5}.$$

From this inequality, we get that the only possibilities for (a_l, p) are

$$(3,2), (1,2), (7,2), (3,3), \text{ and } (1,3).$$

However for $p = 2$, we get 7 as a prime factor in the numerator for $a_l = 3$ or 1, which is not possible, and for $p = 3$, 13 appears as a prime factor in all the cases. Altogether the only remaining possibility is $(a_l, p) = (7, 2)$. (We also note that 2 splits over $l = \mathbb{Q}(\sqrt{-7})$.)

Now it is clear that D_0 should be commutative as otherwise D is non-commutative and so $m = 5$, in which case $n = 1$, and $\mathbb{G}_0$ is anisotropic over F, which is a contradiction. □

Corollary 23. *For $(m, a_l) = (5, 7)$, there are at most four possibilities for D.*

Proof. By Brauer-Hasse-Noether theorem, any division algebra over a number field is a cyclic algebra. Moreover, it can be classified with its local Hasse invariants. By the above proposition, D splits over any prime except those which divide $2 = \mathfrak{p}_0 \cdot \bar{\mathfrak{p}}_0$. Since D admits an involution of second kind,

$$\mathrm{inv}_{\mathfrak{p}_0}(D) + \mathrm{inv}_{\bar{\mathfrak{p}}_0}(D) = 0$$

in $\mathbb{Q}/\mathbb{Z}$. So for a given $\mathrm{inv}_{\mathfrak{p}_0}(D)$, there is a unique D. On the other hand, $\exp([D]) = \mathrm{ind}(D) = 5$. Hence

$$\mathrm{inv}_{\mathfrak{p}_0}(D) \in \left\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right\},$$

and we are done. □

8.3

Let p be a prime, which is unramified over l, and does not split over l. If $\mathbb{G}$ is not quasi-split over p, then m is even. So under our assumptions m is either 6 or 8.

Let $m = 8$. As it is discussed in the proof of Lemma 2, $e'(\mathbf{P}_p)$ is divisible by either

$$\frac{p^8-1}{p+1}, \text{ or } \frac{(p^8-1)(p^7+1)(p^6-1)}{(p^3+1)(p^2-1)(p+1)}.$$

In either case by Lemma 8.1, it has a prime factor larger than 7, which gives us a contradiction similar to the discussion in Section 8.2.

Let $m = 6$. Again as we have seen in the proof of Lemma 2, $e'(\mathbf{P}_p)$ is divisible by either

$$\frac{p^6-1}{p+1}, \text{ or } \frac{(p^6-1)(p^5+1)(p^4-1)}{(p^3+1)(p^2-1)(p+1)}.$$

The second term always has a prime factor larger than 7 by Lemma 8.1[part i], and the first term has such a prime factor if $p > 2$ by Lemma 8.1[part ii]. Hence the only possibility for p is 2. Since p is supposed to be a prime over l, the only possibilities for a_l are either 3 or 31.

Proposition 24. *In the previous setting, if p is a prime over l, and $\mathbb{G}$ is not quasi-split over $\mathbb{Q}_p$, then $m = 6$, a_l is either 3 or 31, and $p = 2$. Furthermore in these cases all the maximal parahorics containing $\mathbf{P}_p$ are special.*

Proof. By the above discussion, the only possibility for m is 6, and p is definitely 2. On the other hand, since p is supposed to be a prime over l, the only possibilities for a_l are either 3 or 31. The second part of the proposition is also a direct consequence of the above analysis, coupling with the fact that the first terms in the above formulas are associated with the special parahorics. □

8.4

Let p be a ramified prime over l. If $\mathbb{G}$ is not quasi-split over $\mathbb{Q}_p$, then, as above, m is equal to 6 or 8.

Let $m = 8$. Similar to the analysis in Section 8.3, we use the proof of Lemma 2, to say that $e'(\mathbf{P}_p)$ is divisible by either

$$p^4-1, \ \frac{(p^8-1)}{(p^2-1)}(p^3-1), \ \frac{(p^8-1)(p^6-1)}{(p^4-1)(p^2-1)}(p^2-1), \text{ or } \frac{(p^8-1)}{(p^2-1)}(p-1).$$

By Lemma 8.1, except the first term, the others have a prime factor larger than 7. Hence with a similar analysis as in Section 8.2, they are not acceptable.

Let $m = 6$. Similar to the previous cases, from the proof of Lemma 2, we know that $e'(\mathbf{P}_p)$ is divisible by either

$$p^3-1, \ p^6-1, \text{ or } \frac{p^6-1}{p+1}.$$

When $p \neq 2$, except the first term, the rest, have a prime factor larger than 7, and so they are not acceptable. On the other hand, p is ramified over l. Thus it divides D_l. By Proposition 18, we know that $D_l \in \{3, 4, 7, 31\}$. Notice that $3^3 - 1$, $7^3 - 1$, and $31^3 - 1$ have prime factors lager than 7.

Proposition 25. *In the previous setting, if p is ramified over l, and $\mathbb{G}$ is not quasi-split over $\mathbb{Q}_p$, then (m, a_l, p) is either $(8, 3, 3)$ or $(6, 1, 2)$. Furthermore when $m = 8$, $\mathbf{P}_p$ is a special parahoric with only one vertex in its type.*

Proof. It is a direct corollary of the above analysis. □

8.5

Here we will study the Hermitian form h (same notation as in Section 3.2). We start with the case $(m, a_l) = (5, 7)$, where we have an l-central division algebra. Corollary 23 gives us the local Hasse invariants, and by a theorem of Landherr [Sc85, Chapter 10, Theorem 2.4.], D has an involution τ of second kind. In this case, h is a map $D \times D \to D$ which is $\mathbb{Q}$-linear in both of the variables, and

$$\begin{array}{ll} i) & h(xd, y) = \tau(d)h(x, y), \\ ii) & h(x, yd) = h(x, y)d, \\ iii) & h(x, y) = \tau(h(y, x)), \end{array}$$

for any x, y, d in D. Thus h is uniquely determined by $a_h = h(1, 1)$.

Proposition 26.

i) By changing a_h, without loss of generality we can take any involution of second kind on D.

ii) For a given second kind involution on D, there is an Hermitian form over D which has signature ind(D) *over the archimedean place.*

iii) For a given second kind involution on D, any two Hermitian forms over D which are anisotropic over the archimedean place determine the same unitary group.

iv) There are exactly two possible $\mathbb{G}$ simply connected $\mathbb{Q}$-absolutely almost simple groups which are anisotropic over $\mathbb{R}$, and coming from a hermitian form over D.

Proof. i) Let τ_1 and τ_2 be two involution of second kind on D. Then $\tau_1 \circ \tau_2$ is an l-automorphism of D. Hence by Skolem-Noether, there is $d \in D$ such that

$$\tau_2(x) = d^{-1}\tau_1(x)d,$$

for any x in D. Since $\tau_2 \circ \tau_2 = id$, $d^{-1}\tau_1(x)d = \tau_1(dxd^{-1}) = \tau_1(d)^{-1}\tau_1(x)\tau_1(d)$. Hence $\tau_1(d) = \lambda d$ for some λ in l. On the other hand, $d = \tau_1 \circ \tau_1(d) = N_{l/\mathbb{Q}}(\lambda)d$, and so $N_{l/\mathbb{Q}}(\lambda) = 1$. By Hilbert's Theorem 90, there is μ in l, such that $\lambda = \mu\bar{\mu}^{-1}$. Let $a \in D$ such that $a = \tau_1(a)$. Then

$$\tau_1(x)ax = a \Leftrightarrow \tau_2(x) \cdot \mu^{-1}d^{-1}a \cdot x = \mu^{-1}d^{-1}a.$$

Moreover $\tau_2(\mu^{-1}d^{-1}a) = \mu^{-1}d^{-1}a$. Therefore we get the same $\mathbb{Q}$-algebraic group, which proves the first part of proposition.

ii) See [Sc85, Theorem 6.9, 6.11].

iii) Any Hermitian form and any of its scalar multiples determine the same unitary group. On the other hand, index of D is odd in our case, so without loss of generality we can assume that determinant of our Hermitian form is 1. On the other hand, by [Sc85, Corollary 6.6.], two hermitian forms with the same dimension, determinant, and signature over the archimedean place are isomorphic, which finishes the proof.

iv) To see the last part is enough to note that (D, τ, a_h) and (D^{op}, τ, a_h) determine isomorphic unitary groups. $\square$

In the other possible cases, there is no division algebra, and we have a hermitian form over a global field. Such a form is uniquely determined by its dim, det, and its sign over the archimedean places [Sc85, Chapter 10, Corollary 6.6]. In particular, without loss of generality we can assume that $h = \mathrm{diag}(b, 1, \cdots, 1)$ for some positive rational number b. We even can and will assume that b is a positive square-free rational integer.

If m is odd, then unitary group of $\mathrm{diag}(b, 1, \cdots, 1)$ and $\mathrm{diag}(b^2, b, \cdots, b)$ are equal. The determinant of the later is 1, and its signature is the same as the identity matrix over the archimedean places. Hence without loss of generality, we can assume that $h = I_m$.

Let $m = 8$. As a result of Proposition 24 and Proposition 25, either the hermitian form splits over all the finite places, or it does not split only over 3. If it splits over all the finite places, then b is in the image of the norm map over any finite place. By our assumption it is also positive. Hence at each place it is isomorphic to I_8. Therefore by Hasse-Minkowski [Sc85, Chapter 5, Lemma 7.4] and the fact that any unitary form is determined uniquely with the quadratic form $q_h(x) = h(x, x)$, we conclude that h is isomorphic to I_8.

Now we claim that if a hermitian form on $\mathbb{Q}(\sqrt{-3})^8$ splits over any prime except 3, then it also splits over 3. Let q_h be the corresponding quadratic form to h, as before. First we show that b is odd since q_h splits over $\mathbb{Q}_2$. Assume the contrary. $b = 2b'$, where b' is odd. Since q_h splits over $\mathbb{Q}_2$, we have

$$\langle 2b' \rangle + 7\langle 1 \rangle + \langle 6b' \rangle + 7\langle 3 \rangle = 0.$$

As $8\langle 1 \rangle = 0$ in $W(\mathbb{Q}_2)$ the Witt ring of $\mathbb{Q}_2$, we get

$$\langle 1, 3 \rangle = \langle 2b' \rangle \otimes \langle 1, 3 \rangle,$$

which happens if and only if $2b'$ is represented by $\langle 1, 3 \rangle$ over $\mathbb{Q}_2$. Looking at it modulo 8, we get a contradiction with the fact that b' is odd.

Now, let us also recall some of the well-known facts on $W(\mathbb{Q})$ the Witt ring of $\mathbb{Q}$. Let ∂_p be a homomorphism from $W(\mathbb{Q})$ to $W(\mathfrak{f}_p)$ defined as follows

$$\partial_p \langle a \rangle = 0, \qquad \partial_p \langle pa \rangle = \langle a \rangle,$$

for each integer a relatively prime to p. Combining these homomorphisms we get one homomorphism $\partial : W(\mathbb{Q}) \to \oplus W(\mathfrak{f}_p)$. It is well-known [MiHu73] that

$$0 \to \mathbb{Z}\langle 1\rangle \hookrightarrow W(\mathbb{Q}) \xrightarrow{\partial} \oplus W(\mathfrak{f}_p) \to 0 \tag{14}$$

is a short exact sequence. By the definition of ∂_p, it factors through $W(\mathbb{Q}_p)$. In particular, since h splits over any prime $p \neq 3$, $\partial_p(q_h) = 0$. On the other hand,

$$q_h = \langle 1, 3\rangle \otimes (\langle b\rangle + 7\langle 1\rangle).$$

So

$$\partial_p(q_h) = \langle 1, 3\rangle \otimes \partial_p\langle b\rangle \quad \text{in} \quad W(\mathfrak{f}_p).$$

Hence if $p \neq 3$ is a prime factor of b, then $\langle 1, 3\rangle = 0$ in $W(\mathfrak{f}_p)$. Hence by quadratic reciprocity, if $p \neq 2$, $p \equiv 1 \pmod 3$. Since b is odd, either $b \equiv 1 \pmod 3$ or $b/3 \equiv 1 \pmod 3$. Let us examine ∂_3. By the definition,

$$\partial_3(q_h) = \partial_3\langle b\rangle + \partial_3\langle 3b\rangle + 7\langle 1\rangle.$$

By the above discussion, $\partial_3(q_h) = 8\langle 1\rangle = 0$ in $W(\mathfrak{f}_3)$. Altogether q_h is in the kernel of ∂, and its dimension is 16. Thus $q_h = 16\langle 1\rangle$ in $W(\mathbb{Q})$. In particular, it also splits over $\mathbb{Q}_3$, which proves our claim.

Let $m = 6$, and as before let q_h be the corresponding quadratic form for h. By (14), if h splits at all the finite places, q_h is a multiple of $\langle 1\rangle$. For $m = 6$, $\dim q_h = 12$, and so $q_h = 12\langle 1\rangle$, which contradicts the fact that $12\langle 1\rangle$ is not zero in $W(\mathbb{Q}_2)$. Hence by Proposition 24 and Proposition 25, $a_l \neq 7$, and moreover 2 is the only prime over which q_h is not trivial. In this case, we also claim that $\partial_p q_h = 0$ for any p. For odd primes, there is nothing to prove. Over $p = 2$, since q_h does not split over 2, $q_h = 2\langle 1, a_l\rangle$. Thus by the definition ∂_2 maps it to zero in $W(\mathfrak{f}_2)$. Therefore by (14), $q_h = 12\langle 1\rangle$. On the other hand, it is the quadratic form associated to the hermitian form $\mathrm{diag}(b, 1, \cdots, 1)$ over $\mathbb{Q}(\sqrt{-a_l})$. Hence we have

$$\langle 1, a_l\rangle \otimes \langle b, 1, \cdots, 1\rangle = 12\langle 1\rangle.$$

By Euler's theorem, $4\langle 1\rangle = 4\langle a_l\rangle$, and so

$$4\langle 1\rangle = \langle 1, a_l\rangle \otimes \langle b, 1\rangle. \tag{15}$$

From Equation (15), one can easily see that $b = 1$ (resp. $b = 2$) works for $a_l = 1$ (resp. $a_l = 3$). However we claim that $a_l = 31$ is not possible.

To show this claim, using Equation (15), it is enough to show that the quaternion algebra $(-1, -1)$ is not isomorphic to the algebra $(-31, -b)$ for any positive square free b. Assume the contrary, so for any $p \in V(\mathbb{Q})$,

$$\left(\frac{-1, -1}{\nu_p}\right) = \left(\frac{-31, -b}{\nu_p}\right) \tag{16}$$

By Equation (16) and Weil's reciprocity law, we have

$$\left(\frac{-1,-1}{\nu_2}\right) = -1 = \left(\frac{-31,-1}{\nu_2}\right) \cdot \prod_{p|b} \left(\frac{-31,p}{\nu_2}\right) \tag{17}$$

On the other hand, since 31 can be represented by $2\langle 1\rangle$,

$$\langle 1, 1, 31, 31\rangle = 0 \tag{18}$$

in $W(\mathbb{Q}_2)$. Similarly, as -31 represented by $\langle 1, -2\rangle$ in $\mathbb{Q}_2$, we get

$$\left(\frac{-31,2}{\nu_2}\right) = 1. \tag{19}$$

For odd primes, we have

$$\begin{aligned} \left(\frac{-31,p}{\nu_2}\right) &= \left(\frac{-1,p}{\nu_2}\right)\left(\frac{31,p}{\nu_2}\right) \\ &= (-1)^{\frac{p-1}{2}} \cdot (-1)^{\frac{31-1}{2}\cdot\frac{p-1}{2}} \\ &= 1. \end{aligned} \tag{20}$$

Equations (17), (18), (19), and (20) give a contradiction, which finishes proof of our claim.

Theorem 27. *As in the previous setting,* $\mathbb{G}$ *is isomorphic to* $\mathbb{SU}_{h,D^n}$*, where* D *is an* l*-central division algebra,* $l = \mathbb{Q}(\sqrt{-a_l})$ *for some* a_l*,* h *is hermitian form, and* $m = n \cdot \mathrm{ind}(D)$*. Moreover when* m *is larger than 4, the only possibilities for the above parameters are*

	m	a_l	D	h
$\mathbf{G_1}$	8	3	l	I_8
$\mathbf{G_2}$	7	3	l	I_7
$\mathbf{G_3}$	6	3	l	$\mathrm{diag}(2, 1, \cdots, 1)$
$\mathbf{G_4}$	6	1	l	I_6
$\mathbf{G_5}$	5	3	l	I_5
$\mathbf{G_6}$	5	1	l	I_5
$\mathbf{G_7}$	5	7	l	I_5
$\mathbf{G_8}$	5	7	D	1

where in the case $\mathbf{G_8}$ *the only nonzero Hasse invariants of* D *are over* $\mathfrak{p}_0 = \frac{1+\sqrt{-7}}{2}$ *and* $\bar{\mathfrak{p}}_0 = \frac{1-\sqrt{-7}}{2}$*. Moreover* $\mathrm{inv}_{\mathfrak{p}_0} D + \mathrm{inv}_{\bar{\mathfrak{p}}_0} D = 0$*.*

Proof. It is a direct consequence of Theorem 14, Proposition 18, Corollary 23, Proposition 26, and the above discussion. ☐

9 Type of parahorics.

In this section, we describe the possible types of $\mathbf{P}_p$ parahorics of maximal type.

9.1

We start with the primes which split over l, i.e. $p = \mathfrak{p} \cdot \bar{\mathfrak{p}}$. By Theorem 27, we know that except for the case $\mathbf{G_8}$ and $p = 2$, $\mathbb{G}(\mathbb{Q}_p)$ is isomorphic to $\mathrm{SL}_m(\mathbb{Q}_p)$ and hence any maximal parahoric subgroup of $\mathbb{G}(\mathbb{Q}_p)$ is isomorphic to $\mathrm{SL}_m(\mathbb{Z}_p)$. In the $\mathbf{G_8}$ case, $\mathbb{G}(\mathbb{Q}_2)$ is a compact subgroup, and so $\mathbf{P}_2 = \mathbb{G}(\mathbb{Q}_2)$. In the other cases, since $\mathbf{P}_p$ is parahoric of maximal type, m_p number of elements of its type Θ_p divides m. Moreover its index in a maximal subgroup containing it is

$$\frac{\prod_{i=1}^{m}(p^i - 1)}{\prod_{i=1}^{m_p}(p^i - 1)^{m/m_p}}. \tag{21}$$

By Lemma 21, $p^m - 1$ has a primitive factor larger than 7 if $m \in \{5, 7, 8\}$. In which case, if $m_1 < m$, $e'(\mathbf{P}_p)$ has a prime factor larger than 7, and we get a contradiction by a similar argument as in Section 8.2. Thus for these dimensions $\mathbf{P}_p$ is a maximal parahoric. For $m = 6$, by Equation (21), $p^5 - 1$ appears in the numerator and not in the denominator, and by Lemma 21 it has a prime factor larger than 7. Therefore again $\mathbf{P}_p$ is a maximal parahoric.

9.2

Here assume that p is a prime over l, and moreover $\mathbb{G}$ is quasi-split over $\mathbb{Q}_p$. In this case, by the virtue of the formula given in the second item of the proof of Lemma 2 and Lemma 21, $e'(\mathbf{P}_p)$ has a prime factor larger than 7 if $m \neq 6$ and Θ_p contains a non-hyper-special vertex. If $m = 6$, we can use the same formula in the proof of Lemma 2, and this time use the primitive factor of $p^{10} - 1$ to get a similar result. There is only one maximal type containing hyper-special vertices which is not hyper-special itself. This occurs only for even dimension with Θ_p containing exactly two hyper-special vertices. In which case,

$$e'(\mathbf{P}_p) = \frac{\prod_{i=2}^{m}(p^i - (-1)^i)}{\prod_{i=1}^{m/2}(p^i - 1)}.$$

Again we can apply Lemma 21, and conclude that p^5+1 has a prime factor larger than 7, and so does $e'(\mathbf{P}_p)$, which is a contradiction as we have seen in Section 8.2. Altogether, we conclude that for this kind of prime, $\mathbf{P}_p$ is hyper-special.

9.3

Let p be an inert prime over l, and assume that $\mathbb{G}$ is not quasi-split over $\mathbb{Q}_p$. Then by Proposition 24 and Theorem 27, $(m, a_l) = (6, 3)$, and $p = 2$. Furthermore by Proposition 24, all the vertices of Θ_p the type of $\mathbf{P}_p$ are special. Hence type of the only possible non-special parahoric contains both of the special vertices. In which case,

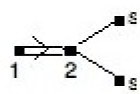
$$e'(\mathbf{P}_2) = \frac{(2^5+1)(2^4-1)(2^3+1)(2^2-1)}{(2^2-1)(2-1)},$$

that gives us a contradiction as the numerator has a prime factor larger than 7. Therefore in this case, $e'(\mathbf{P}_2) = 21$.

9.4

Let p be a ramified prime over l, and assume that $\mathbb{G}$ is quasi-split over $\mathbb{Q}_p$. Since $\mathbf{P}_p$ is of maximal type Θ_p, either Θ_p contains one vertex, or it consists of two special vertices and the dimension is even. We consider them case-by-case. For $(m, a_l, p) = (8, 3, 3)$, 41 appears as a prime factor of the numerator of $e'(\mathbf{P}_p)$ if $\mathbf{P}_p$ is not special. When $(m, a_l, p) = (7, 3, 3)$ and $\mathbf{P}_p$ is not special, 13 is a prime factor of the numerator of $e'(\mathbf{P}_p)$. Thus in these cases, $e'(\mathbf{P}_p)$ is one.

Let $(m, a_l, p) = (6, 3, 3)$. If either $\Theta_3 = \{2\}$ or it consists of two special vertices, then $e'(\mathbf{P}_3)$ has 13 as a prime factor of its numerator. Hence the only possible non-special type is $\Theta_3 = \{1\}$, in which case $e'(\mathbf{P}_3) = 28$.



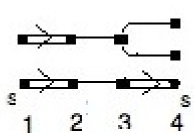
$m = 6$

$m = 5$

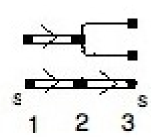
Let $(m, a_l, p) = (5, 3, 3), (5, 1, 2)$, and $(5, 7, 7)$. When $m = 5$, there is only one non-special maximal type. One can easily compute $e'(\mathbf{P}_3)$ in each case, and see that it is equal to 10, 5, and 50, respectively.

9.5

Again let p be a ramified prime over l. But this time $\mathbb{G}$ is not quasi-split over $\mathbb{Q}_p$. By Proposition 25, (m, a_l, p) is either $(8, 3, 3)$ or $(6, 1, 2)$. Furthermore when $m = 8$, only type $\Theta_3 = \{1\}$ is allowed, in which case, $e'(\mathbf{P}_3) = 80$.



$m = 8$



$m = 6$

When $m = 6$, all the maximal types have one vertex, and $e'(\mathbf{P}_2)$ is 7, 63, or 21 if $\Theta_2 = \{1\}, \{2\}$, or $\{3\}$, respectively.

9.6

Proposition 28. *Let $\{\mathbf{P}_p\}$ be as before an admissible family of parahoric subgroups and $\{\Theta_p\}$ be its type. Let $e'(\mathbf{P}_p)$ be the factors defined in Section 5.2. Also let $T = \{p \in V_f(\mathbb{Q}) | \ e'(\mathbf{P}_p) \neq 1\}$. Then*

Label	T	$(p, \Theta_p, e'(\mathbf{P}_p))$
$\mathbf{G_1}$	$\varnothing$	
$\mathbf{G_2}$	$\varnothing$	
$\mathbf{G_3}$	$\{2\} \subseteq \bullet \subseteq \{2,3\}$	$(2, s, 21), (3, \{1\}, 28)$
$\mathbf{G_4}$	$\{2\}$	$(2, \{1\}, 7), (2, \{2\}, 63), (2, \{3\}, 21)$
$\mathbf{G_5}$	$\bullet \subseteq \{3\}$	$(3, \{1\}, 10)$
$\mathbf{G_6}$	$\bullet \subseteq \{2\}$	$(2, \{1\}, 5)$
$\mathbf{G_7}$	$\bullet \subseteq \{7\}$	$(7, \{1\}, 50)$
$\mathbf{G_8}$	$\{2\} \subseteq \bullet \subseteq \{2,7\}$	$(2, \varnothing, 315), (7, \{1\}, 50)$

where $\bullet$ stands for T. In particular, $\#\Theta_p = 1$ for any p.

Proof. It is a direct result of the above discussion. □

Corollary 29. *The only automorphism of $\mathcal{D}_p$ local Dynkin diagram which fixes Θ_p is identity. Moreover $e'(\mathbf{P}_p) = 1$ when (m, a_l, p) is either $(6,3,3)$ or $(5,7,7)$.*

Proof. By Proposition 28, it is enough to show that $e'(\mathbf{P}_3) = 1$ for $(m, a_l) = (6,3)$ to prove the first claim. To show this, we use the original formula

$$\frac{1}{\#\mu(\mathbb{Q}) \cdot \#\overline{\Gamma}_0 \cap \overline{K}} = \frac{n \cdot \mathcal{R}(l/\mathbb{Q}, m)}{\#\overline{\Gamma}_0/\overline{\Lambda}} \cdot \prod e'(\mathbf{P}_p). \tag{22}$$

Here we know that $n = m$ and all the prime factors of $\overline{\Gamma}_0/\overline{\Lambda}$ are also prime factors of m. Moreover $\mathcal{R}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q}, 6)^{-1} = 2^{10} \cdot 3^7 \cdot 5 \cdot 7$ and $\mathcal{R}(\mathbb{Q}(\sqrt{-7})/\mathbb{Q}, 5)^{-1} = 3^2 \cdot 5 \cdot 7$. Hence if $e'(\mathbf{P}_3) \neq 1$ and $(m, a_l) = (6,3)$, then there is an extra 7 in the numerator of the right hand side of the above equality. Similarly if $e'(\mathbf{P}_7) \neq 1$ and $(m, a_l) = (5,7)$, then there is an extra 2 in the numerator of the right hand side. Both of them give us contradiction, which finishes our proof. □

10 Stabilizer of a vertex of $\mathcal{B}$.

10.1

Using the volume formula, we will compute the number of elements of $\overline{\Gamma}_0$ which stabilize a vertex in the Bruhat-Tits building, for any lattice described by Theorem 27 together with Proposition 28 and Corollary 29.

First we compute number of elements of $\overline{\Gamma}_0/\overline{\Lambda}$. By Theorem 1, it is enough to compute $\#\delta(\mathrm{Ad}(\mathbb{G})(k))^\circ_\Theta$. On the other hand, by Corollary 29,

$$\delta(\mathrm{Ad}(\mathbb{G})(k))^\circ_\Theta = \ker(\xi^\circ) \cap \delta(\mathrm{Ad}(\mathbb{G})(k)).$$

So by Section 4.3, we have to compute $\#l_{\xi^\circ}/(l^\times)^m$. We follow a similar argument as in Section 4.3 to describe $l_{\xi^\circ}/(l^\times)^m$. However here we also use the fact that $h_l = 1$ for all the fields l under consideration.

Let $x(l^\times)^m \in l_{\xi^\circ}/(l^\times)^m$. Without loss of generality, we can assume that x is in $\mathcal{O}_l$. Since $\mathcal{O}_l$ is UFD and l is a quadratic extension of $\mathbb{Q}$, we can write

$$x = u \prod \mathfrak{p}^{i_1}\overline{\mathfrak{p}}^{i_2} \cdot \prod p'^{i'} \cdot \prod \mathfrak{p}''^{i''},$$

where u is a unit in $\mathcal{O}_l$, $\mathfrak{p}$, p', and $\mathfrak{p}''$ are prime elements in $\mathcal{O}_l$, $\overline{\mathfrak{p}}$ is the Galois conjugate of $\mathfrak{p}$, $\mathfrak{p}^{-1}\overline{\mathfrak{p}}$ is not a unit in $\mathcal{O}_l$, p' is rational, and $\mathfrak{p}''/\overline{\mathfrak{p}''}$ is a unit in $\mathcal{O}_l$. By the definition l_{ξ° (see Section 4.3), we have

$$N_{l/\mathbb{Q}}(x) = N(u) \cdot \prod N(\mathfrak{p})^{i_1+i_2} \cdot \prod p'^{2i'} \cdot \prod N_{l/\mathbb{Q}}(\mathfrak{p}'')^{i''}$$

is the m-th power of a rational number. Hence m divides $i_1 + i_2$, $2i'$, and i''. We also know that $x(l^\times)^m$ acts trivially on all the local Dynkin diagrams except possibly at the p_0. Thus by the discussion in Section 4.2

$$x(l^\times)^m = u \cdot \mathfrak{p}_0^{i_1} \cdot \overline{\mathfrak{p}}_0^{\,i_2} \cdot (l^\times)^m.$$

If either $\mathbb{G}$ is not quasi-split over the ramified prime, or m is odd, then the local Dynkin diagram over the ramified place has a trivial group of isometries. So

$$\#\overline{\Gamma}_0/\overline{\Lambda} = \#l_{\xi^\circ}/(l^\times)^m = m \cdot \#\mu_m(l). \tag{23}$$

Otherwise, $l = \mathbb{Q}(\sqrt{-3})$ and $(-1)(l^\times)^m$ acts non-tirvially on the local Dynkin diagram over the ramified place. Hence

$$\#\overline{\Gamma}_0/\overline{\Lambda} = \#l_{\xi^\circ}/(l^\times)^m = m \cdot \#\mu_{gcd(m,3)}(l). \tag{24}$$

Proposition 30. *As in the previous setting, we have*

Label	T	$(\Theta_p, e'(\mathbf{P}_p))$	$\frac{\#\overline{\Gamma}_0/\overline{\Lambda}}{m}$	$\#\mu(\mathbb{Q}) \cdot \#\overline{\Gamma}_0 \cap \overline{K}$
$\mathbf{G_1}$	$\varnothing$		1	$2^{15} \cdot 3^9 \cdot 5^2$
$\mathbf{G_2}$	$\varnothing$		1	$2^{10} \cdot 3^8 \cdot 5$
$\mathbf{G_3}$	$\{2\}$	$(s, 21)$	3	$2^{10} \cdot 3^7 \cdot 5$
$\mathbf{G_4}$	$\{2\}$	$(\{1\}, 7), (\{2\}, 63), (\{3\}, 21)$	1	$2^{14} \cdot 3^4, 2^{14} \cdot 3^2, 2^{14} \cdot 3^3$
$\mathbf{G_5}$	$\bullet \subseteq \{3\}$	$(\{1\}, 10)$	1	$2^7 \cdot 3^5 \cdot 5, 2^6 \cdot 3^5$
$\mathbf{G_6}$	$\varnothing$		1	$2^{11} \cdot 3^2$
$\mathbf{G_7}$	$\varnothing$		1	$3^2 \cdot 5 \cdot 7$
$\mathbf{G_8}$	$\{2\}$	$(\varnothing, 315)$	1	1

Proof. It is a direct consequence of Proposition 28, Corollary 29, and Equations (13), (22), (23), and (24). □

10.2

Since working with the simply connected cover is more convenient than working with the adjoint form, we will reformulate Proposition 30 for the simply connected form and Λ_0. Note that

$$\mathrm{Ad}(\Lambda \cap K) = \overline{\Lambda} \cap \overline{K}.$$

On the other hand, $\overline{\Gamma}_0 \cap \overline{K}/\overline{\Lambda} \cap \overline{K}$ can be identified with a subgroup of $\overline{\Gamma}_0/\overline{\Lambda}$ which is isomorphic to $\delta(\mathrm{Ad}(\mathbb{G})(k))^{\circ}_{\Theta}$. By Corollary 29 and the definition of $\overline{K}$, we can identify $\overline{\Gamma}_0 \cap \overline{K}/\overline{\Lambda} \cap \overline{K}$ with a subgroup of

$$\delta(\mathrm{Ad}(\mathbb{G})(k))_{\xi} = \ker(\xi) \cap \delta(\mathrm{Ad}(\mathbb{G})(k)).$$

By a similar argument as in Section 10.1, we know that

$$\#\delta(\mathrm{Ad}(\mathbb{G})(k))_{\xi} = \begin{cases} \#\mu_{gcd(m,3)}(l) & \text{if } 2|m, a_l = 3 \ \&\ \mathbb{G} \text{ quasi-split/ } p = 3; \\ \#\mu_m(l), & \text{otherwise.} \end{cases}$$

Hence in all the cases except possibly for type G_3, $\overline{\Gamma}_0 \cap \overline{K} = \overline{\Lambda} \cap \overline{K}$. For type G_3, $\overline{\Gamma}_0 \cap \overline{K}/\overline{\Lambda} \cap \overline{K}$ either is trivial or has three elements. On the other hand,

$$\#\Lambda \cap K = \#\mu(\mathbb{Q}) \cdot \#\mathrm{Ad}(\Lambda \cap K).$$

Thus altogether, we have

Label	T	$(\Theta_p, e'(\mathbf{P}_p))$	$\#\Lambda \cap K$
$\mathbf{G_1}$	$\varnothing$		$2^{15} \cdot 3^9 \cdot 5^2$
$\mathbf{G_2}$	$\varnothing$		$2^{10} \cdot 3^8 \cdot 5$
$\mathbf{G_3}$	$\{2\}$	$(s, 21)$	$2^{10} \cdot 3^6 \cdot 5$ or $2^{10} \cdot 3^7 \cdot 5$
$\mathbf{G_4}$	$\{2\}$	$(\{1\}, 7), (\{2\}, 63), (\{3\}, 21)$	$2^{13} \cdot 3^4, 2^{13} \cdot 3^2, 2^{13} \cdot 3^3$
$\mathbf{G_5}$	$\bullet \subseteq \{3\}$	$(\{1\}, 10)$	$2^7 \cdot 3^5 \cdot 5, 2^6 \cdot 3^5$
$\mathbf{G_6}$	$\varnothing$		$2^{11} \cdot 3^2$
$\mathbf{G_7}$	$\varnothing$		$3^2 \cdot 5 \cdot 7$
$\mathbf{G_8}$	$\{2\}$	$(\varnothing, 315)$	1

11 Final check.

At this point we only know type of $(\mathbf{P}_p)$'s up to an automorphism of the local Dynkin diagrams. In this section, first we show that we are allowed to take any coherent family of parahoric subgroups with that restriction on their type.

Then we will have all the information about the possible lattices. At the final step, we have to check if they act transitively on the vertices of the Bruhat-Tits building or not. To do so, we have to compute order of $\Lambda \cap K$ directly and see if we get the same number as in the previous section. This means that for any of the possible lattices we check if its covolume is the same as the inverse of the order of the stabilizer of a vertex. It is clear that the equality holds if and only if our candidate acts transitively on the vertices of the building. We will explain how we shall find the order of these finite groups using MAGMA.

11.1

Proposition 31. *Let Θ be an admissible p_0-global type, and $(\mathbf{P}_p)_{p\neq p_0}$ a coherent family of parahoric subgroups of type Θ. Then*

$$(\mathrm{Ad}(\mathbb{G}))(\mathbb{A}) = \mathrm{Ad}(\mathbb{G})(\mathbb{Q}) \cdot \prod_{p\in\{\infty,p_0\}} \mathrm{Ad}(\mathbb{G})(\mathbb{Q}_p) \prod_{p\in V_f\setminus\{p_0\}} \overline{\mathbf{P}}_p,$$

where $\overline{\mathbf{P}}_p = \{g_p \in \mathrm{Ad}(\mathbb{G})(\mathbb{Q}_p) |\ g_p(\mathbf{P}_p) = \mathbf{P}_p\}$.

Proof. Let (g_p) be an element in $\mathrm{Ad}(\mathbb{G})(\mathbb{A})$. For any p, we map g_p to $H^1(\mathbb{Q}_p,\mu)$ via the boundary map δ_p. By the discussion in Section 4.2 and using the fact that $\mathcal{O}_l$ is UFD, we can identify elements of $\mathrm{Aut}(\mathcal{D}_p)$ with some elements of

$$\ker(H^1(\mathbb{Q}_p, R_{l/\mathbb{Q}}(\mu)) \to H^1(\mathbb{Q}_p,\mu)),$$

as follows,

$$\begin{cases} \mathfrak{p}^i({l_\mathfrak{p}}^\times)^m \times \overline{\mathfrak{p}}^{(-i)}({l_{\overline{\mathfrak{p}}}}^\times)^m & p = \mathfrak{p}\overline{\mathfrak{p}} \text{ splits}/\ l; \\ p^{m/2}({l_p}^\times)^m & 2|m \text{ and } p \text{ prime}/\ l; \\ (-1)({l_\mathfrak{p}}^\times)^m & 2|m,\ \mathbb{G} \text{ quasi-split}/\ l, \text{ and } p \text{ ramified}/\ l. \end{cases}$$

Hence $x(l^\times)^m$ maps to these elements where

$$x = \prod \mathfrak{p}^{i_\mathfrak{p}} \cdot \overline{\mathfrak{p}}^{(-i_\mathfrak{p})} \cdot \prod p^{\eta_p m/2} \cdot (-1)^{\varepsilon_\mathfrak{p}},$$

$i_\mathfrak{p}, \eta_p$, and $\varepsilon_\mathfrak{p}$ are coming from the action on the Dynkin diagram. On the other hand, it is easy to see that $x(l^\times)^m$ is in $l_0/(l^\times)^m$ (see Section 4.1 for its definition). Hence there is an element g of $\mathrm{Ad}(\mathbb{G})(\mathbb{Q})$ which is mapped to $x(l^\times)^m$ by the boundary map (see Section 4.1). Thus $g^{-1}g_p$ acts trivially on the local Dynkin diagrams for any prime p. In particular, $g^{-1}g_p(\mathbf{P}_p)$ has the same type as $\mathbf{P}_p$. Thus there is $\tilde{g}_p \in \mathbb{G}(\mathbb{Q}_p)$ such that $\mathrm{Ad}(\tilde{g}_p)g^{-1}g_p(\mathbf{P}_p) = \mathbf{P}_p$. Clearly we can assume that $\tilde{g}_p = 1$ for all p's except finitely many. For the archimedean place and p_0, let $\tilde{g}_p = 1$. So $(\tilde{g}_p) \in \mathbb{G}(\mathbb{A})$. By strong approximation, there are $\tilde{g} \in \mathbb{G}(\mathbb{Q})$ and

$$(\tilde{g}'_p) \in \prod_{p\in\{\infty,p_0\}} \mathbb{G}(\mathbb{Q}_p) \cdot \prod_{p\in V_f\setminus\{p_0\}} \mathbf{P}_p \subseteq \mathbb{G}(\mathbb{A})$$

such that, $(\tilde{g}_p) = (\tilde{g}'_p)\cdot\tilde{g}$. Hence for any $p \neq p_0$, $\mathrm{Ad}(\tilde{g})g^{-1}g_p(\mathbf{P}_p) = \mathbf{P}_p$, which completes the proof.

□

Corollary 32. *Let Θ^1 and Θ^2 be two admissible p_0-global types in the same orbit of $\prod \mathrm{Aut}(\mathcal{D}_p)$, and $\{\mathbf{P}^1_p\}$ and $\{\mathbf{P}^2_p\}$ be two families of coherent parahoric subgroups of type Θ^1 and Θ^2, respectively. Then the corresponding lattices $\overline{\Gamma}^1$ and $\overline{\Gamma}^2$ are conjugate in* $\mathrm{Ad}(\mathbb{G})(\mathbb{Q})$.

Proof. Since in our cases ξ is surjective, using the assumptions we can conclude that there is $(g_p) \in \mathrm{Ad}(\mathbb{G})(\mathbb{A})$ such that $g_p(\mathbf{P}_p^1) = \mathbf{P}_p^2$ for any $p \neq p_0$. By Proposition 31, there are $g \in \mathrm{Ad}(\mathbb{G})(\mathbb{Q})$ and

$$(g'_p) \in \prod_{p\in\{\infty,p_0\}} \mathrm{Ad}(\mathbb{G})(\mathbb{Q}_p) \cdot \prod_{p\in V_f\setminus\{p_0\}} \overline{\mathbf{P}}_p^1,$$

such that $g_p = gg'_p$ for any p. Thus for any $p \neq p_0$, $g(\mathbf{P}_p^1) = \mathbf{P}_p^2$, and so $g(\Lambda^1) = \Lambda^2$, where

$$\Lambda^1 = \mathbb{G}(\mathbb{Q}) \cap \prod_{p\in V_f\setminus\{p_0\}} \mathbf{P}_p^1 \quad \& \quad \Lambda^2 = \mathbb{G}(\mathbb{Q}) \cap \prod_{p\in V_f\setminus\{p_0\}} \mathbf{P}_p^2.$$

Therefore $\mathrm{Ad}(\Lambda^1)$ and $\mathrm{Ad}(\Lambda^2)$ are conjugate of each other, and so are $\overline{\Gamma}^1$ and $\overline{\Gamma}^2$ their normalizers. □

11.2

Here we show that the case $\mathbf{G_8}$ gives rise to four families of simply transitive actions on the vertices of Bruhat-Tits buildings. Let us recall that, in this case, $\mathbb{G}$ is isomorphic to either $\mathbb{SU}_{a_1,D_1}$ or $\mathbb{SU}_{a_2,D_2}$, where D_i are $\mathbb{Q}(\sqrt{-7})$-central simple algebras of degree 5 such that their only nonzero Hasse invariants are over $\mathfrak{p}_0 = \frac{1+\sqrt{-7}}{2}$ and $\overline{\mathfrak{p}_0} = \frac{1-\sqrt{-7}}{2}$ and we have

$$\mathrm{inv}_{\mathfrak{p}_0}(D_1) = -\,\mathrm{inv}_{\overline{\mathfrak{p}_0}}(D_1) = \frac{1}{5} \text{ and } \mathrm{inv}_{\mathfrak{p}_0}(D_2) = -\,\mathrm{inv}_{\overline{\mathfrak{p}_0}}(D_2) = \frac{2}{5}.$$

By Proposition 26, for any i, we can construct a unique desirable $\mathbb{G}$. In both of the cases, by [PY08, Lemma 4], $\mathbb{G}(\mathbb{Q})$ is torsion free. In particular, $\Lambda \cap K = \{1\}$. Hence by Proposition 30 and Section 10.2, in each of these cases choosing a p_0-admissible family of parahoric subgroups gives rise to a lattice which acts transitively on the vertices of the Bruhat-Tits building of $\mathrm{PGL}_5(\mathbb{Q}_{p_0})$, where p_0 is any prime over which $\mathbb{G}$ splits. Let us also recall that for each of the above cases, there are only two possible p_0-admissible family of parahoric subgroups; namely except over 2 or 7 the other parahoric subgroups are hyper-special, over 7 we can choose either of the special parahoric subgroups and over 2 we take the full compact group $\mathbb{G}(\mathbb{Q}_2)$.

Altogether, for any odd prime p_0 which is congruent to $1, 2$ or 4 modulo 7, we get four vertex-simply-transitive actions on the Bruhat-Tits building of $\mathrm{PGL}_5(\mathbb{Q}_{p_0})$.

11.3

In this section, we will find the order of the desired finite groups, i.e. the possible $\Lambda \cap K$. By the definition of Λ (see Theorem 1), $\Lambda \cap K = \mathbb{G}(\mathbb{Q}) \cap \prod_p \mathbf{P}_p$. To find the order of this group, technically first we describe $\mathbb{Z}_p$-schemes $\mathcal{H}_p$ which are corresponded to the parahoric subgroups, i.e. $\mathcal{H}_p(\mathbb{Z}_p) = \mathbf{P}_p$ for any p. Then we "glue" them together, i.e. find a $\mathbb{Z}$-scheme $\mathcal{H}$ such that $\mathcal{H} \simeq \mathcal{H}_p$ over $\mathbb{Z}_p$ for any p. As a result, $\Lambda \cap K = \mathcal{H}(\mathbb{Z})$. This process essentially boils down to describing a positive definite hermitian form h' and finding the number of elements of $\mathrm{SL}_m(\mathcal{O}_l)$ which preserves h'. To this end, we first look at the quadratic form $q_{h'}$ associated to h' over $\mathbb{Z}^{2m}$. Using MAGMA [BCP97], we find the group of symmetries of $q_{h'}$ and elements which commute with l_ω, where $\mathcal{O}_l = \mathbb{Z}[\omega]$ and l_ω is the linear map associated to multiplication by ω in $\mathcal{O}_l^m = \mathbb{Z}^{2m}$.

Here is a more detailed description of our process. For a given $\mathbb{G}$, i.e. a_l, m, and h. We will do the following four steps.

1- For a non-splitting prime p over l, we give a hermitian form h_p on $l_\mathfrak{p}^m = l \otimes_\mathbb{Q} \mathbb{Q}_p^m$, such that the corresponded special unitary group is isomorphic to $\mathbb{G}$ over $\mathbb{Q}_p$, and moreover,

$$\{g \in \mathrm{SL}_m(\mathcal{O}_\mathfrak{p}) \mid \rho(g)(h_p) = h_p\}$$

is mapped to a parahoric of the same type as $\mathbf{P}_p$.

2- For a non-splitting prime p over l, we find $g_p\mathrm{GL}_m(\mathcal{O}_\mathfrak{p}) \in \mathrm{GL}_m(l_\mathfrak{p})/\mathrm{GL}_m(\mathcal{O}_\mathfrak{p})$, such that $\rho(g_p)(h_p) = h$. For a splitting prime p over l, by Proposition 28, without loss of generality we can assume that $g_p \in \mathrm{GL}_m(\mathbb{Z}_p)$.

3- We find $g \in \mathrm{GL}_m(l)$, such that $g\mathrm{GL}_m(\mathcal{O}_\mathfrak{p}) = g_p\mathrm{GL}_m(\mathcal{O}_\mathfrak{p})$, where $\mathfrak{p}$ is a prime in $\mathcal{O}_l$ which divides p.

4- Let $h' = \rho(g^{-1})(h)$,

$$q_{h'} = \begin{bmatrix} \mathrm{Re}(h') & \mathrm{Re}(wh') \\ \mathrm{Re}(wh')^t & N(w)\mathrm{Re}(h') \end{bmatrix},$$

and

$$l_\omega = \begin{bmatrix} & -N(\omega)I_m \\ I_m & \mathrm{Tr}(\omega)I_m \end{bmatrix}.$$

We find the group of $2m$ by $2m$ integer matrices which preserve $q_{h'}$ and commute with l_ω, using MAGMA. By looking at the generators of this group we find the image of determinant map from $\mathrm{GL}_m(\mathcal{O}_l)$ if needed.

Note that with these choices of g and h_p's, after identifying $\mathbb{G}$ with the $\mathbb{Q}$-special unitary group associated with h', for any non-split prime p,

$$\{x \in \mathrm{SL}_m(\mathcal{O}_\mathfrak{p}) | \ \rho(x)(h') = h'\}$$

is a parahoric subgroup of the desired type in $\mathbb{G}(\mathbb{Q}_p)$. In particular,

$$\Lambda \cap K = \{x \in \mathrm{SL}_m(\mathcal{O}_l) | \ \rho(x)(h') = h'\}$$

gives us a precise description of this intersection.

11.3.1

The first step is a relatively easy consequence of Bruhat-Tits theory and Proposition 30. Namely, by Proposition 30, for any p, we know the possible types of $\mathbf{P}_p$. In our cases, for any p, $\#\Theta_p = 1$ (see Proposition 30). So it is relatively easy to find a hermitian form h_p with the desired properties. For instance, let

$$J_k = \begin{cases} \begin{bmatrix} & I_{k/2} \\ I_{k/2} & \end{bmatrix} & \text{if } k \text{ is even,} \\ \\ \begin{bmatrix} 1 & & \\ & & I_{(k-1)/2} \\ & I_{(k-1)/2} & \end{bmatrix} & \text{if } k \text{ is odd,} \end{cases}$$

and consider an unramified quadratic extension of local fields E/F. Then

$$\{g \in \mathrm{SL}_k(\mathcal{O}_E) |\ \rho(g)(J_k) = J_k\}$$

is a hyper-special parahoric subgroup of $\mathbb{SU}_{J_k,E^k}(F)$. One can see this, by looking at its Zariski-closure in $\mathbb{SL}_{2k,\mathcal{O}_F}$ and noticing that its special fiber is semisimple; in fact it is $\mathbb{SU}_{J_k,\mathfrak{e}^k}$ where $\mathfrak{e}$ is the residue field of E. It is worth mentioning that we are using Bruhat-Tits theory to finish the argument [T79, Section 3.5].

Let us describe special parahoric subgroups of $\mathbb{SU}_{J_k,E^k}(F)$, where E/F is a ramified quadratic extension of local fields. We notice up to the symmetries of the local Dynkin diagram there is only one such special parahoric if k is even and there are two of them if k is odd. If k is even, let $h = \rho(\mathrm{diag}(\pi_E I_{k/2}, I_{k/2}))(J_k)$, where π_E is a traceless uniformizer. We claim that

$$\{g \in \mathrm{SL}_k(\mathcal{O}_E) |\ \rho(g)(h) = h\}$$

is a special parahoric subgroup. One can easily verify this claim through a similar argument as the previous case. For a similar kind of argument and the odd dimension, we refer to [T79, Section 3.10].

It is straightforward to repeat a similar line of argument for any other case. Now we just summarize the result in all the cases as a proposition.

Proposition 33. *In the above setting, except for the $\mathbf{G_8}$ case, we have the following possibilities for h_p. For any (m, a_l, h), h_p is the hermitian form associated to J_m if p is not mentioned in the following list. For $(m, a_l, h) = (8, 3, I_8)$ and $p = 3$,*

$$(\mathbf{G_1^I}) \qquad h_p = \rho(\mathrm{diag}(\sqrt{-3}I_4, I_4))(J_8).$$

For $(m, a_l, h) = (7, 3, I_7)$, $h_2 = -J_7$, and h_3 is equal to either

$$(\mathbf{G_2^I}) \quad J_7, \text{ or } \qquad (\mathbf{G_2^{II}}) \quad \rho(\mathrm{diag}(1, \sqrt{-3}I_3, I_3))(J_7).$$

For $(m, a_l, h) = (6, 3, \mathrm{diag}(2, 1, \cdots, 1))$ *and* $p = 2$ *(resp.* $p = 3$*),*

$$(\mathbf{G_3^I}) \qquad h_p = \begin{bmatrix} 2 & & & \\ & 1 & & \\ & & & I_2 \\ & & I_2 & \end{bmatrix} \qquad (\text{resp. } \rho(\mathrm{diag}(\sqrt{-3}I_3, I_3))(J_6).)$$

For $(m, a_l, h) = (6, 1, I_6)$ *and* $p = 2$*,* h_p *is equal to either* $(\mathbf{G_4^I})$ X*,*

$$(\mathbf{G_4^{II}}) \ \rho(\mathrm{diag}(I_2, (1+i)I_2, I_2))(X), \text{ or } \ (\mathbf{G_4^{III}}) \ \rho(\mathrm{diag}(I_2, 1+i, I_3))(X),$$

where $X = \begin{bmatrix} I_2 & & \\ & & I_2 \\ & I_2 & \end{bmatrix}.$

For $(m, a_l, h) = (5, 3, I_5)$ *and* $p = 3$*,* h_p *is equal to either* $(\mathbf{G_5^I})$ J_5*,*

$$(\mathbf{G_5^{II}}) \ \rho(\mathrm{diag}(1, \sqrt{-3}I_2, I_2))(J_5), \text{ or } \ (\mathbf{G_5^{III}}) \ \rho(\mathrm{diag}(1, \sqrt{-3}, I_3))(J_5).$$

For $(m, a_l, h) = (5, 1, I_5)$ *and* $p = 2$*,* h_p *is equal to either*

$$(\mathbf{G_6^I}) \ J_5, \text{ or } \quad (\mathbf{G_6^{II}}) \ \rho(\mathrm{diag}(1, (1+i)I_2, I_2))(J_5).$$

For $(m, a_l, h) = (5, 7, I_5)$ *and* $p = 7$*,* h_p *is equal to either*

$$(\mathbf{G_7^I}) \ J_5, \text{ or } \quad (\mathbf{G_7^{II}}) \ \rho(\mathrm{diag}(1, \sqrt{-7}I_2, I_2))(J_5).$$

11.3.2

Here we will go through the possibilities for h_p's for any prime p, and find $g_p\mathrm{GL}_m(\mathcal{O}_\mathfrak{p})$ as described in the second step. Before going to each case separately, let us describe $g_p\mathrm{GL}_m(\mathcal{O}_\mathfrak{p})$'s for almost all primes.

Lemma 34. *In the above setting, if* p *a non-splitting prime over* l *that does not divide* $2a_l$*, then there is* $g_p \in \mathrm{GL}_m(\mathcal{O}_p)$ *such that* $\rho(g_p)(h_p) = h$*.*

Proof. Let

$$B_m = \begin{cases} \begin{bmatrix} \frac{1}{2}I_{m/2} & -\frac{1}{2}I_{m/2} \\ I_{m/2} & I_{m/2} \end{bmatrix} & \text{if } m \text{ is even,} \\ \begin{bmatrix} 1 & & \\ & \frac{1}{2}I_{m/2} & -\frac{1}{2}I_{m/2} \\ & I_{m/2} & I_{m/2} \end{bmatrix} & \text{if } m \text{ is odd.} \end{cases}$$

Then $\rho(B_m)(\mathrm{diag}(I_{\lceil m/2 \rceil}, -I_{\lfloor m/2 \rfloor})) = J_m$.

On the other hand, for any prime p which does not divide $2a_l$, there are $\overline{x}_1, \overline{x}_2 \in \mathfrak{f}_p$ such that $\overline{x}_1^2 + a_l\overline{x}_2^2 = -1$ (resp $2(\overline{x}_1^2 + a_l\overline{x}_2^2) = 1$). Thus, by virtue of Hensel's lemma, there are $x_1, x_2 \in \mathbb{Z}_p$ such that $x_1^2 + a_lx_2^2 = -1$ (resp. $2(x_1^2 + a_lx_2^2) = 1$). Therefore there is y_p a diagonal matrix in $\mathrm{GL}_m(\mathcal{O}_p)$, such that $\rho(y_p)(h) = \mathrm{diag}(I_{\lceil m/2 \rceil}, -I_{\lfloor m/2 \rfloor})$.

By the above discussion, and Proposition 33, we have that $\rho(B_m y_p)(h) = J_m = h_p$. So $g_p = y_p^{-1} B_m^{-1}$ satisfies all the desired conditions. $\square$

($\mathbf{G_1^I}$) By Lemma 34, we should only understand g_2 and g_3. Let us start with $p = 2$. It is clear that $(\overline{x}_1, \overline{x}_2) = (2, 1)$ is a solution of $\overline{x}_1^2 + 3\overline{x}_2^2 = -1$ in $\mathbb{Z}/8\mathbb{Z}$. Hence by virtue of Hensel's lemma, there are x_1 and x_2 in $\mathbb{Z}_2$ such that $x_1^2 + 3x_2^2 = -1$. So

$$h_2 = J_8 = \rho(B_8 \mathrm{diag}(I_4, (x_1 + \sqrt{-3}x_2)I_4))(h).$$

Hence $g_2 = \mathrm{diag}(I_4, (-x_1 + \sqrt{-3}x_2)I_4)B_8^{-1}$ brings h_2 to h. Now we will go one step further, and find a representative of $g_2\mathrm{GL}_8(\mathbb{Z}_2[\frac{1+\sqrt{-3}}{2}])$ in $\mathrm{GL}_8(\mathbb{Q}[\sqrt{-3}])$. Indeed using the fact that $x_1 \equiv 2 \pmod 4$ and $x_2 \equiv 1 \pmod 4$, it is easy to check that

$$g_2\mathrm{GL}_8\left(\mathbb{Z}_2\left[\frac{1+\sqrt{-3}}{2}\right]\right) = \begin{bmatrix} \frac{1}{2}I_4 & \\ \frac{-2+\sqrt{-3}}{2}I_4 & 2I_4 \end{bmatrix} \mathrm{GL}_8\left(\mathbb{Z}_2\left[\frac{1+\sqrt{-3}}{2}\right]\right).$$

Now consider $p = 3$. $(\overline{x}_1, \overline{x}_2) = (1, 1)$ is a solution of $\overline{x}_1^2 + \overline{x}_2^2 = -1$ in $\mathfrak{f}_3$. So there are x_1 and x_2 in $\mathbb{Z}_3$ such that $x_1^2 + x_2^2 = -1$. Hence

$$h_3 = \rho\left(\mathrm{diag}(\sqrt{-3}I_4, I_4) \cdot B_8 \cdot \mathrm{diag}(I_4, \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}, \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix})\right)(h),$$

and as a consequence, we have to find a representative of $g_3\mathrm{GL}_8(\mathbb{Z}_3[\sqrt{-3}])$ in $\mathrm{GL}_8(\mathbb{Q}[\sqrt{-3}])$, where

$$g_3 = \left(\mathrm{diag}(\sqrt{-3}I_4, I_4) \cdot B_8 \cdot \mathrm{diag}(I_4, \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}, \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix})\right)^{-1}.$$

Using the fact that $x_1 \equiv x_2 \equiv 1 \pmod 3$, it is easy to check that

$$g_3\mathrm{GL}_8(\mathbb{Z}_3[\sqrt{-3}]) = \begin{bmatrix} \frac{1}{\sqrt{-3}}I_2 & & & \\ & \frac{1}{\sqrt{-3}}I_2 & & \\ \frac{1}{\sqrt{-3}}Z(1,-1) & & I_2 & \\ & \frac{1}{\sqrt{-3}}Z(1,-1) & & I_2 \end{bmatrix} \mathrm{GL}_8(\mathbb{Z}_3[\sqrt{-3}]),$$

where $Z(y_1, y_2) = \begin{bmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{bmatrix}$.

($\mathbf{G_2^I}$) By Lemma 34, we only have to find g_2 and g_3. Similar to the previous case, one can easily find that

$$g_2\mathrm{GL}_7\left(\mathbb{Z}_2\left[\frac{1+\sqrt{-3}}{2}\right]\right) = \begin{bmatrix} 1 & & \\ & \frac{1}{2}I_3 & \\ & \frac{-2+\sqrt{-3}}{2}I_3 & 2I_3 \end{bmatrix} \mathrm{GL}_7\left(\mathbb{Z}_2\left[\frac{1+\sqrt{-3}}{2}\right]\right).$$

Now let $p = 3$.

$$h_3 = -J_7 = \rho(B_7)(\mathrm{diag}(-I_4, 3)) = \rho(B_7 \cdot \mathrm{diag}(Z(x_1, x_2), Z(x_1, x_2), I_3))(h),$$

where (x_1, x_2) is a solution of $x_1^2 + x_2^2 = -1$ in $\mathbb{Z}_3$. So it clear that

$$g_3\mathrm{GL}_7(\mathbb{Z}_3[\sqrt{-3}]) = \mathrm{GL}_7(\mathbb{Z}_3[\sqrt{-3}]).$$

($\mathbf{G_2^{II}}$) By Lemma 34, again we only have to find g_2 and g_3. g_2 is the same as $\mathbf{G_2^{I}}$. For $p = 3$, we proceed similar to the previous case, and we get

$$h_3 = \rho(\mathrm{diag}(1, \sqrt{-3}I_3, I_3) \cdot B_7 \cdot \mathrm{diag}(Z(x_1, x_2), Z(x_1, x_2), I_3))(h),$$

where x_1 and x_2 are as in $\mathbf{G_2^{I}}$ case. We can and will assume that $x_1 \equiv x_2 \equiv 1 \pmod 3$, and then use it to check that

$$g_3\mathrm{GL}_7(\mathbb{Z}_3[\sqrt{-3}]) = \begin{bmatrix} \frac{1}{\sqrt{-3}} & & & \\ -\frac{1}{\sqrt{-3}} & 1 & & \\ & & \frac{1}{\sqrt{-3}}I_2 & \\ & & & I_3 \end{bmatrix} \begin{bmatrix} I_4 & \\ Y_1 & I_3 \end{bmatrix} \mathrm{GL}_7(\mathbb{Z}_3[\sqrt{-3}]),$$

where

$$Y_1 = \begin{bmatrix} \frac{-1}{\sqrt{-3}} & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{-3}} & \frac{-1}{\sqrt{-3}} \\ 0 & 0 & \frac{1}{\sqrt{-3}} & \frac{-1}{\sqrt{-3}} \end{bmatrix}.$$

($\mathbf{G_3^{I}}$) We only have to study $p = 2$ and $p = 3$. We start with h_2.

$$h_2 = \rho(\mathrm{diag}(I_2, B_4))(\mathrm{diag}(2, I_3, -1_2) = \rho(\mathrm{diag}(I_2, B_4)\mathrm{diag}(I_4, (2+\sqrt{-3}x)I_2)(h),$$

where x is a solution of $3x^2 = -5$ in $\mathbb{Z}_2$.By virtue of Hensel's lemma such a solution exist, as it is the case in $\mathbb{Z}/8\mathbb{Z}$. We can further assume that $x \equiv 1 \pmod 4$. Now it is easy to check that

$$g_2\mathrm{GL}_6\left(\mathbb{Z}_2\left[\frac{1+\sqrt{-3}}{2}\right]\right) = \begin{bmatrix} I_2 & & \\ & \frac{1}{2}I_2 & \\ & \frac{-2+\sqrt{-3}}{2}I_2 & 2I_2 \end{bmatrix} \mathrm{GL}_6\left(\mathbb{Z}_2\left[\frac{1+\sqrt{-3}}{2}\right]\right).$$

Now we study h_3. We know that

$$h_3 = \rho(\mathrm{diag}(\sqrt{-3}I_3, I_3)B_6)(\mathrm{diag}(I_3, -I_3)).$$

Let x be a solution of $x^2 = -2$ in $\mathbb{Z}_3$. Then $\rho(\mathrm{diag}(Z(x, 1), x))(-I_3) = \mathrm{diag}(I_2, 2)$. Hence we have

$$h_3 = \rho(\mathrm{diag}(\sqrt{-3}I_3, I_3)B_6\mathrm{diag}(I_3, Z(x, 1), x^{-1}))(\mathrm{diag}(I_5, 2)).$$

Let σ be the permutation matrix corresponded to the following permutation $(6, 5, 4, 3, 2, 1)$. So

$$h_3 = \rho(\mathrm{diag}(\sqrt{-3}I_3, I_3)B_6\mathrm{diag}(I_3, Z(x, 1), x^{-1})\sigma)(h).$$

One can check that

$$g_3\mathrm{GL}_6(\mathbb{Z}_3[\sqrt{-3}]) = \mathrm{diag}(\frac{1}{\sqrt{-3}}I_3, I_3)\begin{bmatrix} I_3 & \\ Y_2 & I_3 \end{bmatrix}\mathrm{GL}_6(\mathbb{Z}_3[\sqrt{-3}]),$$

where

$$Y_2 = \frac{1}{\sqrt{-3}}\begin{bmatrix} 1 & & \\ & 1 & -1 \\ & 1 & 1 \end{bmatrix}.$$

($\mathbf{G_4^I}$) In this case, by Lemma 34, we only have to study h_2. We have

$$h_2 = \rho(\mathrm{diag}(I_2, B_4))(\mathrm{diag}(I_4, -I_2)).$$

On the other hand, by virtue of Hensel's lemma, there are x_1, x_2, x_3 and x_4 in $\mathbb{Z}_2$ such that

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = -1.$$

Let $H(y_1, y_2, y_3, y_4) = \begin{bmatrix} y_1 + \sqrt{-1}\,y_2 & y_3 + \sqrt{-1}\,y_4 \\ -y_3 + \sqrt{-1}\,y_4 & y_1 - \sqrt{-1}\,y_2 \end{bmatrix}$. Hence we have

$$h_2 = \rho(\mathrm{diag}(I_2, B_4)\mathrm{diag}(I_4, H(x_1, x_2, x_3, x_4)))(h).$$

We can and will assume that $x_1 = 2$, $x_3 = x_4 = 1$, and $x_2 \equiv 1 \pmod 4$. One can check that

$$g_2\mathrm{GL}_6(\mathbb{Z}_2[\sqrt{-1}]) = \begin{bmatrix} I_2 & & \\ & \frac{1}{2}I_2 & \\ & \frac{1}{2}H(-2,1,1,1) & 2I_2 \end{bmatrix}\mathrm{GL}_6(\mathbb{Z}_2[\sqrt{-1}]).$$

($\mathbf{G_4^{II}}$) In this case, again by Lemma 34, we only have to study h_2. Borrowing notations from the previous case, we have

$$h_2 = \rho(\mathrm{diag}(I_2, (1+i)I_2, I_2)\mathrm{diag}(I_2, B_4)\mathrm{diag}(I_4, H(x_1, x_2, x_3, x_4))(h).$$

One can check that

$$g_2\mathrm{GL}_6(\mathbb{Z}_2[\sqrt{-1}]) = \begin{bmatrix} I_2 & & \\ & \frac{1}{2}I_2 & \\ & \frac{1}{2}H(-2,1,1,1) & (1-i)I_2 \end{bmatrix}\mathrm{GL}_6(\mathbb{Z}_2[\sqrt{-1}]).$$

($\mathbf{G_4^{III}}$) Here again, we only have to calculate $g_2\mathrm{GL}_6(\mathbb{Z}_2[\sqrt{-1}])$. As before we start with h_2.

$$h_2 = \rho(\mathrm{diag}(I_2, 1+i, I_3)\ \mathrm{diag}(I_2, B_4)\ \mathrm{diag}(I_4, H(x_1, x_2, x_3, x_4))(h).$$

It can be checked that

$$g_2\mathrm{GL}_6(\mathbb{Z}_2[\sqrt{-1}]) = \begin{bmatrix} I_2 & & \\ & \frac{1}{2}I_2 & \\ & \frac{1}{2}H(-2,1,1,1) & 2Y_3 \end{bmatrix}\mathrm{GL}_6(\mathbb{Z}_2[\sqrt{-1}]),$$

where $Y_3 = \mathrm{diag}((1+i)^{-1}, 1)$.

($\mathbf{G_5^I}$) The only primes which should be studied are 2 and 3. $p = 2$ is almost identical to $\mathbf{G_3^I}$, and we have

$$g_2 \mathrm{GL}_5\left(\mathbb{Z}_2\left[\frac{1+\sqrt{-3}}{2}\right]\right) = \begin{bmatrix} 1 & & \\ & \frac{1}{2}I_2 & \\ & \frac{-2+\sqrt{-3}}{2}I_2 & 2I_2 \end{bmatrix} \mathrm{GL}_5\left(\mathbb{Z}_2\left[\frac{1+\sqrt{-3}}{2}\right]\right).$$

Now let $p = 3$. We have

$$h_3 = \rho(\mathrm{diag}(1, B_4)\ \mathrm{diag}(I_3, Z(x_1, x_2))(h),$$

where (x_1, x_2) is a solution of $x_1^2 + x_2^2 = -1$ in $\mathbb{Z}_3$. Hence

$$g_3 \mathrm{GL}_5(\mathbb{Z}_3[\sqrt{-3}]) = \mathrm{GL}_5(\mathbb{Z}_3[\sqrt{-3}]).$$

($\mathbf{G_5^{II}}$) Again we have two primes to look at, and $p = 2$ is the same as $\mathbf{G_5^I}$. For h_3, we know that

$$h_3 = \rho(\mathrm{diag}(1, \sqrt{-3}I_2, I_2)\ \mathrm{diag}(1, B_4)\ \mathrm{diag}(I_3, Z(x_1, x_2))(h),$$

where x_1 and x_2 are as in the previous case. We can and will assume that $x_1 \equiv x_2 \equiv 1 \pmod 3$. One can check that

$$g_3 \mathrm{GL}_5(\mathbb{Z}_3[\sqrt{-3}]) = \begin{bmatrix} 1 & & \\ & \frac{1}{\sqrt{-3}}I_2 & \\ & \frac{1}{\sqrt{-3}}Z(1,-1) & I_2 \end{bmatrix} \mathrm{GL}_5(\mathbb{Z}_3[\sqrt{-3}]).$$

($\mathbf{G_5^{III}}$) As in the previous case, by Lemma 34, we only have to study $p = 2$ and $p = 3$, and the case of $p = 2$ is identical with $\mathbf{G_5^I}$. On the other hand, we know that

$$h_3 = \rho(\mathrm{diag}(1, \sqrt{-3}, I_3)\ \mathrm{diag}(1, B_4)\ \mathrm{diag}(I_3, Z(x_1, x_2))(h),$$

where x_1 and x_2 are as in the previous two cases. One can check that

$$g_3 \mathrm{GL}_5(\mathbb{Z}_3[\sqrt{-3}]) = \begin{bmatrix} 1 & & \\ & Y_4 & \\ & Y_5 & I_2 \end{bmatrix},$$

where $Y_4 = \mathrm{diag}(\frac{1}{\sqrt{-3}}, 1)$ and $Y_5 = \begin{bmatrix} \frac{1}{\sqrt{-3}} & 0 \\ \frac{1}{\sqrt{-3}} & 0 \end{bmatrix}$.

($\mathbf{G_6^I}$) In this case, we only have to describe g_2. We know that

$$h_2 = \rho(\mathrm{diag}(1, B_4)\ \mathrm{diag}(I_3, H(x_1, x_2, x_3, x_4))(h),$$

where (x_1, x_2, x_3, x_4) is a solution of $x_1^2 + x_2^2 + x_3^2 + x_4^2 = -1$ in $\mathbb{Z}_2$. Indeed we can and will assume that $x_1 = 2$, $x_3 = x_4 = 1$, and $x_2 \equiv 1 \pmod 4$. One can check that

$$g_2 \mathrm{GL}_5(\mathbb{Z}_2[\sqrt{-1}]) = \begin{bmatrix} 1 & & \\ & \frac{1}{2}I_2 & \\ & \frac{1}{2}H(-2,1,1,1) & 2I_2 \end{bmatrix} \mathrm{GL}_6(\mathbb{Z}_2[\sqrt{-1}]).$$

($\mathbf{G_6^{II}}$) Following the previous case, we have to find g_2, and we have

$$h_2 = \rho(\mathrm{diag}(1, (1+i)I_2, I_2)\ \mathrm{diag}(1, B_4)\ \mathrm{diag}(I_3, H(x_1, x_2, x_3, x_4)))(h),$$

where are x_i's are as in the previous case. One can see that, in this case,

$$g_2\mathrm{GL}_5(\mathbb{Z}_2[\sqrt{-1}]) = \begin{bmatrix} 1 & & \\ & \frac{1}{2}I_2 & \\ & \frac{1}{2}H(-2,1,1,1) & (1-i)I_2 \end{bmatrix} \mathrm{GL}_6(\mathbb{Z}_2[\sqrt{-1}]).$$

($\mathbf{G_7^{I}}$) By Lemma 34, we only have to study $p = 2$ and $p = 7$. However in this case, $p = 2$ splits over l, and so we only have to find g_7. We know that

$$h_7 = \rho(\mathrm{diag}(1, B_4)\ \mathrm{diag}(I_3, Z(x_1, x_2)))(h),$$

where (x_1, x_2) is a solution of $x_1^2 + x_2^2 = -1$ in $\mathbb{Z}_7$. Thus, it is clear that

$$g_7\mathrm{GL}_5(\mathbb{Z}_7[\sqrt{-7}]) = \mathrm{GL}_5(\mathbb{Z}_7[\sqrt{-7}]).$$

($\mathbf{G_7^{II}}$) As in the previous case, we only have to find g_7. Similarly, we know that

$$h_7 = \rho(\mathrm{diag}(1, \sqrt{-7}I_2, I_2)\ \mathrm{diag}(1, B_4)\ \mathrm{diag}(I_3, Z(x_1, x_2)))(h),$$

where x_i's are as in the previous case. Further we can and will assume that $x_1 \equiv 3 \pmod 7$ and $x_2 \equiv 2 \pmod 7$. I can be checked that

$$g_7\mathrm{GL}_5(\mathbb{Z}_7[\sqrt{-7}]) = \begin{bmatrix} 1 & & \\ & I_2 & \\ & \frac{1}{\sqrt{-7}}Z(3,-2) & I_2 \end{bmatrix} \mathrm{GL}_5(\mathbb{Z}_7[\sqrt{-7}]).$$

11.3.3

Now, we will use the given local data, and list matrices in $\mathrm{GL}_m(l)$ which represent the described local cosets. In all the cases, there are at most two non-trivial local matrices, and all of them are lower triangular matrices. When $a_l = 1$, there is only one non-trivial local matrix for which we have a representative in $\mathrm{GL}_m(\mathcal{O}[\frac{1}{1+i}])$. Thus it also satisfies the other local conditions. By this or a similar argument, we get the following l-matrix representatives with the desired local conditions for the following cases.

type	$g \in \mathrm{GL}_m(l)$	type	$g \in \mathrm{GL}_m(l)$
$\mathbf{G_2^I}$	$\begin{bmatrix} 1 & & \\ & \frac{1}{2}I_3 & \\ & \frac{-2+\sqrt{-3}}{2}I_3 & 2I_3 \end{bmatrix}$	$\mathbf{G_4^I}$	$\begin{bmatrix} I_2 & & \\ & \frac{1}{2}I_2 & \\ & \frac{1}{2}Y_6 & 2I_2 \end{bmatrix}$
$\mathbf{G_4^{II}}$	$\begin{bmatrix} I_2 & & \\ & \frac{1}{2}I_2 & \\ & \frac{1}{2}Y_6 & (1-i)I_2 \end{bmatrix}$	$\mathbf{G_4^{III}}$	$\begin{bmatrix} I_2 & & \\ & \frac{1}{2}I_2 & \\ & \frac{1}{2}Y_6 & 2Y_3 \end{bmatrix}$
$\mathbf{G_5^I}$	$\begin{bmatrix} 1 & & \\ & \frac{1}{2}I_2 & \\ & \frac{-2+\sqrt{-3}}{2}I_2 & 2I_2 \end{bmatrix}$	$\mathbf{G_6^I}$	$\begin{bmatrix} 1 & & \\ & \frac{1}{2}I_2 & \\ & \frac{1}{2}Y_6 & 2I_2 \end{bmatrix}$
$\mathbf{G_6^{II}}$	$\begin{bmatrix} 1 & & \\ & \frac{1}{2}I_2 & \\ & \frac{1}{2}Y_6 & (1-i)I_2 \end{bmatrix}$	$\mathbf{G_7^I}$	I_7
$\mathbf{G_7^{II}}$	$\begin{bmatrix} 1 & & \\ & \frac{1}{\sqrt{-7}}I_2 & \\ & \frac{1}{\sqrt{-7}}Z(3,-2) & I_2 \end{bmatrix}$		

where $Y_3 = \mathrm{diag}((1+i)^{-1}, 1)$ and $Y_6 = H(-2, 1, 1, 1)$.

For the other cases, we will write the local matrices as product of a unipotent matrix and a diagonal matrix. Then use method of Chinese remainder argument to find the needed l-matrix representative. We get the following matrices.

$$(\mathbf{G_1^I}) \quad g = \begin{bmatrix} I_2 & & & \\ & I_2 & & \\ Y_7 & & I_2 & \\ & Y_7 & & I_2 \end{bmatrix} \mathrm{diag}\left(\frac{1}{2\sqrt{-3}}I_4, 2I_4\right), \text{ where}$$

$$Y_7 = \begin{bmatrix} 10 - 3\sqrt{-3} & -4 \\ 4 & 10 - 3\sqrt{-3} \end{bmatrix}.$$

$(\mathbf{G_2^{II}})\quad g = \begin{bmatrix} 1 & & \\ v & I_3 & \\ w & Y_8 & I_3 \end{bmatrix} \operatorname{diag}(\frac{1}{\sqrt{-3}}, \frac{1}{2}, \frac{1}{2\sqrt{-3}} I_2, 2I_3)$, where

$$v = -4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, w = (10-3\sqrt{-3})v, Y_8 = \begin{bmatrix} 10-3\sqrt{-3} & & \\ & -10+3\sqrt{-3} & -4 \\ & 4 & -10+3\sqrt{-3} \end{bmatrix}.$$

$(\mathbf{G_3^{I}})\quad g = \begin{bmatrix} I_2 & & \\ Y_9 & I_2 & \\ Y_{10} & Y_{11} & I_2 \end{bmatrix} \operatorname{diag}(\frac{1}{\sqrt{-3}} I_2, \frac{1}{2\sqrt{-3}}, \frac{1}{2}, 2I_2)$, where

$$Y_9 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, Y_{10} = \begin{bmatrix} & -2 \\ 6-3\sqrt{-3} & -2 \end{bmatrix}, Y_{11} = \begin{bmatrix} 2-3\sqrt{-3} & \\ -4 & 6-3\sqrt{-3} \end{bmatrix}.$$

$(\mathbf{G_5^{II}})\quad g = \begin{bmatrix} 1 & & \\ & I_2 & \\ & Y_7 & I_2 \end{bmatrix} \operatorname{diag}(1, \frac{1}{2\sqrt{-3}} I_2, 2I_2)$, where Y_7 is as above.

$(\mathbf{G_5^{III}})\quad g = \begin{bmatrix} 1 & & \\ & I_2 & \\ & Y_{12} & I_2 \end{bmatrix} \operatorname{diag}(1, \frac{1}{2\sqrt{-3}}, \frac{1}{2}, 2I_2)$, where

$$Y_{12} = \begin{bmatrix} -2+\sqrt{-3} & \\ 4 & -2+\sqrt{-3} \end{bmatrix}.$$

In each case, by the choice of g, $\overline{\Gamma}$ acts transitively on the vertices of the associated Bruhat-Tits building if and only if

$$\#\{x \in SL_m(\mathcal{O}_l) \mid \rho(x)(\rho(g^{-1})(h)) = \rho(g^{-1})(h)\} \tag{25}$$

is equal to the value of $\#\Lambda \cap K$ as we have already computed in Section 10.2. So to complete the panorama, one has to compute (25), which can be done as described in the forth step, and we execute in the next section.

11.3.4

In this section, we will summarize the results of programming with MAGMA as described in the forth step. Namely, at each case, we looked at the associated quadratic form over $\mathbb{Z}^{2m}$, found its group of symmetries, a generating set and image of the determinant map if needed to find the number of those with determinant one. In some of the cases, it is clear that the determinant is onto the group of roots of unity of $\mathcal{O}_l$, e.g. when m is odd and $l = \mathbb{Q}[\omega]$ or $l = \mathbb{Z}[i]$. Consequently, we get the following table, which combined with the results of Section 10.2, finishes our proof of theorem A and theorem B.

Proposition 35. *Let* Λ*,* K*, and* $\mathbf{G_i^j}$ *be as above. Then*

Label	*Full group of symmetries*	$\#\Lambda \cap K$
$\mathbf{G_1^I}$	$2^{15} \cdot 3^{10} \cdot 5^2$	$2^{15} \cdot 3^9 \cdot 5^2$
$\mathbf{G_2^I}$	$2^{11} \cdot 3^9 \cdot 5 \cdot 7$	$2^{10} \cdot 3^8 \cdot 5 \cdot 7$
$\mathbf{G_2^{II}}$	$2^{11} \cdot 3^9 \cdot 5$	$2^{10} \cdot 3^8 \cdot 5$
$\mathbf{G_3^I}$	$2^6 \cdot 3^6$	$2^6 \cdot 3^5$ or $2^5 \cdot 3^5$
$\mathbf{G_4^I}$	$2^{15} \cdot 3^2 \cdot 5$	$2^{\bullet} \cdot 3^2 \cdot 5$
$\mathbf{G_4^{II}}$	$2^{16} \cdot 3^2$	$2^{14} \cdot 3^2$
$\mathbf{G_4^{III}}$	$2^{16} \cdot 3$	$2^{\bullet} \cdot 3$
$\mathbf{G_5^I}$	$2^8 \cdot 3^6 \cdot 5$	$2^7 \cdot 3^5 \cdot 5$
$\mathbf{G_5^{II}}$	$2^8 \cdot 3^6 \cdot 5$	$2^7 \cdot 3^5 \cdot 5$
$\mathbf{G_5^{III}}$	$2^7 \cdot 3^6$	$2^6 \cdot 3^5$
$\mathbf{G_6^I}$	$2^{12} \cdot 3^2 \cdot 5$	$2^{10} \cdot 3^2 \cdot 5$
$\mathbf{G_6^{II}}$	$2^{13} \cdot 3^2$	$2^{11} \cdot 3^2$
$\mathbf{G_7^I}$	$2^8 \cdot 3 \cdot 5$	$2^7 \cdot 3 \cdot 5$ or $2^8 \cdot 3 \cdot 5$
$\mathbf{G_7^{II}}$	$2^5 \cdot 3^2 \cdot 5 \cdot 7$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$ or $2^5 \cdot 3^2 \cdot 5 \cdot 7$

Appendix A: Table d=2.

$D_k = 5$

$\delta_{l/k}$	h_l	r_l
5	1	10
9	1	6
16	1	4
41	1	2
49	1	2
61	1	2
73	1	2
64	1	2
109	1	2
117	2	6
121	2	2

$D_k = 8$

$\delta_{l/k}$	h_l	r_l
4	1	8
5	1	4
8	1	4
9	1	6
9	1	6
13	1	4
17	1	2
17	1	2
20	1	4
25	1	2
27	1	6
29	1	4
32	1	2
32	1	2
33	1	2
33	1	2
36	2	2
36	2	4
37	1	4

$D_k = 12$

$\delta_{l/k}$	h_l	r_l
1	1	12
3	1	6
4	1	6
4	1	6
7	1	6
7	1	6
12	1	6
13	2	6

$D_k \geq 13$

D_k	$\delta_{l/k}$	h_l	r_l
13	9	2	6
13	13	1	2
17	8	1	2
21	1	1	6
24	1	1	6
24	1	1	6
24	3	1	6
24	4	2	2
24	4	2	4
24	5	2	4
28	1	1	4
28	2	1	2
33	1	1	6
40	1	1	2
44	1	1	4
52	1	1	4
56	1	1	2
56	1	1	2
56	1	2	2
57	1	1	6
60	1	2	2
60	1	2	4
60	1	2	6

Appendix B: Values of Zeta and L-functions.

As it is mentioned before, using Bernoulli numbers, we compute values of zeta function at negative odd integer numbers, and we get the following.

i	1	3	5	7	9	11
$\zeta(-i)$	$\frac{-1}{12}$	$\frac{1}{120}$	$\frac{-1}{252}$	$\frac{1}{240}$	$\frac{-1}{132}$	$\frac{691}{32760}$.

Here we provide the Mathematica program which gives us a bad prime factor of the corresponded L-functions, together with the results. These prime factors do not appear in the denominator of the other zeta or L-function factors of $\mathcal{R}(l/\mathbb{Q}, m)$. Wherever an entry is 0, it means that the numerator does not have a large enough prime factor.

For a given number, first we establish if it is discriminant of a complex quadratic field or not. Then define the character χ associated to this quadratic field l. Finally we introduce the related exponential function of the generalized Bernoulli numbers, compute the L-function via the generalized Bernoulli numbers, and give a large enough prime factor of the numerator.

....

```
For[i = 2, i <135, i++,
    bool1:=Mod[i, 4] == 0 && ! (MoebiusMu[i/4] == 0)
           && ! (Mod[i/4, 4] == 3);
    bool2 := Mod[i, 4] == 3 && ! (MoebiusMu[i] == 0);
    If [ bool1 || bool2, a := If[bool1, i/4, i];
       Chi[s_] := If[OddQ[s], JacobiSymbol[-i, s],
                   OddPart := s/2^ (FactorInteger[s][[1]][[2]]);
                   (If[! (OddQ[a]), 0, If[Mod[a, 8] == 7, 1, -1]])^
                   (FactorInteger[s][[1]][[2]])*
                   JacobiSymbol[-i, OddPart]];
       F[z_] := Sum[Chi[k]*z*E^ (k*z)/(E^ (i*z) - 1), {k, 1, i - 1}];
       B = Series[F[z], z, 0, 12];
       For[b = 1, b < 11, b = b + 2;
          L = FactorInteger[SeriesCoefficient[B, b]*(b - 1)!];
          ABadPrimeFactor = 0;
          NotFound = True;
          For[counter = 1, counter < Length[L] + 1, counter++,
             If[NotFound && L[[counter]][[2]] > 0 && 11 < L[[counter]][[1]] &&
                GCD[L[[counter]][[1]], i] == 1 && PrimeQ[L[[counter]][[1]]],
                ABadPrimeFactor = L[[counter]][[1]]; NotFound = False ]; ];
          Print["&", ABadPrimeFactor]; ];
       Print["\\\\"]; ]; ];
```

......................The Mathematica program ....

a	$L_{l/\mathbb{Q}}(-2)$	$L_{l/\mathbb{Q}}(-4)$	$L_{l/\mathbb{Q}}(-6)$	$L_{l/\mathbb{Q}}(-8)$	$L_{l/\mathbb{Q}}(-10)$
3	0	0	0	809	1847
1	0	0	61	277	19
7	0	0	73	8831	73
2	0	19	307	83579	23
11	0	17	17	4999	43
15	0	31	941	821063	1682150401
19	0	269	53	13	41
5	59	137	23	1116041413	149
23	0	71	18517	41	63659
6	23	797	249089	13	43
31	0	0	41	337	17
35	0	107	31	19	61
39	0	457	30509	552942737	3480906042721
10	79	39521	1579	109493813441	1217
43	83	29	76565663	202075601281	13
47	0	59	5099	13640153	3671
51	67	2297	116111407	17235782633	37
13	409	263	55257133	251	16747
55	0	10687	103	193	997121
14	0	61	7963537	73	607
59	67	14813	47	19	7877706624037007
67	251	19	1367650871	151	3272681
17	13	19211	23	95621	1039
71	17	2267	4021907	30007358867	13
79	31	19	1879	17	23
83	31	37	167	151	107
21	43	53	46411	3511	22079
87	0	37	59	89	3633269
22	13	239	5171	7507	19
91	0	154877	1000621267	11827264167629	19
95	0	42013	199	59	271
103	17	7193	157	163	34471
26	311	39161	1409	197	6451
107	97	51341	317	263	89
111	151	19	197	1031593	34369
115	491	19	6949	13	13057719994803998711
29	107	17	17	37	24123887717272745429
119	31	37	10957	1209625671061	3889
30	127	53	6091875511	421	53
123	17	19	31	9947794325893	19
127	0	83	19	31	43
131	17	6967	4603	619	79829759
33	10333	8189899	17	23	59

Appendix C: Siegel-Klingen theorem.

In section 6, we had to find the exact value of $\mathcal{R}(l/k, m)$ for certain k, l and m. To do so, we used PARI, and we had to have a bound for the denominator of $\mathcal{R}(l/k, m)$. Here we give a new proof of Siegel-Klingen theorem, which also provides a bound for the denominator of $\mathcal{R}(l/k, m)$. J-P. Serre in [Se71] had already mentioned relation between co-volume of S-arithmetic lattices and rationality of zeta-values (Siegel's theorem).[2] He used Euler-Ponicarè measures to get Siegel's theorem. Here we proceed with a similar approach, in the p-adic setting, and as a result we also get rationality of certain L-function values (Klingen's theorem).

Let k be a totally real number field, and l a totally complex quadratic extension of k. Consider the hermitian space (l^m, I_m), and $\mathbb{G}$ the corresponding absolutely almost simple, simply connected unitary k-group. Let $(\mathbf{P}_\mathfrak{p})_{\mathfrak{p}\in V_f(k)}$ be a coherent family of parahorics with maximum volume among the parahoric subgroups of the corresponded group. For any prime $\mathfrak{p}$ which splits over l, as we have seen in section 3, one can construct $\Lambda_\mathfrak{p}$ a lattice in $\mathrm{SL}_m(k_\mathfrak{p})$. By Equation (8), Lemma 2, and Lemma 3, we have that

$$\mathrm{vol}(\mathrm{SL_m(k_\mathfrak{p})}/\Lambda_\mathfrak{p}) = \mathcal{R}(\mathrm{l/k,m}) \cdot \prod \mathrm{e}'(\mathbf{P}_{\mathfrak{p}'}).$$

By the choice of $\mathbf{P}_{\mathfrak{p}'}$'s, whenever $\mathbb{G}$ is quasi-split over a place, $e'(\mathbf{P}_{\mathfrak{p}'}) = 1$. On the other hand, if m is odd, over any prime we get a quasi-split group. When m is a multiple of 4, then over any prime the determinant of the split hermitian from is equal to one. Hence $\mathbb{G}$ is again quasi-split over any prime. So altogether we have

$$\mathrm{vol}(\mathrm{SL_m(k_\mathfrak{p})}/\Lambda_\mathfrak{p}) = \mathcal{R}(\mathrm{l/k,m}), \qquad \text{when } m \text{ is odd or } 4|\mathrm{m}.$$

When m is congruent to 2 modulo 4, $\mathbb{G}$ is quasi-split over a place whenever -1 is in the image of norm map. In particular, over all the unramified places, it is quasi-split. Over a ramified place, by Lemma 2, $e'(\mathbf{P}_{\mathfrak{p}'})$ is either one or $q_\mathfrak{p}^{m/2} - 1$.

On the other hand, we know that $\Lambda_\mathfrak{p}$'s are co-compact lattices, and have a finite set of vertices as a fundamental domain in the associated Bruhat-Tits building. Since the first congruence subgroup of $\mathrm{SL}_m(\mathcal{O}_\mathfrak{p})$ is a pro-p group where p is an odd rational prime divisible by $\mathfrak{p}$, the intersection of $\Lambda_\mathfrak{p}$ with the stabilizer of any vertex is a finite group whose order divides $\#\mathrm{SL}_m(\mathfrak{f}_\mathfrak{p})$ times a power of p. As a consequence

$$\mathrm{vol}(\mathrm{SL_m(k_\mathfrak{p})}/\Lambda_\mathfrak{p}) \in \frac{1}{\#\mathrm{SL_m}(\mathfrak{f}_\mathfrak{p})}\mathbb{Z}[1/\mathrm{p}].$$

Moreover, if $\nu_\mathfrak{p}(p)$, the $\mathfrak{p}$-valuation of p, is at most $p-2$, then the first congruence subgroup of $\mathrm{SL}_m(\mathcal{O}_\mathfrak{p})$ is torsion free. In particular, when $\mathfrak{p}$ is unramified over

[2] The second author would like to thank Professor A. Rapinchuk for pointing out this reference to him.

$\mathbb{Q}$ or p is larger than $\dim_{\mathbb{Q}} k + 2$, the first congruence subgroup is torsion free. In that case

$$\mathrm{vol}(\mathrm{SL_m}(\mathrm{k}_{\mathfrak{p}})/\Lambda_{\mathfrak{p}}) \in \frac{1}{\#\mathrm{SL_m}(\mathfrak{f}_{\mathfrak{p}})}\mathbb{Z}.$$

Hence if m is either odd or a multiple of 4,

$$\mathcal{R}(l/k, m) \cdot \mathfrak{W}_m \in \mathbb{Z},$$

where $\mathfrak{W}_m = \text{g.c.d.}_{\mathfrak{p}}(\#\mathrm{SL}_m(\mathfrak{f}_{\mathfrak{p}}))$, $\mathfrak{p}$ splits over l, and $\nu_{\mathfrak{p}}(p) < p - 1$. When m is congruence to 2 modulo 4, then by a similar argument

$$\mathcal{R}(l/k, m) \cdot \mathfrak{W}_m \cdot \prod_{\mathfrak{p}\ ramify/l} (q_{\mathfrak{p}}^{m/2} - 1) \in \mathbb{Z}.$$

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