MATH 109: THE FIRST EXAM.
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NAME: Solution and Remarks

PID: ....

(1) Write your Name and PID on the front of your exam sheet.
(2) No calculators or other electronic devices are allowed during this exam.
(3) Show all of your work; no credit will be given for unsupported answers.
(4) Read each question carefully to avoid spending your time on something that you are not supposed to (re)prove.
(5) Ask me when you are unsure if you are allowed to use certain fact or not.

Problem	Score out of 10
1	
2	
3	
Total out of 30	

Date: 04/29/2013.

(1) Which one of the following propositional forms is NOT equivalent to $P \Rightarrow (Q \vee R)$? Justify your answer.
  (a) $(P \wedge (\neg Q) \wedge (\neg R)) \Rightarrow O$.
  (b) $(P \wedge (\neg Q)) \Rightarrow R$.
  (c) $(P \Rightarrow Q) \wedge (P \Rightarrow R)$.
  (d) $(\neg P) \vee Q \vee R$.
(You have to only prove why your chosen propositional form is not equivalent to $P \Rightarrow (Q \vee R)$. You do NOT need to argue why the rest are equivalent.)

(c) $(P \Rightarrow Q) \wedge (P \Rightarrow R)$ is NOT equivalent to $P \Rightarrow (Q \vee R)$.

P	Q	R	$P \Rightarrow Q$	$P \Rightarrow R$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$	$Q \vee R$	$P \Rightarrow (Q \vee R)$
T	T	F	T	F	F	T	T

So they do not have the same truth-tables. Hence they are NOT equivalent.

Remark. • Proof by contradiction says

$$P \Rightarrow (Q \vee R) \equiv (P \wedge \neg(Q \vee R)) \Rightarrow O$$
$$\equiv (P \wedge (\neg Q) \wedge (\neg R)) \Rightarrow O \quad \text{(De Morgan's law)}$$

• One method of proving a disjunction is

Hypothesis	Goal
P	$Q \vee R$
$P \wedge (\neg Q)$	R

which means $P \Rightarrow (Q \vee R) \equiv (P \wedge (\neg Q)) \Rightarrow R$.

• We know that $T \Rightarrow S \equiv (\neg T) \vee S$. So $P \Rightarrow (Q \vee R) \equiv (\neg P) \vee Q \vee R$.

Remark. One can use $T \Rightarrow S \equiv (\neg T) \vee S$, de Morgan's law and ... to prove the above equivalencies:

$$P \Rightarrow (Q \vee R) \equiv (\neg P) \vee Q \vee R$$

$$(P \wedge (\neg Q) \wedge (\neg R)) \Rightarrow O \equiv \neg(P \wedge (\neg Q) \wedge (\neg R)) \vee O \equiv (\neg P) \vee Q \vee R.$$

$$(P \wedge (\neg Q)) \Rightarrow R \equiv \neg(P \wedge (\neg Q)) \vee R \equiv (\neg P) \vee Q \vee R.$$

(2) Prove that there are no integers m and n such that $7m + 21n = 15$.

If not, there are integers m and n such that

$$7m + 21n = 15.$$

$$\Rightarrow \quad 7m + 7 \times 3n - 7 \times 2 = 1$$

$$\Rightarrow \quad 7(m + 3n - 2) = 1$$

$$\Rightarrow \quad m + 3n - 2 = \frac{1}{7} \text{ is an integer}$$

as m and n are integers.

This contradicts the fact that there are no integers larger than $\underline{0}$ and less than $\underline{1}$.

$(0 < 1 < 7 \Rightarrow 0 < \frac{1}{7} < 1.)$

(3) Let $a_1 = 1$ and
$$a_{n+1} = 1 + \frac{1}{1+\frac{1}{a_n}}$$
for any positive integer n. Prove that for any positive integer n we have that $a_n < a_{n+1}$. (Hint: if $0 < b < c$, then $\frac{1}{c} < \frac{1}{b}$. And you can assume $a_n > 0$ without proof.)

We proceed by induction on $\underline{n}$.

<u>Base of the induction.</u> $n=1$. We have to show $a_1 < a_2$.

$$\left.\begin{array}{l} a_1 = 1 \text{ and } a_2 = 1 + \frac{1}{1+\frac{1}{1}} = \frac{3}{2} \\ 2 < 3 \end{array}\right\} \Rightarrow a_1 < a_2.$$

<u>The inductive step.</u> We would like to prove that

$$a_k < a_{k+1} \Rightarrow a_{k+1} < a_{k+2}.$$

$$a_k < a_{k+1} \Rightarrow 0 < \frac{1}{a_{k+1}} < \frac{1}{a_k}$$

(as we are told $a_n > 0$ for any positive integer $\underline{n}$.)

$$\Rightarrow 0 < 1 + \frac{1}{a_{k+1}} < 1 + \frac{1}{a_k}$$

$$\Rightarrow \frac{1}{1+\frac{1}{a_k}} < \frac{1}{1+\frac{1}{a_{k+1}}}$$

$$\Rightarrow 1 + \frac{1}{1+\frac{1}{a_k}} < 1 + \frac{1}{1+\frac{1}{a_{k+1}}}$$

$$\Rightarrow a_{k+1} < a_{k+2}.$$

Remark ① One can show that $a_n = \frac{f_{2n}}{f_{2n-1}}$ where $\{f_n\}$ is the Fibonacci sequence.

② If we set $a_1 = 2$ instead of 1, then $\{a_n\}$ is decreasing, i.e. $a_{n+1} < a_n$ for any positive integer n.

③ Let $a_1 = a$. If $0 < a < \frac{1+\sqrt{5}}{2}$, then $a_n < a_{n+1}$ for any positive integer n.

If $\frac{1+\sqrt{5}}{2} < a$, then $a_{n+1} < a_n$ for any positive integer n.

④ Using this problem and Remark ②, one can show

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

is well-defined and is equal to $\frac{1+\sqrt{5}}{2}$. This is a **continued fraction**.