Math 109 Midterm 1 Review

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1 Propositional forms and truth-table.

1. Which one of the following propositional forms is NOT equivalent to $P \Rightarrow Q$? Justify your answer.
   (a) $(P \wedge (\neg Q)) \Rightarrow O$.
   (b) $(\neg Q) \Rightarrow (\neg P)$.
   (c) $\neg(P \wedge Q)$.
   (d) $(\neg P) \vee Q$.
2. Show, without using truth tables, that the propositional form $(P \wedge Q) \vee (P \wedge \neg Q)$ is equivalent to P.
3. Find a propositional form whose truth table is the following.

P	Q	R	S	
T	T	T	T	F
T	T	T	F	F
T	T	F	T	T
T	T	F	F	T
T	F	T	T	F
T	F	T	F	F
T	F	F	T	F
T	F	F	F	F
F	T	T	T	F
F	T	T	F	T
F	T	F	T	F
F	T	F	F	F
F	F	T	T	F
F	F	T	F	F
F	F	F	T	T
F	F	F	F	F

2 Direct proof, case-by-case proof and proof by contradiction.

1. Prove that for all integers n, $4(n^2 + n + 1) - 3n^2$ is a perfect square.
2. Show that if a product of two positive real numbers is greater than 100, then at least one of the numbers is greater than 10.
3. Recall that a rotation matrix is a matrix of the form $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ for some angle θ. Prove that $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is not a rotation matrix. (You may use trig identities without proof.)
4. Prove that if $d|l_1$ and $d|l_2$, then $d|s_1 l_1 + s_2 l_2$ for any integers s_1, s_2.

5. Let k be a positive integer. Prove that $2k-1$ and $2k+1$ have no common divisor larger than 1.

6. Prove that if n^2 is even, then n is even. (Hint: An even number is one that can be written in the form $2k$ for some integer k.)

7. For a real number x, let $\lfloor x \rfloor$ be the integer part of x, i.e. it is the largest integer less than or equal to x.

   (a) Prove that $\lfloor -x \rfloor = -\lfloor x \rfloor$ if and only if x is an integer. (You may use the fact that for a real number x and an integer n we have $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$.)

   (b) Prove that $\lfloor x+m \rfloor = \lfloor x \rfloor + m$ for any real number x and any integer m.

   (c) Prove that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$ for any real number x. (Hint: let $y = x - \lfloor x \rfloor$ and consider three cases separately (i) $0 \leq y < 1/3$, (ii) $1/3 \leq y < 2/3$, and (iii) $2/3 \leq y < 1$.)

3 Constructing a proof backwards and inequalities.

1. Prove that for any positive real numbers x and y we have

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

2. Let a, b, c and d be real numbers such that $a > b$ and $c > d$. Prove that

$$ac + bd > ad + bc.$$

3. Let x and y be positive integers. Prove that

$$\frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}}.$$

4. Let a, b and c be three real numbers. Prove that

$$ab + ac + bc \leq a^2 + b^2 + c^2.$$

4 Proof by induction.

1. Give a recursive definition for each sequence:

   a) 1, 4, 7, 10, 13, 16, ...

   b) -1, 1, -1, 1, -1, 1, ...

   c) 1, 4, 9, 16, 25, 36, ...

   d) 1, 4, 8, 13, 19, 26, ...

   e) 8, 16, 32, 64, 128, 256, ...

2. Show that $n^3 - n$ is divisible by 3 for every positive integer n.

3. Prove that for any positive integer n,

   a) $\displaystyle\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

b) $\displaystyle\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

c) $\displaystyle\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

4. If the Fibonacci numbers are defined by $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$, prove that for all $n \geq 0$, $\displaystyle\sum_{i=0}^{n} f_i^2 = f_n f_{n+1}$.

5. Let $a_1 = 1$ and $a_{n+1} = \frac{3a_n+1}{2a_n+1}$ for any positive integer n. Prove that

   (a) For any positive integer n, we have that $a_n < a_{n+1}$.

   (b) For any positive integer n, we have that $a_n < \frac{1+\sqrt{3}}{2}$.

6. Let $a_1 = 1$ and $a_{n+1} = \sqrt{1+a_n}$ for any positive integer n. Prove that

   (a) For any positive integer n, we have that $a_n < a_{n+1}$.

   (b) For any positive integer n, we have that $a_n < \frac{1+\sqrt{5}}{2}$.

7. Let $u_0 = 0$, $u_1 = 1$ and $u_{n+1} = 2u_n + 2u_{n-1}$. Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$. Prove that

   (a) For any positive integer n we have

   $$A^n = \begin{bmatrix} 2u_{n-1} & u_n \\ 2u_n & u_{n+1} \end{bmatrix}.$$

   (b) For any positive integers m and n, we have that

   $$u_{m+n} = 2u_{n-1}u_m + u_n u_{m+1}.$$

   (Hint: $A^{m+n} = A^m A^n$.)

   (c) Use part (b) and induction to prove that $u_m | u_{mk}$ for any positive integers m and k.