1 [Expansion of Midterm Problem 1.]

Let R be a semisimple ring. Then

$$R = \bigoplus_{i=1}^{k} I_i ,$$

where I_i are minimal left R-mod. After relabeling I_i, we can and will assume that

$$R = \bigoplus_{i=1}^{\ell} \left(\bigoplus_{j=1}^{n_i} I_{i,j} \right),$$

such that $I_{i,j} \simeq I_{i,j'}$ for any $1 \le j \le j' \le n_i$

and $I_{i,j} \not\simeq I_{i',j'}$ for any $i \neq i'$.

(By the way, this implies that $R \simeq \bigoplus_{i=1}^{\ell} \left(I_{i,1}\right)^{n_i}$.)

(a) Let $\phi \in \mathrm{Hom}_R(I_{i,1}, R)$. Prove that

$$\mathrm{Im}(\phi) \subseteq M_i := \bigoplus_{j=1}^{n_i} I_{i,j} .$$

(b) Prove that if $\phi \in \mathrm{End}_R(R)$, then, for any i,

$$\phi(M_i) \subseteq M_i ,$$

where $M_i = \bigoplus_{j=1}^{n_i} I_{i,j}$. (<u>Hint</u>: Use part (a).)

(c) Prove $\mathrm{End}_R(R) \simeq \bigoplus_{i=1}^{\ell} \mathrm{End}_R(M_i)$ as two rings.

(d) Prove that $\mathrm{End}_R(M_i) \simeq M_{n_i}(D_i)$ where $D_i = \mathrm{End}_R(I_{i,1})$ is a division ring.

(e) Prove that $R \simeq \bigoplus_{i=1}^{\ell} M_{n_i}(D_i^{op})$.

(<u>Hint</u>: $\mathrm{End}_R(R) \simeq R^{op}$.)

(For a given ring $(A,+,\cdot)$, its opposite ring $(A^{op},+,\bullet)$ is a ring with the same underlying additive group and its multiplication is defined as follows

$$x \bullet y := y \cdot x.$$

<u>Exp 1.</u> A commutative $\Rightarrow A = A^{op}$.

<u>Exp 2.</u> $A = M_n(F) \Rightarrow A \simeq A^{op}$
$x \mapsto x^T$ where x^T is the transpose of x.

<u>Exp 3.</u> $\tau : A \to A$ is called an involution if

① $\tau^2 = id_A$.

② $\tau(x+y) = \tau(x) + \tau(y)$.

③ $\tau(xy) = \tau(y)\cdot\tau(x)$.

If A has an involution, then $A \simeq A^{op}$.

Exp 4. If D is a division ring, then D^{op} is also a division ring.

Exp 5. $M_n(D)^{op} \simeq M_n(D^{op})$ [Similar to Exp 2.])

[As a result of this exercise, you see that any (left) semisimple ring is isomorphic to

$$M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus \cdots \oplus M_{n_k}(D_k)$$

where D_i are division rings. One can prove that n_i and D_i are unique (up to isomorphism.) This is called Artin-Wedderburn theorem.]

② [Expansion of Midterm Problem 3.]

ⓐ Let R be a commutative ring and

$$GL_n(R) = \{A \in M_n(R) \mid \exists B \in M_n(R) : AB = BA = I_n\}.$$

Prove that $A \in GL_n(R)$ if and only if $\det(A)$ is a unit in R.

(You can find the following useful:

① For any commutative ring R, one can define

$$\det : M_n(R) \longrightarrow R$$

(2) $\det(I_n) = 1$ and $\det(AB) = \det(A)\det(B)$.

(3) The (i,j) minor A_{ij} of A is the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting the i^{th} row and the j^{th} column. The adjoint $\operatorname{adj}(A)$ of A is an $n\times n$ matrix whose (i,j) entry is

$$(-1)^{i+j} A_{ji}.$$

Then $A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det(A)\, I_n$.)

(b) Let R be a commutative ring and $A, B \in M_n(R)$.

Prove that $\operatorname{Im}(A) \subseteq \operatorname{Im}(B) \iff \exists X \in M_n(R) : A = BX$.

$\left(\operatorname{Im}(C) = \{ C\vec{v} \mid \vec{v} \in R^n \} = \sum_{i=1}^{n} R\vec{c}_i \text{ where } C = [\vec{c}_1\, \vec{c}_2 \cdots \vec{c}_n]. \right)$

(c) Let K be a field and $R \subseteq K$ be a subring.

Assume $A, B \in M_n(R)$ and $\det(A) \neq 0$. Prove that

$$\operatorname{Im}(A) = \operatorname{Im}(B) \iff \exists P \in GL_n(R) : A = BP.$$

(d) Let R be a commutative ring and $P \in M_n(R)$. Prove that $P \in GL_n(R)$ if and only if the columns of P form an R-basis of R^n.

(e) Let R be a commutative ring and $A \in M_n(R)$.

Prove that $\det(A) R^n \subseteq \operatorname{Im}(A)$.

(f) Let K be a field and $R \subseteq K$ be a PID.

Assume that $A \in M_n(R)$ and $\det(A) \neq 0$. Prove that

$\exists\, \vec{v}_1, \dots, \vec{v}_n \in R^n$ and $q_1 | q_2 | \cdots | q_n$ s.t.

(i) $R^n = R\vec{v}_1 \oplus \cdots \oplus R\vec{v}_n$.

(ii) $\operatorname{Im}(A) = Rq_1 v_1 \oplus \cdots \oplus Rq_n \vec{v}_n$.

Conclude that $\exists\, P_1 \in GL_n(R)$ s.t.

$$\operatorname{Im}(A) = \operatorname{Im}\left(P_1 \begin{bmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{bmatrix}\right).$$

Now using (c), show that $\exists\, P_1, P_2 \in GL_n(R)$ s.t.

$$A = P_1 \begin{bmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{bmatrix} P_2$$

(g) Let F be a field. If $A \in M_n(F[x])$ and $\det(A) \neq 0$, then

$$\dim_F \left(F[x]^n / \operatorname{Im}(A) \right) = \deg(\det A).$$

③ Prove that the following are equivalent:

ⓐ R is a (left) semisimple ring.

ⓑ Any left R-module is semisimple.

ⓒ Any left R-module is projective.

ⓓ Any left R-module is injective.

(You do not have to show that ⓐ and ⓑ are equivalent.)

④ ⓐ Let R be a ring and M be an R-module. Prove that

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv)

(i) M is free.

(ii) M is projective.

(iii) M is flat.

(iv) M is torsion-free, i.e. $ax \neq 0$ if $0 \neq x \in M$ and $a \in R$ is NOT a zero-divisor.

ⓑ Prove that if R is a PID and M is a f.g. R-mod, then (i), (ii), (iii) and (iv) are equivalent.

⑤ An abelian group G is called divisible if for any $a \in G$ and any $n \in \mathbb{Z} \setminus \{0\}$ $nx = a$ has a solution in G.

(a) Prove that a $\mathbb{Z}$-mod G is injective if and only if G is divisible.

(b) Let G and H be $\mathbb{Z}$-modules and $\phi \in \mathrm{Hom}_{\mathbb{Z}}(G, H)$. If G is divisible, then $\phi(G)$ is also divisible.

(c) If G_i are divisible, then $\bigoplus_{i \in I} G_i$ and $\prod_{i \in I} G_i$ are also divisible.

(d) Prove that any abelian group can be embedded into a divisible abelian group.

(Hint: $\bigoplus_{i \in I} \mathbb{Z}/K \hookrightarrow \bigoplus_{i \in I} \mathbb{Q}/K$.)

(e) Let J be a divisible abelian group. Prove that $\mathrm{Hom}_{\mathbb{Z}}(R, J)$ is an injective R-module.

(f) Prove that any R-module can be embedded into an injective module.

(Hint: $\forall M$: R-mod $\exists J$: $\mathbb{Z}$-mod & divisible s.t.

$M \hookrightarrow \mathcal{J}$ as $\mathbb{Z}$-modules $\Rightarrow$

$$M \simeq \mathrm{Hom}_R(R,M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R,M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R,\mathcal{J}).)$$

⑥ Let P_1 and P_2 be projective R-modules. Prove that if

$$0 \to Q_1 \to P_1 \xrightarrow{\pi_1} M \to 0$$

and

$$0 \to Q_2 \to P_2 \xrightarrow{\pi_2} M \to 0$$

are short exact sequences, then

$$P_1 \oplus Q_2 \simeq P_2 \oplus Q_1.$$

(<u>Hint</u>. Consider $N = \{(x_1, x_2) \in P_1 \oplus P_2 \mid \pi_1(x_1) = \pi_2(x_2)\}$.)