1. [Infinite Galois Theory]

Let $E/_F$ be a normal and separable extension.

Let $\Omega := \{ F \subseteq K \subseteq E \mid K/_F : \text{finite, normal and separable} \}$

ⓐ Prove that $E = \bigcup_{K \in \Omega} K$.

ⓑ Let $\Phi : \mathrm{Aut}(E/_F) \longrightarrow \prod_{K \in \Omega} \mathrm{Gal}(K/_F)$

$$\Phi(\tau) := (\tau|_K)_{K \in \Omega}.$$

Prove that Φ is a well-defined, injective group homomorphism.

ⓒ Let $\varprojlim \mathrm{Gal}(K/_F) := \{ (\tau_K) \in \prod_{K \in \Omega} \mathrm{Gal}(K/_F) \mid$

$$\tau_K|_{K'} = \tau_{K'} \text{ if } K' \subseteq K \ \& \ K, K' \in \Omega \}.$$

Prove that $\mathrm{Im}(\Phi) = \varprojlim \mathrm{Gal}(K/_F)$ and conclude that $\mathrm{Aut}(E/_F) \simeq \varprojlim \mathrm{Gal}(K/_F)$.

ⓓ Take the discrete topology on the finite sets $\mathrm{Gal}(K/F)$ for any $K \in \Omega$ and prove that with respect to the product topology $\varprojlim \mathrm{Gal}(K/F)$ is a closed subset of $\prod_{K \in \Omega} \mathrm{Gal}(K/F)$.

ⓔ With respect to the above topology prove that, for any $K \in \Omega$,

$$G_K := \{\tau \in \mathrm{Aut}(E/F) \mid \tau|_K = \mathrm{id}_K\}$$

is an open subgroup of $\mathrm{Aut}(E/F)$. And any neighborhood of the identity contains G_K for some $K \in \Omega$.

[As it is discussed in class, it is called the Krull topology.]

ⓕ Prove that, for any field $F \subseteq K \subseteq E$, G_K is a closed subgroup of $\mathrm{Aut}(E/F)$.

ⓖ Prove that, for any subgroup G of $\mathrm{Aut}(E/F)$,

$$G_{\mathrm{Fix}(G)} = \overline{G},$$

where $G_{\mathrm{Fix}(G)}$ is defined as in ⓔ and $\overline{G}$ is the closure of G in $\mathrm{Aut}(E/F)$.

[<u>Hint</u>: By ⓕ, $\overline{G} \subseteq G_{\mathrm{Fix}(G)}$.

Let $\mathcal{O}$ be an open subset of $\mathrm{Aut}(E/F)$ which does not intersect G. By ⓔ, we can assume that $\mathcal{O} = \tau_0 G_K$ for some $K \in \Omega$.

Let $\pi_K : \mathrm{Aut}(E/F) \longrightarrow \mathrm{Gal}(K/F)$ be $\pi_K(\tau) = \tau|_K$.

We have ① $\pi_K(\tau_0) \notin \pi_K(G)$

② $\mathrm{Fix}(G) \cap K = \mathrm{Fix}(\pi_K(G))$.

③ $\{\sigma \in \mathrm{Gal}(K/F) \mid \sigma|_{\mathrm{Fix}(\pi_K(G))} = \mathrm{id}_{\mathrm{Fix}(\pi_K(G))}\} = \pi_K(G)$.

Hence $\tau_0 \notin G_{\mathrm{Fix}(G)}$.]

ⓗ Prove that there is a correspondence between closed subgroups of $\mathrm{Aut}(E/F)$ and fields $F \subseteq K \subseteq E$.

[$G \longmapsto \mathrm{Fix}(G)$ and $K \longmapsto G_K$.]

2. [Separable and purely inseparable extensions]

Let F be a field of characteristic $p>0$. Recall that an irreducible polynomial $f(x) \in F[x]$ is separable if and only if $f'(x) \neq 0$.

ⓐ Prove that, if $f(x) \in F[x]$ is irreducible, then

$$f(x) = f_{sep}(x^{p^k})$$

where $f_{sep}(x) \in F[x]$ is an irreducible polynomial.

[Hint: ① $g'(x) = 0 \Longleftrightarrow \exists\, h(x) \in F[x] : g(x) = h(x^p)$.

② $h(x^p)$ irreducible $\Rightarrow h(x)$ irreducible.]

ⓑ Let E/F be an algebraic extension. Prove that the following statements are equivalent:

(i) $\forall \alpha \in E$, $\alpha^{p^k} \in F$ for some k.

(ii) $\forall \alpha \in E$, $m_{\alpha,F}(x) = x^{p^k} - a$ for some $a \in F$.

(iii) $\forall \alpha \in E \setminus F$, α is NOT separable over F.

[Hint: (i) $\Rightarrow m_{\alpha,F}(x) \mid x^{p^k} - \alpha^{p^k} = (x-\alpha)^{p^k}$

$\Rightarrow m_{\alpha,F}(x) = (x-\alpha)^n$, in particular $\alpha^n \in F$.

• $\alpha \neq 0 \Rightarrow \text{Ord}(\alpha F^{\times}) \mid p^k \Rightarrow \text{Ord}(\alpha F^{\times}) = p^{\ell} \mid n$.

• $(x-\alpha)^{p^{\ell}} = x^{p^{\ell}} - \alpha^{p^{\ell}} \mid m_{\alpha,F}(x) \rightsquigarrow$ (ii).

(iii) $\overset{?}{\Rightarrow}$ (i) Use (a).]

[E/F is called a purely inseparable extension if the above properties hold.]

[Observe that E/F separable and purely inseparable imply that $E=F$.]

(c) Let E/F be an algebraic extension. Let's recall that the separable closure

$$E_{sep} := \{\alpha \in E \mid m_{\alpha,F}(x) \text{ separable}\}$$

of F in E is a field. Prove that E/E_{sep} is purely inseparable.

(d) Prove that, if E/F and K/E are algebraic separable extensions, then K/F is an algebraic separable extension.

[Hint: $E \subseteq K_{sep} \Rightarrow K/K_{sep}$ is separable and purely inseparable $\Rightarrow K = K_{sep}$.]

(e) F is called a perfect field if any algebraic extension E/F is separable. Prove that F is perfect if and only if $F^p = F$.

[Hint: ($\Rightarrow$) $a \in F \setminus F^p \Rightarrow x^p - a \in F[x]$ is irreducible.

($\Leftarrow$) If $g(x) = a_n x^n + \cdots + a_0 \in F[x]$, then $g(x^p) = h(x)^p$ where $h(x) = b_n x^n + \cdots + b_0$ and $b_i^p = a_i$.]

3. Let $\tau \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $F = \mathrm{Fix}(\tau)$. Prove that, if E/F is (finite and) Galois, then $\mathrm{Gal}(E/F)$ is cyclic.

4. Let F be a maximal subfield of $\overline{\mathbb{Q}}$ which does not contain $\sqrt{3}$. Prove that, if E/F is (finite and) Galois, then $\mathrm{Gal}(E/F)$ is cyclic.

[Hint: Let $G = G_F \subseteq G_{\mathbb{Q}} := \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$, $N = G_{\mathbb{Q}[\sqrt{3}]}$ and $\pi : G_{\mathbb{Q}} \to G_{\mathbb{Q}}/N \simeq \mathrm{Gal}(\mathbb{Q}[\sqrt{3}]/\mathbb{Q})$.

$\mathbb{Q}[\sqrt{3}] \not\subseteq F \Rightarrow G_F \not\subseteq N \Rightarrow |\pi(G_F)| = 2.$

$\left(F \subsetneq E \Rightarrow \mathbb{Q}[\sqrt{3}] \subseteq E\right) \Rightarrow \left(\underbrace{H \subsetneq G_F}_{\text{closed}} \Rightarrow |\pi(H)| = 1\right)$

$\Rightarrow$ If $\pi(g) \neq 1$, then $\overline{\langle g \rangle} = G_F$.]