FINAL MATH 200B.

NAME: ....

1. (5 points) Determine the number of conjugacy classes of $\mathrm{GL}_2(\mathfrak{f}_q)$. (Hint: use linear algebra)

2. (10 points) Prove that any projective module is flat.

3. (10 points) Let E/F be a (finite) Galois extension. Prove that as E-algebras

$$E \otimes_F E \simeq E \oplus \cdots \oplus E,$$

where the right hand side is the sum of $[E : F]$-many terms.

4. Let E/F be a normal finite extension and $\mathrm{char}(F) = p > 0$.

(a) (5 points) Prove that there is $f(x) \in F[x]$ such that E is the splitting field of $f(x)$ over F.
(b) (5 points) Let $F_s := \{a \in E \mid a \text{ is separable over } F\}$. Prove that F_s is a field and F_s/F is Galois.
(c) (10 points) Prove that the restriction map induces an isomorphism between $\mathrm{Aut}(E/F)$ and $\mathrm{Aut}(F_s/F)$.

5. (10 points) Let $f(x) \in F[x]$ be an irreducible and separable polynomial. Assume $\deg(f) = p$ is prime and G is the Galois group of f. Prove that $p \mid |G|$ and $p^2 \nmid |G|$.

6.(a) (10 points) Let $f(x) \in \mathfrak{f}_p[x]$ be an irreducible polynomial of degree d. Prove that

$$d|n \qquad \text{if and only if} \qquad f(x)|x^{p^n} - x.$$

(b) (5 points) Let $A_d := \{f(x) \in \mathfrak{f}_p[x] \mid f(x) \text{ is irreducible, monic and of degree } d\}$ and $a_d := |A_d|$. Prove that

$$\sum_{d|n} d a_d = p^n.$$

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