Outline of Solutions

$\boxed{5a}$ Prove by induction on n that $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$,

where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Base Case: $n=1 \mapsto$ It is clear.

Induction Step:

$$A^{k+1} = A^k \cdot A$$

$$= \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{(by induction hypothesis.)}$$

$$= \begin{bmatrix} f_{k+1} + f_k & f_{k+1} \\ f_k + f_{k-1} & f_k \end{bmatrix}$$

$$= \begin{bmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{bmatrix} \quad \text{(by the def.)}$$

$\boxed{5b}$ $A^{m+n} = A^m \cdot A^n \Rightarrow$ (using part $\underline{\underline{a}}$)

$$\begin{bmatrix} f_{m+n+1} & f_{m+n} \\ f_{m+n} & f_{m+n-1} \end{bmatrix} = \begin{bmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{bmatrix} \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

Compare the 2,1 entries:

$$f_{m+n} = f_{m+1} \cdot f_n + f_m \cdot f_{n-1}.$$

$\boxed{5c}$ By induction on $\underline{k}$, we prove that for any natural number n, $f_n \mid f_{nk}$.

<u>Base Case</u>: $k=1 \rightsquigarrow f_n \mid f_n$ (It is clearly true.)

<u>Induction Step</u>:

$$f_{n(k+1)} = f_{nk+n}$$

$$= f_{nk+1} \cdot f_n + f_{nk} \cdot f_{n-1} \quad \text{(using part b.)}$$

$$\overset{f_n}{\equiv} f_{nk} \cdot f_{n-1}$$

$$\overset{f_n}{\equiv} 0 \quad \text{(Induction hypoth.)}$$

$\boxed{6}$ First by induction on n we prove

$$2 \le a_n < 2+\sqrt{3}.$$

<u>Base Case</u>: $n=0 \rightsquigarrow 2 \le a_0 = 2 < 2+\sqrt{3}$ ✓

<u>Induction Step</u>:

$$2 \le a_k < 2+\sqrt{3} \implies 8 \le 4a_k < 8+4\sqrt{3}$$

$$\Longrightarrow 7 \le 4a_k - 1 < 7 + 4\sqrt{3}$$
$$= (2+\sqrt{3})^2.$$

$$\Longrightarrow \sqrt{7} \le \sqrt{4a_k - 1} < 2 + \sqrt{3}$$

$$\Longrightarrow 2 < \sqrt{7} \le a_{k+1} < 2 + \sqrt{3}.$$

. Now we'd like to prove

$$a_n \le a_{n+1} = \sqrt{4a_n - 1}.$$

Since $\underline{a_n}$ is positive,

$$a_n \le \sqrt{4a_n - 1} \iff a_n^2 \le 4a_n - 1$$
$$\iff a_n^2 - 4a_n + 1 \le 0.$$

$$\iff a_n^2 - 4a_n + 4 \le 3$$

$$\iff (a_n - 2)^2 \le 3$$

$$\iff 2 - \sqrt{3} \le a_n \le 2 + \sqrt{3}$$

Since $2 \le a_n < 2 + \sqrt{3}$, by the above argument,

$$a_n \le a_{n+1}$$

$\boxed{7a}$ $A_1 \Delta A_2 \Delta \cdots \Delta A_{n+1} := (A_1 \Delta \cdots \Delta A_n) \Delta A_{n+1}$

[Alternatively: • $B_1 := A_1$

• $B_{k+1} := B_k \Delta A_{k+1}$.]

$\boxed{7b}$ Before proving the claim by induction on n, let us show that

$$\chi_{A\Delta B} = \chi_A + \chi_B - 2\chi_{A\cap B}.$$

First solution:

$x\in A$	$x\in B$	$x\in A\cap B$	$x\in A\Delta B$	LHS	RHS
T	T	T	F	0	$1+1-2=0$
T	F	F	T	1	$1+0-0=1$
F	T	F	T	1	$0+1-0=1$
F	F	F	F	0	$0+0-0=0$

Second Solution:

$$\chi_{A\Delta B} = \chi_{(A\cup B)\setminus(A\cap B)}$$

$$= \chi_{A\cup B} - \chi_{A\cap B} \quad (\text{since } A\cap B \subseteq A\cup B)$$

$$= \chi_A + \chi_B - \chi_{A\cap B} - \chi_{A\cap B} \quad (\text{from homework.})$$

$$= \chi_A + \chi_B - 2\chi_{A\cap B}$$

Now we prove by induction on n that

$$\chi_{A_1 \Delta \cdots \Delta A_n} \overset{2}{\equiv} \sum_{i=1}^{n} \chi_{A_i}.$$

<u>Base Case</u> : <u>$n=2$</u>

$$\chi_{A_1 \Delta A_2} = \chi_{A_1} + \chi_{A_2} - 2\chi_{A_1 \cap A_2}$$

$$\overset{2}{\equiv} \chi_{A_1} + \chi_{A_2}. \quad \checkmark$$

<u>Induction Step.</u>

$$\chi_{A_1 \Delta \cdots \Delta A_{k+1}} = \chi_{(A_1 \Delta \cdots \Delta A_k) \Delta A_{k+1}}$$

(Using the base case) $\overset{2}{\equiv} \chi_{A_1 \Delta \cdots \Delta A_k} + \chi_{A_{k+1}}$

(by the induction hypothesis.) $\overset{2}{\equiv} (\chi_{A_1} + \cdots + \chi_{A_k}) + \chi_{A_{k+1}}.$

$\boxed{8a}$ <u>Yes</u> : $f(x_1) = f(x_2) \Rightarrow \dfrac{ax_1+b}{cx_1+d} = \dfrac{ax_2+b}{cx_2+d}$

$$\Rightarrow (ax_1+b)(cx_2+d) = (ax_2+b)(cx_1+d)$$

$$\Rightarrow acx_1x_2 + bcx_2 + adx_1 + bd =$$
$$acx_1x_2 + bcx_1 + adx_2 + bd$$

$$\Rightarrow (ad-bc)(x_1-x_2) = 0$$

$$\Rightarrow \quad x_1 = x_2 \quad (\text{since } ad-bc \neq 0)$$

$\boxed{8b}$ $y \in Im(f) \iff \exists x \in \mathbb{R} \setminus \{-\frac{d}{c}\},\ y = \frac{ax+b}{cx+d}$

$$\iff (cx+d)\,y = ax+b \wedge x \neq \frac{-d}{c}$$

$$\iff cyx + dy = ax+b \wedge x \neq \frac{-d}{c}$$

$$\iff (cy-a)\,x = -dy+b \wedge x \neq \frac{-d}{c}$$

$\boxed{c \neq 0}$

If $y \neq \frac{a}{c}$, then $x = \dfrac{-dy+b}{cy-a}$ satisfies the above equality. Let us also check that it is NOT equal to $\frac{-d}{c}$:

If not, $\frac{-d}{c} = \dfrac{-dy+b}{cy-a} \Rightarrow -dcy + ad = -dcy + bc$

$\Rightarrow ad-bc = 0$ which is a contradiction.

Thus $\mathbb{R} \setminus \{a/c\} \subseteq Im(f)$.

On the other hand, we claim that $\frac{a}{c} \notin Im(f)$.

If not, then for some $x \in \mathbb{R} \setminus \{-d/c\}$ we have $(c \cdot \frac{a}{c} - a)\,x = -d \cdot \frac{a}{c} + b \Rightarrow$

$$0 = -\frac{ad}{c} + b \Rightarrow$$

$$ad - bc = 0,$$

which is a contradiction.

So if $c \neq 0$, $\mathrm{Im}(f) = \mathbb{R} \setminus \{a/c\}$.

$\boxed{c=0}$ By the above argument,

$$y \in \mathrm{Im}(f) \iff -ax = -dy + b$$

Since $ad - bc \neq 0$ and $c = 0$, $a \neq 0$.

Hence $x = \frac{d}{a} y - \frac{b}{a}$, which means

$$\mathrm{Im}(f) = \mathbb{R}.$$

$\boxed{8c}$ $\boxed{c \neq 0}$ By parts a & b, f induces a bijection from $\mathbb{R} \setminus \{\frac{-d}{c}\}$ to $\mathbb{R} \setminus \{\frac{a}{c}\}$. One can easily extends it to a bijection from $\mathbb{R} \cup \{☺\}$ to $\mathbb{R} \cup \{☺\}$, as follows

$$\tilde{f}\left(\frac{-d}{c}\right) = ☺ \quad \text{and} \quad \tilde{f}(☺) = \frac{a}{c}$$

[alternatively, $\tilde{f}\left(\frac{-d}{c}\right) = \frac{a}{c}$ and $\tilde{f}(☺) = ☺$]

$\boxed{c=0}$ By parts a & b, f is a bijection from $\mathbb{R}$ to $\mathbb{R}$. Let $\tilde{f}(☺) = ☺$.