

SUM-PRODUCT PHENOMENA: p -ADIC CASE.

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ABSTRACT. The sum-product phenomena over a finite extension K of \mathbb{Q}_p is explored. The main feature of the results is the fact that the implied constants are independent p .

1. INTRODUCTION

1.1. Bounded generation phenomena and the general approach towards proving them. Let R be a unital commutative ring. Given two subsets A and B of R , we define the *sum set*

$$A + B := \{a + b \mid a \in A, b \in B\},$$

the *difference set*

$$A - B := \{a - b \mid a \in A, b \in B\},$$

and the *product set*

$$A \cdot B := \{ab \mid a \in A, b \in B\}.$$

We also define the l -fold sum set and the l -fold product set of a subset A of R :

$$(1) \quad \sum_l A := \{a_1 + \cdots + a_l \mid a_i \in A\} \quad \text{and} \quad \prod_l A := \{a_1 \cdots a_l \mid a_i \in A\}.$$

Starting with a subset A of the ring R , the subring generated by A is denoted by $\langle A \rangle$; that means $\langle A \rangle$ is the smallest subset of R which contains A and $\langle A \rangle \cdot \langle A \rangle \subseteq \langle A \rangle$ and $\langle A \rangle - \langle A \rangle = \langle A \rangle$. In order to have a measurement on how fast A generates the subring $\langle A \rangle$, we define

$$(2) \quad \langle A \rangle_l := \sum_l \prod_l A - \sum_l \prod_l A;$$

and so, if $0, 1 \in A$, then $\langle A \rangle = \bigcup_{l=1}^{\infty} \langle A \rangle_l$.

The remarkable *sum-product phenomenon* in a finite field, proved by Bourgain, Katz, and Tao [BKT04] (see Lemma 43), implies that: for any $\varepsilon > 0$ there is a positive integer C such that for a subset A of a finite field \mathfrak{f} , if $\log |A| \geq \varepsilon \log |\mathfrak{f}|$ and $0, 1 \in A$, then $\langle A \rangle_C = \langle A \rangle$. Based on this result, Helfgott [Hel05] proved a *product* theorem in $\mathrm{SL}_2(\mathfrak{f}_p)$ where \mathfrak{f}_p is the finite field of prime order p ; this result implies that for any $\varepsilon > 0$ there is a positive integer C such that, for any symmetric generating set A of $\mathrm{SL}_2(\mathfrak{f}_p)$ of cardinality at least $|\mathrm{SL}_2(\mathfrak{f}_p)|^\varepsilon$, we have $\prod_C A = \mathrm{SL}_2(\mathfrak{f}_p)$. One can view these results as examples of *bounded generation phenomena*. Later the product theorem had been extended to all the finite simple groups of Lie type (see [Hel11] for $\mathrm{SL}_3(\mathfrak{f}_q)$ case and either [BGT11] or [PS16] for the general case); and this product theorem implies a bounded generation result for such groups. Within the proof of the mentioned sum-product result for finite fields, the *vector space* structure of such fields had been used and it was proved that for any $\varepsilon > 0$ there is a positive integer C such that, for any subset A of a finite field \mathfrak{f} , if $\log |A| \geq \varepsilon \log |\mathfrak{f}|$, then there are $\alpha_1, \dots, \alpha_C \in \mathfrak{f}$ such that $\alpha_1 A + \cdots + \alpha_C A = \mathfrak{f}$; this result can be viewed as yet another example of bounded generation phenomena. Based on these examples, one can philosophize and vaguely formulate a *naïve* bounded generation phenomenon that says: if a finite algebraic structure \mathfrak{A} is *rich enough*, then any

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generic subset A of \mathfrak{A} of cardinality at least $|\mathfrak{A}|^\varepsilon$ generates \mathfrak{A} in C steps, where C only depends on ε . There is, however, one algebraic obstruction to the proposed bounded generation phenomenon: having a bounded generation for \mathfrak{A} passes to all of its factors; that means if $\pi : \mathfrak{A} \rightarrow \pi(\mathfrak{A})$ is a surjective homomorphism of \mathfrak{A} and A generates \mathfrak{A} in C steps, then $\pi(\mathfrak{A})$ can be generated by $\pi(A)$ in C steps as well. So one would need to have $\log |\pi(A)| \geq \varepsilon' \log |\pi(\mathfrak{A})|$ for some $\varepsilon' > 0$ independent of π . Notice that this obstruction does not show up in a finite field \mathfrak{f} or $\mathrm{SL}_2(\mathfrak{f}_p)$ as they do not have a lot of factors. One can see the subtlety of this issue already in the ring $\mathbb{Z}/p^n\mathbb{Z}$. This ring has n factors: for any $1 \leq i \leq n$, let $\pi_{p^i} : \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^i\mathbb{Z}$ be the natural quotient map. And so for a subset A of $\mathbb{Z}/p^n\mathbb{Z}$ that contains $0, 1$ and satisfies $\log |A| \geq \varepsilon \log |\mathbb{Z}/p^n\mathbb{Z}|$, one cannot expect to get $\langle A \rangle_C = \mathbb{Z}/p^n\mathbb{Z}$ for a constant C which depends only on ε unless $\log |\pi_{p^i}(A)| \geq \varepsilon' \log |\mathbb{Z}/p^i\mathbb{Z}|$ for any $1 \leq i \leq n$ and some $\varepsilon' := \varepsilon'(\varepsilon) > 0$. By fixing p and varying n , we get the following p -adic interpretation of the above mentioned case: for a subset A of the ring \mathbb{Z}_p of p -adic integers and a number $0 < \delta < 1$, let $\mathcal{N}_\delta(A)$ be the smallest positive integer n such that there are balls B_1, \dots, B_n of radius δ (with respect to the standard metric on \mathbb{Z}_p) such that $A \subseteq B_1 \cup \dots \cup B_n$. Notice that $\mathcal{N}_{p^{-i}}(A) = |\pi_{p^i}(A)|$ as any ball of radius p^{-i} is an additive coset of $p^i\mathbb{Z}_p$ in \mathbb{Z}_p ; and so $\log |\pi_{p^i}(A)| \geq \varepsilon' \log |\mathbb{Z}/p^i\mathbb{Z}|$ is equivalent to $\log \mathcal{N}_{p^{-i}}(A) / \log(p^i) \geq \varepsilon'$. On the other hand, let us recall that the lower box dimension of a subset A of a metric space is defined to be $\liminf_{\delta \rightarrow 0^+} \log \mathcal{N}_\delta(A) / \log(1/\delta)$; since for us the analysis in a given scale δ is important, we call $\log \mathcal{N}_\delta(A) / \log(1/\delta)$ the *box dimension of A at the scale δ* . Therefore, for a subset A of \mathbb{Z}_p with lower box dimension $\varepsilon > 0$, we have that, if n is a large enough integer depending on ε , then the box dimension $\log \mathcal{N}_{p^{-n}}(A) / \log(p^n)$ of A at the scale p^{-n} is at least $\varepsilon/2$. Now using a *regularization* argument (see [Bou08, Section 4] or Lemma 38), one can find a subset A' of $(A - A) \cap p^{n_0}\mathbb{Z}_p$ such that $\log \mathcal{N}_{p^{-n}}(A') / \log(p^{n-n_0}) \geq \varepsilon/4$ for any $n \geq n_0(\varepsilon)$. So after rescaling A' one can apply the proposed bounded generation and get that $\langle A \rangle_C$ contains a \mathbb{Z}_p -segment of size L where both C and L depend only on the lower box dimension ε . This type of *bounded generation* seems to be the *right* property to look for in various cases; that means if A is a subset with lower box dimension ε of an algebraic structure \mathfrak{A} which is *rich enough*, then A generates a *large substructure* \mathfrak{B} of \mathfrak{A} in C steps, where both the *largeness* of \mathfrak{B} and the positive integer C are supposed to depend only on ε . For instance Bourgain's proof (see [Bou03]) of the Katz-Tao discretized ring conjecture (see [KT01]) implies this phenomenon for the ring \mathbb{R} .

To prove a bounded generation result, using in part tools from additive combinatorics (for instance see the influential article [Gow98] and the nice book on this subject [TV06]; the method of the proof of the main theorem of [EM03] has been used in the subsequent articles on this subject, too), one often proves the weighted version; that means one starts with the probability counting measure \mathcal{P}_A on the set A and then consider the push-forward of $\mathcal{P}_A \times \dots \times \mathcal{P}_A$ under the map $f_C : \prod_{i=1}^{\Theta(C^{O(1)})} \mathfrak{A} \rightarrow \mathfrak{A}$, where f_C is given by the C step generation in \mathfrak{A} by its algebraic operations. For instance, when $\mathfrak{A} = G$ is a group, $f_C : \prod_{i=1}^{2C} G \rightarrow G$, $f_C(g_1, \dots, g_{2C}) := g_1 g_2^{-1} \dots g_{2C-1} g_{2C}^{-1}$ and we start with a set that contains the identity element; when $\mathfrak{A} = R$ is a ring, $f_C : \prod_{i=1}^C \prod_{j=1}^C R \times R \rightarrow R$, $f_C(a_{ij}, b_{ij}) := \sum_{i=1}^C \prod_{j=1}^C a_{ij} - \sum_{i=1}^C \prod_{j=1}^C b_{ij}$ and we start with a set that contains the zero and the identity elements. In this setting, the *naïve* bounded generation implies that the support of $\mu_A^{[C]\mathfrak{A}} := f_C(\mathcal{P}_A \times \dots \times \mathcal{P}_A)$ is the entire \mathfrak{A} . In the *weighted* version, one would like to show that $\mu_A^{[C]\mathfrak{A}}$ is close to the equidistribution on \mathfrak{A} ; that means $\mu_A^{[C]\mathfrak{A}}$ is close to the probability counting measure on \mathfrak{A} for some positive integer C which only depends on the *box dimension* of A . To get such a result, one often imposes additional assumptions on the set A and proves that $\mu_A^{[O(1)]}$ is *substantially more distributed* compared to \mathcal{P}_A for some positive integer $O(1)$ which only depends on \mathfrak{A} .

There are many ways to quantify how well a probability measure μ is distributed. One way is to use *Fourier analysis*; for instance in the abelian setting, an upper bound on the values of the Fourier transform $\widehat{\mu}$ of μ gives us a way to say how well μ is distributed. This means one needs to get some cancellations in certain *exponential sums* (for instance see [Cha02, Bou08, Bou05], the appendix of [BG09]). In a metric (not necessarily abelian) setting, this can be interpreted as saying that one needs to get an upper bound on $\mu * f$ where f is a function which lives in a *scale* δ (this roughly means f is almost constant in balls of radius δ^2 and almost orthogonal to the characteristic functions of balls of radius $\delta^{1/2}$) (for instance see [BG12, BISG17]). Another way of measuring how well a measure μ is distributed is using its *entropy* $H(\mu)$ (for instance

see [Rud90, Joh92, LMP99]). In this note following a work of Lindenstrauss and Varjú, we use the entropy approach to get the desired bounded generation result, which will be explained in the next section.

It should be pointed out that bounded generation results in rings, such as finite fields \mathbb{f}_q , \mathbb{R} , or \mathbb{Z}_p , have been playing an indispensable role in proving bounded generation results in groups (for instance see [Hel05, Hel11, BGT11, PS16] for the case of finite simple groups of Lie type, [BG08, BG12, dS15] for the case of compact simple Lie groups, and [BG09] for the case of $\mathrm{SL}_n(\mathbb{Z}_p)$). Using the results of this note, first a bounded generation result for semisimple p -adic analytic groups is proved in [SG17] and then in [SG] this result is extended to the case of perfect p -adic analytic groups with abelian unipotent radical. Furthermore these results are *uniform* on the prime p ; in the sense that the implied constants do not depend on p . Using such bounded generation results, the p -adic case of *super-approximation* property is proved (we refer the interested reader to the mentioned articles for the precise formulation of these results).

1.2. Main results. One of the main results of this note is the following *bounded generation* result for a characteristic zero non-Archimedean local field K ; the importance of this result is on the fact that the implied constants are *independent of the characteristic of the residue field of K* .

Theorem 1. *Suppose $0 < \varepsilon \ll 1$, d is a positive integer, and $N \gg_{d,\varepsilon} 1$ is a positive integer. Then there are $0 < \delta := \delta(\varepsilon, d)$, and positive integer $C := C(\varepsilon, d)$, such that for any finite extension K of \mathbb{Q}_p with degree $[K : \mathbb{Q}_p] \leq d$ the following holds: let \mathcal{O} be the ring of integers of K , and \mathfrak{p} be a uniformizing element of K . Suppose $A \subseteq \mathcal{O}$ such that*

$$|\pi_{\mathfrak{p}^N}(A)| \geq |\pi_{\mathfrak{p}}(\mathcal{O})|^{N\varepsilon},$$

where $\pi_{\mathfrak{p}^N} : \mathcal{O} \rightarrow \mathcal{O}/\mathfrak{p}^N\mathcal{O}$ is the canonical quotient map. Then there are positive integers N_1 and N_2 , $a \in \mathcal{O}$, and a subfield K_0 of K with ring of integers \mathcal{O}_0 such that

$$(3) \quad \begin{aligned} \lfloor N\delta \rfloor + N_1 &\leq N_2 \leq NC, && \text{(Scale and thickness)} \\ \pi_{\mathfrak{p}^{N_2}}(\mathcal{O}_0 a) &\subseteq \pi_{\mathfrak{p}^{N_2}}(\langle A \rangle_C), & v_{\mathfrak{p}}(a) &= N_1, && \text{(Bounded generation)} \\ |\pi_{\mathfrak{p}}(\mathcal{O}_0)| &\geq |\pi_{\mathfrak{p}}(\mathcal{O})|^{\varepsilon/4}. && \text{(Box dimension control)} \end{aligned}$$

where $\langle A \rangle_C$ is defined as in (2).

As there are many parameters in Theorem 1, the reader might find the following rough description of the parameters useful.

Think about ε as a lower bound for the box dimension of A at scale $|\mathfrak{p}^N|$: recall that the smallest number of balls of radius δ which cover A is denoted by $\mathcal{N}_{\delta}(A)$; and so $\mathcal{N}_{|\mathfrak{p}^N|} = |\pi_{\mathfrak{p}^N}(A)|$ and we have

$$\log \mathcal{N}_{|\mathfrak{p}^N|}(A) / \log(1/|\mathfrak{p}^N|) \geq \varepsilon.$$

Then Theorem 1 provides us a *large subring of integers* \mathcal{O}_0 and a lower bound C for the *number of steps* needed in order to get an \mathcal{O}_0 segment of *length* $|\mathfrak{p}^{N_1}|$ at the *scale* $|\mathfrak{p}^{N_2}|$; of course the significance of this statement is on the fact that the length $|\mathfrak{p}^{N_1}|$ is *much larger* than the scale $|\mathfrak{p}^{N_2}|$. For instance the inequality in (3) implies that the box dimension of $\mathfrak{p}^{N_1}\mathcal{O}_0$ at scale $|\mathfrak{p}^{N_2}|$ is at least $\varepsilon\delta/4C$ as one can see in the following equation:

$$(4) \quad \frac{\log \mathcal{N}_{|\mathfrak{p}^{N_2}|}(\mathfrak{p}^{N_1}\mathcal{O}_0)}{\log(1/|\mathfrak{p}^{N_2}|)} = \frac{\log_p |\pi_{\mathfrak{p}^{N_2-N_1}}(\mathcal{O}_0)|}{N_2[f : \mathbb{f}_p]} \geq \frac{(N_2 - N_1) \log_p |\pi_{\mathfrak{p}}(\mathcal{O}_0)|}{e(\mathfrak{p}/p)N_2 f(\mathfrak{p}/p)} = \frac{(N_2 - N_1) \log_p |\pi_{\mathfrak{p}}(\mathcal{O}_0)|}{N_2 \log_p(\pi_{\mathfrak{p}}(\mathcal{O}))} \geq \frac{\varepsilon\delta}{4C},$$

where $e(\mathfrak{p}/p)$ is the ramification index and $f(\mathfrak{p}/p)$ is the residue degree of \mathfrak{p} in the extension K/\mathbb{Q}_p (see [Ser79, Page 14]). So δ measures the *thickness* of the attained \mathcal{O}_0 -segment.

It should be pointed out that in [BG09, Proposition 3.3], a weaker form of Theorem 1 is proved where it is assumed that the extension K/\mathbb{Q}_p is *not widely ramified* and the *characteristic of its residue field* is a fixed prime; that means the implied constants depend on the characteristic of the residue field as well.

As in the proof of [BG09, Proposition 3.1], using induction on the rank, one can extend Theorem 1 from the rank 1 case to the rank d_0 , where d_0 is a *fixed* positive integer.

Corollary 2. *For any $0 < \varepsilon \ll 1$ and positive integers d_0 and d , there are $0 < \delta := \delta(\varepsilon, d_0, d)$, and positive integer $C := C(\varepsilon, d_0, d)$, such that for any finite extension K of \mathbb{Q}_p with degree $[K : \mathbb{Q}_p] \leq d$ the following holds: let \mathcal{O} be the ring of integers of K , and \mathfrak{p} be a uniformizing element of K . Suppose $A \subseteq \mathcal{O}^{d_0} := \mathcal{O} \times \cdots \times \mathcal{O}$ such that*

$$|\pi_{\mathfrak{p}^N}(A)| \geq |\mathfrak{f}|^{N\varepsilon},$$

where \mathfrak{f} is the residue field of K . Then

$$\pi_{\mathfrak{p}^{N_2}}(\mathfrak{p}^{N_1} \mathbb{Z} \mathbf{x}) \subseteq \pi_{\mathfrak{p}^{N_2}}(\langle A \rangle_C),$$

for some $\mathbf{x} \in \mathcal{O}^{d_0} \setminus \mathfrak{p} \mathcal{O}^{d_0}$, and integers N_1 and N_2 such that

$$\lfloor N\delta \rfloor + N_1 \leq N_2 \leq NC.$$

Another important corollary of Theorem 1 is its global version; this is a generalization of [Bou08, Corollary, Part I.1] where the case of $k = \mathbb{Q}$ (and $d_0 = 1$) is proved.

Corollary 3. *For any $0 < \varepsilon \ll 1$ and positive integers d and d_0 , there are $0 < \delta := \delta(\varepsilon, d, d_0)$, and positive integer $C := C(\varepsilon, d, d_0)$, such that for any finite extension k of \mathbb{Q} of degree at most d the following holds:*

Let \mathcal{O}_k be the ring of integers of k , and \mathfrak{p} be a non-zero prime ideal of \mathcal{O}_k . Suppose $A \subseteq \mathcal{O}_k^{d_0} := \mathcal{O}_k \times \cdots \times \mathcal{O}_k$ such that

$$|\pi_{\mathfrak{p}^N}(A)| \geq |\mathcal{O}_k/\mathfrak{p}|^{N\varepsilon}.$$

Then

$$\pi_{\mathfrak{p}^{N_2}}(\{i\mathbf{x} \mid i \in \mathbb{Z} \cap \mathfrak{p}^{N_1}\}) \subseteq \pi_{\mathfrak{p}^{N_2}}(\langle A \rangle_C),$$

for some $\mathbf{x} \in \mathcal{O}_k^{d_0} \setminus \mathfrak{p}^d \mathcal{O}_k^{d_0}$, and integers N_1 and N_2 such that

$$\lfloor N\delta \rfloor + N_1 \leq N_2 \leq NC.$$

Proof. For any \mathfrak{p} , let $k_{\mathfrak{p}}$ be the completion of k with respect to the \mathfrak{p} -adic topology. Let $\mathcal{O}_{\mathfrak{p}}$ be the ring of integers of $k_{\mathfrak{p}}$, and $\tilde{\mathfrak{p}}$ be a uniformizing element of $\mathcal{O}_{\mathfrak{p}}$. Then it is well-known that $\mathfrak{p} \mathcal{O}_{\mathfrak{p}} = \langle \tilde{\mathfrak{p}} \rangle$ and the embedding of \mathcal{O}_k into $\mathcal{O}_{\mathfrak{p}}$ induces an isomorphism between $\pi_{\mathfrak{p}^N}(\mathcal{O}_k)$ and $\pi_{\tilde{\mathfrak{p}}^N}(\mathcal{O}_{\mathfrak{p}})$. Now we get the desired result by Corollary 2. \square

It is not clear to the author whether the implied constants in Theorem 1 should depend on the degree or not. In this note the degree is used in a crucial way to analyze subrings of \mathcal{O} . But the implied constants in many results proved here are independent of K as it will be explained in Section 1.4. Here is one such result.

Theorem 4. *For any positive integer t , positive numbers $0 < \varepsilon_1 \ll \varepsilon_2 \ll_t 1$, $0 < \delta \ll_{\varepsilon_1} 1$, any positive integer $C \gg_{\varepsilon_1} 1$, and any finite extension K of \mathbb{Q}_p with large, depending on ε_1 , residue field \mathfrak{f} the following holds: let \mathcal{O} be the ring of integers of K , and \mathfrak{p} be a uniformizing element of K . Suppose $A \subseteq \pi_{\mathfrak{p}^N}(\mathcal{O})$ such that*

- (a) $|\pi_{\mathfrak{p}^i}(A)| \geq |\mathfrak{f}|^{i\varepsilon_1}$ for any $N\delta \leq i \leq N$.
- (b) $0, 1 \in A$ and there are $a_1, a_2 \in A$ such that $a_1 - a_2 \in \pi_{\mathfrak{p}^N}(\mathfrak{p} \mathcal{O} \setminus \mathfrak{p}^2 \mathcal{O})$.

Then either

$$\pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lfloor \varepsilon_2 N \rfloor} \mathcal{O}) \subseteq \langle A \rangle_C,$$

or

$$\pi_{\mathfrak{p}^{\lfloor t\varepsilon' N \rfloor}}(\langle A \rangle_C \cap \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lfloor \varepsilon' N \rfloor} \mathcal{O})) \text{ is a ring, and } \langle A \rangle_C \cap \mathfrak{p}^{\lfloor \varepsilon' N \rfloor} \mathcal{O} \setminus \mathfrak{p}^{\lfloor \varepsilon' N \rfloor + 1} \mathcal{O} \neq \emptyset$$

for some ε' in $[\varepsilon_2^{m(\varepsilon_1)}, \varepsilon_2]$.

The subtlety in Theorem 1 is that A might be in a smaller field. Or even if the field generated by A is the entire K , still in certain scales A might be seen as a subring of a smaller field. Condition (b) in Theorem 4 guarantees that at least the ramification index of the field generated by A is the same as K and it can be detected even in the large scale $|\mathfrak{p}|$.

Question 5. *Does Theorem 1 hold with no restriction on K ? If not, what is the least information needed on K ?*

1.3. A bounded generation conjecture for the quotients of ring of integers of a number field. As it was pointed out in Section 1.1, Theorem 1 was used in [SG17] to deduce a bounded generation statement for semisimple p -adic analytic groups, and then it was extended to perfect groups with abelian unipotent groups in [SG]; and ultimately these results were utilized to prove the p -adic case of super-approximation. If the following bounded generation for number fields holds, then one might be able to prove super-approximation for semisimple groups (or at least absolutely almost simple groups).

Conjecture 6. *Suppose $0 < \varepsilon \ll 1$, d is a positive integer, and $N_0 \gg_{d,\varepsilon} 1$ is a positive integer. Then there are $0 < \delta := \delta(\varepsilon, d)$, and positive integer $C := C(\varepsilon, d)$ such that for any number field k of degree at most d the following holds: let \mathcal{O} be the ring of integers of k . Suppose \mathfrak{a} is an ideal of \mathcal{O} such that $N(\mathfrak{a}) := |\mathcal{O}/\mathfrak{a}| \geq N_0$; and suppose $A \subseteq \mathcal{O}$ such that*

$$|\pi_{\mathfrak{a}}(A)| \geq |\pi_{\mathfrak{a}}(\mathcal{O})|^{\varepsilon}.$$

Then there are an ideal \mathfrak{a}_2 of \mathcal{O} and $a_1 \in \mathcal{O}$ such that

$$\mathfrak{a}^C \subseteq \mathfrak{a}_2, \quad N(\mathfrak{a}_2 : a_1) \geq N(\mathfrak{a})^{\delta} \text{ where } (\mathfrak{a}_2 : a_1) = \{x \in \mathcal{O} \mid xa_1 \in \mathfrak{a}_2\}, \text{ and} \\ \pi_{\mathfrak{a}_2}(\mathbb{Z}a_1) \subseteq \pi_{\mathfrak{a}_2}(\langle A \rangle_C).$$

1.4. A detailed outline of proofs of Theorems 1 and 4; and some of the auxiliary results. In this section a detailed outline of the arguments is given. Here are some of my reasons to include this admittedly long overview (1) many of the statements are fairly technical; but lots of ideas in their proofs can be useful for other problems as well. Having an overview which includes the main ideas of the auxiliary results can help readers to focus on the parts of the note that they find suitable for their purposes; (2) this can help to highlight the needed new ideas introduced here and put them in the perspective of some of the previous related works; (3) this should help the reader to get a more coherent picture of otherwise *locally disconnected* note.

Before we get to the main goal of this section, we start with recalling the setting and some of the basic properties of the ring \mathcal{O} of integers of a finite extension K of \mathbb{Q}_p .

We let \mathfrak{p} be a uniformizing element of \mathcal{O} , and \mathfrak{f} be the residue field (that means $\mathfrak{f} := \mathcal{O}/\mathfrak{p}\mathcal{O}$). For $x \in \mathcal{O}$, we let $v(x)$ be its \mathfrak{p} -adic valuation; that means $v(x)$ is a non-negative integer such that $x \in \mathfrak{p}^{v(x)}\mathcal{O} \setminus \mathfrak{p}^{v(x)+1}\mathcal{O}$. For $x \in \mathcal{O}$, we let $|x| := (1/|\mathfrak{f}|)^{v(x)}$. It is well-known that $d(x, y) := |x - y|$ defines a metric on \mathcal{O} and the ball of radius $|\mathfrak{p}^n|$ centered at 0 is $\mathfrak{p}^n\mathcal{O}$. On the algebraic side, $\{\mathfrak{p}^i\mathcal{O}\}_{i=0}^{\infty}$ is a filtration of \mathcal{O} ; that means it is a family of ideals of \mathcal{O} and $\mathfrak{p}^i\mathcal{O} \cdot \mathfrak{p}^j\mathcal{O} \subseteq \mathfrak{p}^{i+j}\mathcal{O}$. A common technique to study an algebra with a filtration is making use of the associated graded algebra. In our setting this means, we define $\text{gr}_{i,\mathfrak{p}}(\mathcal{O}) := \mathfrak{p}^i\mathcal{O}/\mathfrak{p}^{i+1}\mathcal{O}$ for any non-negative integer i , and let $\text{gr}_{\mathfrak{p}}(\mathcal{O}) := \bigoplus_{i=0}^{\infty} \text{gr}_{i,\mathfrak{p}}(\mathcal{O})$. As $\text{gr}_{i,\mathfrak{p}}(\mathcal{O})$ are abelian groups so is $\text{gr}_{\mathfrak{p}}(\mathcal{O})$. It is well-known that $\text{gr}_{\mathfrak{p}}(\mathcal{O})$ is a graded algebra with respect to the following multiplication: $(x_i + \mathfrak{p}^{i+1}\mathcal{O})(x_j + \mathfrak{p}^{j+1}\mathcal{O}) := x_i x_j + \mathfrak{p}^{i+j+1}\mathcal{O}$, for any $x_i + \mathfrak{p}^{i+1}\mathcal{O} \in \text{gr}_{i,\mathfrak{p}}(\mathcal{O})$ and $x_j + \mathfrak{p}^{j+1}\mathcal{O} \in \text{gr}_{j,\mathfrak{p}}(\mathcal{O})$. In fact, it is well-known that $\text{gr}_{\mathfrak{p}}(\mathcal{O})$ is isomorphic to the ring of polynomials $\mathfrak{f}[t]$ with coefficients in the residue field \mathfrak{f} . Based on this fact and the completeness of \mathcal{O} , we get the following description of its elements: suppose Ω is a subset of \mathcal{O} such that the quotient map $\pi_{\mathfrak{p}} : \mathcal{O} \rightarrow \mathfrak{f}$ induces a bijection from Ω to the set \mathfrak{f}^{\times} of non-zero elements of \mathfrak{f} . Then for any $X \in \mathcal{O}$ there are unique $X_i \in \Omega \cup \{0\}$ such that

$$(5) \quad X = X_0 + \mathfrak{p}X_1 + \mathfrak{p}^2X_2 + \cdots.$$

We call X_i the i -th \mathfrak{p} -adic digit with respect to Ω , and sometimes denote it by $D_{i,\Omega}(X)$ or simply $D_i(X)$ (these digits depend on the choice of Ω , but Ω will be fixed at the beginning of any proof). For instance when

$\mathcal{O} = \mathbb{Z}_p$ and $\Omega := \{1, \dots, p-1\}$, Equation (5) gives us the usual p -base description of the p -adic integers. (At some point we will be working with a different set of digits, but for now a reader can think about $\mathcal{O} = \mathbb{Z}_p$ and $\Omega = \{1, \dots, p-1\}$ in order to get a more concrete understanding of the setting). Let us notice that for any non-negative integer n and $X, Y \in \mathcal{O}$ we have that $\pi_{\mathfrak{p}^n}(X) = \pi_{\mathfrak{p}^n}(Y)$ if and only if $D_i(X) = D_i(Y)$ for any $0 \leq i \leq n-1$. So for $0 \leq i \leq n-1$, we can and will talk about *the i -th \mathfrak{p} -adic digit $D_i(\pi_{\mathfrak{p}^n}(X))$ with respect to Ω* of an element $\pi_{\mathfrak{p}^n}(X)$ of $\mathcal{O}/\mathfrak{p}^n\mathcal{O}$. Hence for any $x \in \mathcal{O}/\mathfrak{p}^n\mathcal{O}$ we have

$$x = D_0(x) + D_1(x)\mathfrak{p} + \dots + D_{n-1}(x)\mathfrak{p}^{n-1} + \mathfrak{p}^n\mathcal{O}.$$

Step 1. *Describing subrings of the ring of integers \mathcal{O} .*

Starting with a subset A of \mathcal{O} , Theorems 1 and 4 are claiming certain bounded generation phenomena within the ring R generated by A in certain scales. So it is only reasonable to start with a description of subrings of \mathcal{O} in a given scale.

Theorem 7. *Let \mathcal{O} be the ring of integers of a finite extension K of \mathbb{Q}_p . Let \mathfrak{p} be a uniformizing element of \mathcal{O} , and $\mathfrak{f} := \mathcal{O}/\mathfrak{p}\mathcal{O}$ be the residue field of K . Suppose R is a closed subring of \mathcal{O} which contains 1. Let C be an integer which is at least 3. Suppose F is an integer and $F \gg_C [K : \mathbb{Q}_p]$. Then there are integers a and b , and a subfield K_0 of K such that*

$$(6) \quad b - a \gg_{C, [K : \mathbb{Q}_p]} F, \quad \text{and} \quad F \geq b \geq Ca,$$

$$(7) \quad \pi_{\mathfrak{p}^b}(\mathcal{O}_0 \cap \mathfrak{p}^a\mathcal{O}) = \pi_{\mathfrak{p}^b}(R \cap \mathfrak{p}^a\mathcal{O}), \quad \text{and} \quad \pi_{\mathfrak{p}^b}(R) \subseteq \pi_{\mathfrak{p}^b}(\mathcal{O}_0)$$

where \mathcal{O}_0 is the ring of integers of K_0 and $\pi_y : \mathcal{O} \rightarrow \mathcal{O}/y\mathcal{O}$ is the natural quotient map for any $y \in \mathcal{O} \setminus \{0\}$.

Theorem 7 essentially says that, if we can only compute the first F -digits of the elements of the subring R , then we can find a *large* segment (proportional with F) of digits where R is the same as ring of integers of a closed subfield.

To prove Theorem 7, first we use the above mentioned philosophy, and prove the graded version (see Proposition 12). Proof of Proposition 12 is combinatorial in nature. Along the way a result for numerical semigroups is proved that might be of independent interest (see Proposition 13).

In general going to a graded structure we might lose a lot of information about the original ring. For instance, starting with a wildly ramified Galois extension K/\mathbb{Q}_p there are non-trivial elements $\sigma \in \text{Gal}(K/\mathbb{Q}_p)$ such that $\sigma(a) \equiv a \pmod{\mathfrak{p}}$ for any $a \in \mathcal{O}$. This means σ induces the trivial automorphism of $\text{gr}_{\mathfrak{p}}(\mathcal{O})$. So we can get subrings of \mathcal{O} that give us the same graded subrings of $\text{gr}_{\mathfrak{p}}(\mathcal{O})$.

Proof of Theorem 7 is a bit delicate which relies on rather well-known techniques from algebraic number theory; for instance a generalization of Hensel's lemma, Krasner's lemma, and basic facts about local fields. Along the way we get that if a closed subring R of \mathcal{O} have the same graded ring as ring of integers \mathcal{O}_0 of a closed subfield, then R is the ring of integers of a closed subfield (see Proposition 19 and Step 1 of proof of Proposition 44).

A reader who is interested in the new techniques related to sum-product results can skip the proof of these statements. These results are used only towards the end of the note in the proof of Theorem 1. But readers should familiarize themselves with the notation and basic properties introduced in Lemma 16 and Corollary 17.

Step 2. *Using conditional entropy to get a Scalar-Sum expansion.* Starting with two subsets A and B of $\pi_{\mathfrak{p}^N}(\mathcal{O})$, we would like to get a lower bound on $|A + B|$. As it was explained earlier, one often proves a *weighted version*: let \mathcal{P}_A and \mathcal{P}_B be the probability counting measures on A and B , respectively. Then $A + B$ is the support of the additive convolution $\mathcal{P}_A * \mathcal{P}_B$ of \mathcal{P}_A and \mathcal{P}_B . So if we show this new measure is *more distributed* than the initial measures, we should get a desired expansion on their supports. In this note, following [LV], we use entropy to quantify how well a measure is distributed. Let X and Y be random

variables with respect to the distribution laws \mathcal{P}_A and \mathcal{P}_B , respectively. Then it is well-known that

$$\log |A + B| \geq H(X + Y)$$

where $H(\bullet)$ is the (Shanon) entropy of the given random-variable (see Definition 23 and Lemma 24 for the definition and some of the basic properties of entropy). As it was explained above, a random variable Z with values in $\pi_{\mathfrak{p}^N}(\mathcal{O})$ can be given in terms of its \mathfrak{p} -adic digits with respect to Ω . So we get random variables $D_i(Z)$ with values in $\Omega \cup \{0\}$ for any $0 \leq i \leq N - 1$; and we have

$$(8) \quad H(Z) = H(D_0(Z), \dots, D_{N-1}(Z)).$$

By (8) and a basic property of conditional entropy (see Lemma 24), we get

$$(9) \quad H(Z) = H(D_0(Z)) + H(D_1(Z)|D_0(Z)) + \dots + H(D_{N-1}(Z)|D_0(Z), \dots, D_{N-2}(Z)).$$

Now we observe that the *carry over* method for addition works in \mathcal{O} as well; this means for $X, Y \in \mathcal{O}$, to determine the m -th \mathfrak{p} -adic digit $D_m(X + Y)$ of $X + Y$, we should add the m -th digits $D_m(X)$ and $D_m(Y)$ of X and Y , and add the *carry over* $f_\Omega(D_0(X), \dots, D_{m-1}(X), D_0(Y), \dots, D_{m-1}(Y))$ from the addition of the first $m - 1$ digits. Moreover, since $\pi_{\mathfrak{p}}$ induces a bijection between $\Omega \cup \{0\}$ and \mathfrak{f} , to find $D_m(X + Y)$ it is necessary and sufficient to find

$$(10) \quad \pi_{\mathfrak{p}}(D_m(X)) + \pi_{\mathfrak{p}}(D_m(Y)) + \pi_{\mathfrak{p}}(f_\Omega(D_0(X), \dots, D_{m-1}(X), D_0(Y), \dots, D_{m-1}(Y))).$$

In particular, the first m -th \mathfrak{p} -adic digits with respect to Ω of $X + Y$ are uniquely determined by the first m \mathfrak{p} -adic digits with respect to Ω of X and Y . Therefore for any $0 \leq m \leq N - 1$ we have

$$(11) \quad \begin{aligned} & H(D_m(X + Y)|D_0(X + Y), \dots, D_{m-1}(X + Y)) \\ & \geq H(D_m(X + Y)|D_0(X), \dots, D_{m-1}(X), D_0(Y), \dots, D_{m-1}(Y)) \\ & = H(\pi_{\mathfrak{p}}(D_m(X)) + \pi_{\mathfrak{p}}(D_m(Y)) + \pi_{\mathfrak{p}}(f_\Omega(D_0(X), \dots, D_{m-1}(X), D_0(Y), \dots, D_{m-1}(Y)))) \\ & \quad |D_0(X), \dots, D_{m-1}(X), D_0(Y), \dots, D_{m-1}(Y)) \\ & = H(\pi_{\mathfrak{p}}(D_m(X)) + \pi_{\mathfrak{p}}(D_m(Y))|D_0(X), \dots, D_{m-1}(X), D_0(Y), \dots, D_{m-1}(Y)). \end{aligned}$$

Based on (9) and (11), in order to get a lower bound on $H(X + Y)$, one needs to get a lower bound on $H(\overline{X} + \overline{Y})$ where \overline{X} and \overline{Y} are two random-variables with values in the residue field \mathfrak{f} . This is the line of thought in [LV] where they deal with the case of $\mathbb{Z}/2^N\mathbb{Z}$; and so $\mathfrak{f} = \mathbb{Z}/2\mathbb{Z}$. In that case, any distribution on $\mathfrak{f} = \mathbb{Z}/2\mathbb{Z}$ can be characterized by one value, say the probability of hitting 1. Based on this and using calculus of single variable functions a desired lower bound for $H(\overline{X} + \overline{Y})$ is attained in [LV].

When the order of the residue field can be arbitrarily large, our method should have some implications for finite fields as well. In [BKT04, Lemma 2.1] in order to prove a sum-product result for finite fields, first a scalar-sum expansion is proved; to be precise it is showed that in average the size of $|\overline{A} + \overline{\alpha}\overline{B}|$ is at least $\min\{|\overline{A}||\overline{B}|/2, |\mathfrak{f}|/10\}$, where $\overline{\alpha}$ is a random-variable with respect to the counting probability measure on \mathfrak{f}^\times . So it seems the following question to be the right property to seek.

Question 8. *Let \mathfrak{f} be a finite field. Suppose \overline{X} , \overline{Y} , $\overline{\alpha}$, and $\overline{Z}_{\mathfrak{f}}$ are random variables with values in \mathfrak{f} ; $\overline{\alpha}$ is distributed with respect to the probability counting measure on the set \mathfrak{f}^\times of non-zero elements of \mathfrak{f} , and $\overline{Z}_{\mathfrak{f}}$ is distributed with respect to the counting probability measure on \mathfrak{f} . Is there a (fixed universal) positive number c such that*

$$H(\overline{X} + \overline{\alpha}\overline{Y}|\overline{\alpha}) \geq \min\{H(\overline{X}) + H(\overline{Y}), H(\overline{Z}_{\mathfrak{f}})\} - c?$$

It is worth pointing out that we know $H(\overline{Z}_{\mathfrak{f}})$ is $\log |\mathfrak{f}|$ and $\overline{Z}_{\mathfrak{f}}$ has the maximum entropy among all the random variables with values in \mathfrak{f} ; in particular $H(\overline{X} + \overline{\alpha}\overline{Y}|\overline{\alpha}) \leq H(\overline{Z}_{\mathfrak{f}})$ (in the setting of Question 8). For now, we do not know the answer to Question 8 for arbitrary random variables \overline{X} and \overline{Y} ; but Lemma 28

implies an affirmative answer to this question when \overline{X} and \overline{Y} are distributed according to the probability counting measures $\mathcal{P}_{\overline{A}}$ and $\mathcal{P}_{\overline{B}}$, respectively. More precisely, Proposition 27 states

$$(12) \quad H(\overline{X} + \overline{\alpha Y} | \overline{\alpha}) \geq -\log \left(\frac{1}{|\overline{A}||\overline{B}|} + \frac{1}{|\mathfrak{f}|} \right) \geq \min\{\log |\overline{A}| + \log |\overline{B}|, \log |\mathfrak{f}|\} - \log 2$$

where \overline{A} and \overline{B} are subsets of \mathfrak{f} , and \overline{X} and \overline{Y} are distributed according to the probability counting measures $\mathcal{P}_{\overline{A}}$ and $\mathcal{P}_{\overline{B}}$, respectively.

In order to be able to use (12) in the \mathfrak{p} -adic setting via (9) and (11), we need to start with *regular* subsets A and B of $\pi_{\mathfrak{p}^N}(\mathcal{O})$ (see Definition 21). Basically a subset A of $\pi_{\mathfrak{p}^N}(\mathcal{O})$ is an (m_0, \dots, m_{N-1}) -regular subset, if for a random-variable X according to the probability counting measure on A and a given first i \mathfrak{p} -adic digits w_0, \dots, w_{i-1} of an element of A , the conditional distribution

$$P(D_i(X) | D_0(X) = w_0, \dots, D_{i-1}(X) = w_{i-1})$$

is the probability counting measure on a set of cardinality m_i . So for an (m_0, \dots, m_{N-1}) -regular subset A , an (l_0, \dots, l_{N-1}) -regular subset B , a random-variable X according to the probability counting measure \mathcal{P}_A , and a random-variable Y according to the probability counting measure \mathcal{P}_B , we have

$$\begin{aligned} H(X + \alpha Y | \alpha) &\geq \sum_{i=0}^{N-1} H(\pi_{\mathfrak{p}}(D_i(X)) + \pi_{\mathfrak{p}}(\alpha) \pi_{\mathfrak{p}}(D_i(Y)) | \alpha, D_j(X), D_j(Y) \text{ for } 0 \leq j \leq i-1) \\ &\geq \sum_{i=0}^{N-1} -\log \left(\frac{1}{m_i l_i} + \frac{1}{|\mathfrak{f}|} \right), \end{aligned}$$

where α is a random-variable according to the probability counting measure on $\pi_{\mathfrak{p}^N}(\Omega)$. And this implies

$$(13) \quad \max_{w \in \Omega} |A + \pi_{\mathfrak{p}^N}(w)B| \geq \prod_{i=0}^{N-1} \left(\frac{1}{m_i l_i} + \frac{1}{|\mathfrak{f}|} \right)^{-1},$$

which is our desired Scalar-Sum expansion (see Proposition 20). Roughly this inequality says that, if we do not get a meaningful Scalar-Sum expansion, the reason is that at any level i either both $\log m_i$ and $\log l_i$ are close to $\log |\mathfrak{f}|$ or both $\log m_i$ and $\log l_i$ are close to 0.

Step 3. *Following Lindenstrauss-Varjú's treatment to get a Scalar-Sum-Product expansion for a regular set.* In this step, we prove that for some $a \in A - A$ and $w \in \Omega$ the set $A + \pi_{\mathfrak{p}^N}(w)aA$ is significantly larger than A where A is a *regular* subset of $\pi_{\mathfrak{p}^N}(\mathcal{O})$ with three other conditions. The key observation behind this step is the fact that, if A is an (m_0, \dots, m_{N-1}) -regular subset of $\pi_{\mathfrak{p}^N}(\mathcal{O})$, then, for any $x \in \mathcal{O}$, $\pi_{\mathfrak{p}^N}(x)A$ is a $(1, \dots, 1, m_0, \dots, m_{N-1-v(x)})$ -regular subset where $v(x)$ is the \mathfrak{p} -adic valuation of x . This implies that, for any $i \in B := \{j \in [0, N-1] \mid m_j > 1\}$, there is $a \in A - A$ such that aA is a $(1, \dots, 1, m_0, \dots, m_{N-1-i})$ -regular subset. Let $T := \{j \in [0, N-1] \mid \log m_j / \log |\mathfrak{f}| \geq 1/2\}$. Now applying the Scalar-Sum expansion proved in the second step (see the inequality in (13)) for the regular sets A and aA for any $a \in A - A$, we get that either $|A + \pi_{\mathfrak{p}^N}(w)aA|$ is significantly larger than $|A|$ for some $w \in \Omega$ and $a \in A - A$, or T is almost invariant under the shifts by elements of B . Then assuming that A has at least box dimension ε for any scale smaller than $(1/|\mathfrak{f}|)^{O(\varepsilon^4 N)}$ and $0, 1 \in B$, we deduce that a shift of B has Schnirlmann density (see Definition 31) at least ε ; and then by a theorem of Mann (see Theorem 32) we reach to a contradiction.

Let us emphasize that the key point of the argument is where we say *the set T of indexes where m_i is at least $\sqrt{|\mathfrak{f}|}$ is almost invariant under shifts by integers j such that $m_j > 1$* (see Lemma 34); and we deduce this claim using the inequality in (13) for the regular sets A and aA for suitable $a \in A - A$.

Another remark is that the crucial condition $m_1 > 1$ is why we get a result with no dependence on K .

Step 4. *Proving a Scalar-Sum-Product expansion: removing the regularity assumption.* Finally at this step we get a satisfactory expansion result:

Theorem 9 (Scalar-Sum-Product expansion). *For any $\varepsilon > 0$, $0 < \delta \ll \varepsilon^5$, and any finite extension K of \mathbb{Q}_p with large, depending on ε , residue field \mathfrak{f} the following holds:*

Let \mathcal{O} be the ring of integers of K , and \mathfrak{p} be a uniformizing element of K . Let $\Omega \subseteq \mathcal{O}$, and suppose $\pi_{\mathfrak{p}}$ induces a bijection between $\Omega \subseteq \mathcal{O}$ and \mathfrak{f}^\times . Suppose $A \subseteq \pi_{\mathfrak{p}^N}(\mathcal{O})$ such that

- (a) $|A| \leq |\mathfrak{f}|^{N(1-\varepsilon)}$,
- (b) $|\pi_{\mathfrak{p}^i}(A)| \geq |\mathfrak{f}|^{i\varepsilon}$ for any $N\delta \leq i \leq N$.
- (c) *there are $a_{01}, a_{02}, a_{11}, a_{12} \in A$ such that $a_{01} - a_{02} \in \pi_{\mathfrak{p}^N}(\mathcal{O} \setminus \mathfrak{p}\mathcal{O})$, $a_{11} - a_{12} \in \pi_{\mathfrak{p}^N}(\mathfrak{p}\mathcal{O} \setminus \mathfrak{p}^2\mathcal{O})$.*

Then

$$\max_{\omega \in \Omega} |\langle A \rangle_{\mathfrak{p}} + \pi_{\mathfrak{p}^N}(\omega) \langle A \rangle_{\mathfrak{p}}| \geq |A| |\mathfrak{f}|^{N\delta}.$$

Condition (a) allows us to have enough space to expand. Condition (b) says that the box dimension of the lift $\pi_{\mathfrak{p}^N}^{-1}(A) \subseteq \mathcal{O}$ of A at any scale smaller than $(1/|\mathfrak{f}|)^{N\delta}$ is at least ε . This is a rather (needed) technical assumption which will be eventually removed; but removing this condition results to having a weaker conclusion, which has a meaning only in \mathcal{O} and not in $\pi_{\mathfrak{p}^N}(\mathcal{O})$. Condition (c) tells us something about the valuation of elements of $A - A$: it is equivalent to say that there are $a_0, a_1 \in A - A$ such that $v(a_0) = 0$ and $v(a_1) = 1$. As before this crucial condition helps us get a result that works with no dependency on the field K .

The key idea is a *regularization process* that has been used in most of the previous works related to either a sum-product or a product result in a *multi-scaled space* (for instance see [BG09, Bou03, Bou08] or [SG17, Section 2.2]). In this process, we construct a rooted regular tree with N levels; the vertices in the i -th row are elements of $\pi_{\mathfrak{p}^i}(\mathcal{O})$, and the parent of $\pi_{\mathfrak{p}^i}(x)$ is $\pi_{\mathfrak{p}^{i-1}}(x)$ for $1 \leq i \leq N$. We view A as a subset of the vertices at the N -th level, and consider the rooted sub-tree induced by A . Through this process, each time we choose a subset A_i of A such that first $|A_i| \geq |A|/(\log |\mathfrak{f}|)^i$ and second the last i -th levels of the rooted sub-tree induced by A_i are *regular*; that means there are equal number of paths from any vertex at the $(N - i)$ -th level to the N -th level. After obtaining this regular *large* subset and changing it a little bit, we apply the Scalar-Sum-Product expansion for regular sets and deduce the desired result.

Step 5. *Proving a bounded generation result.* At this step we prove:

Theorem 10. *For any $0 < \varepsilon_1 \ll \varepsilon_2 \ll 1$, a positive integer m , $0 < \delta \ll_{m, \varepsilon_1} 1$, positive integers $1 \ll_{m, \varepsilon_1} C$ (number of needed sum-product) and $1 \ll_{\varepsilon_1} k$ (number of needed scalars) and any finite extension K of \mathbb{Q}_p with large, depending on ε_1 , residue field \mathfrak{f} the following holds:*

Let \mathcal{O} be the ring of integers of K , and \mathfrak{p} be a uniformizing element of K . Let $\Omega \subseteq \mathcal{O}$, and suppose $\pi_{\mathfrak{p}}$ induces a bijection between $\Omega \subseteq \mathcal{O}$ and \mathfrak{f}^\times . Suppose $A \subseteq \pi_{\mathfrak{p}^N}(\mathcal{O})$ such that

- (a) $|\pi_{\mathfrak{p}^i}(A)| \geq |\mathfrak{f}|^{i\varepsilon_1}$ for any $N\delta \leq i \leq N$.
- (b) *there are $a_{01}, a_{02}, a_{11}, a_{12} \in A$ such that $a_{i1} - a_{i2} \in \pi_{\mathfrak{p}^N}(\mathfrak{p}^i\mathcal{O} \setminus \mathfrak{p}^{i+1}\mathcal{O})$.*

Then

$$(14) \quad \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon_2^m N \rceil} \mathcal{O}) \subseteq \langle A \rangle_C + \pi_{\mathfrak{p}^N}(\omega_1) \langle A \rangle_C + \cdots + \pi_{\mathfrak{p}^N}(\omega_k) \langle A \rangle_C,$$

for some $\omega_i \in \prod_k (\Omega \cup \{1\})$.

The conditions (a) and (b) are similar to the technical conditions (b) and (c) of Theorem 9. As here we are seeking a *bounded generation* result and not an *expansion* result, no upper bound on $|A|$ is needed (see condition (a) of Theorem 9).

A quick explanation of the parameters involved in Theorem 10: ε_1 gives us a lower bound for the box dimension of $\pi_{\mathfrak{p}^N}^{-1}(A)$ at scales smaller than $(1/|\mathfrak{f}|)^{N\delta}$; so a smaller δ imposes more conditions on A ; the *thickness* of the \mathcal{O} -segment generated in C steps sum-product and k steps scalar-sum is roughly $(1/|\mathfrak{f}|)^{\varepsilon_2^m N}$;

so for a smaller ε_2 and a larger m we get a thicker \mathcal{O} -segment. An important point to raise is that the number k of the needed scalar-sum steps is *independent* of m . This is crucial when we want to get a bounded generation result using only sum and product.

To prove Theorem 10, first we show the case of $m = 1$ (see Proposition 39). To show this case, we use the Scalar-Sum-Product expansion result, Theorem 9, repeatedly to get a subset of $\pi_{\mathfrak{p}}(\mathcal{O})$ with arbitrarily large *box dimension*; that means to get a subset A' such that $|A'| \geq |\pi_{\mathfrak{p}}(\mathcal{O})|^{1-\delta}$ for a small fixed positive number δ . Then we get the desired bounded generation result using Fourier analysis. This is a common feature of most of the proofs on this type of results; for instance this part of a bounded generation result in groups is usually done by proving a kind of *mixing property* (see Sarnak-Xue [SX91] and Gower's notion of quasi-randomness [Gow08]).

In the second step, we appeal to the associated graded algebra $\text{gr}_{\mathfrak{p}}(\mathcal{O}) := \bigoplus_{i=0}^{\infty} \mathfrak{p}^i \mathcal{O} / \mathfrak{p}^{i+1} \mathcal{O}$ (as we have pointed out earlier this algebra is isomorphic to $\mathfrak{f}[t]$) in order to gain more information on $\langle A \rangle_C$ for some integer $C := C(\varepsilon)$. To be more precise, to any subset \tilde{A} of \mathcal{O} and any non-negative integer i , we associate *the i -th grade* $\text{gr}_{i,\mathfrak{p}}(\tilde{A}; \mathcal{O}) := ((\tilde{A} \cap \mathfrak{p}^i \mathcal{O}) + \mathfrak{p}^{i+1} \mathcal{O}) / \mathfrak{p}^{i+1} \mathcal{O}$ of \tilde{A} (and of course this can be done for any subset A of $\pi_{\mathfrak{p}^N}(\mathcal{O})$ as well). And for $A \subseteq \pi_{\mathfrak{p}^N}(\mathcal{O})$ we let $J(A) := \{i \in [0, N-1] \mid \text{gr}_i(A) \neq 0\}$. Notice that

$$J(A) = \{v(x) \mid \pi_{\mathfrak{p}^N}(x) \in A\} \cap [0, N-1]$$

where v is the \mathfrak{p} -adic valuation, and condition (b) of Theorem 10 is equivalent to saying $0, 1$ are in $J(A - A)$. Let us also observe that the graded algebra structure of $\text{gr}_{\mathfrak{p}}(\mathcal{O})$ implies $J(A_1 A_2) \supseteq J(A_1) + J(A_2)$ for any two subsets A_1 and A_2 of \mathcal{O} . Having these in mind, using Lemma 33 (which is based on the Mann theorem on sets with positive Schnirlmann density) we deduce that $J(\langle A \rangle_{3\lceil 1/\varepsilon_1 \rceil}) \supseteq (\lceil 1/\varepsilon_1 \rceil \delta N, N) \cap \mathbb{Z}$ if A satisfies properties (a) and (b) of Theorem 10.

Finally to prove Theorem 10, we use the $m = 1$ case for the parameters $\varepsilon_1^{\text{new}} := \varepsilon_1$, $\varepsilon_2^{\text{new}} := \varepsilon_2$, $N^{\text{new}} := \lceil \varepsilon_2^{m-1} N \rceil$, and $A^{\text{new}} := \pi_{\mathfrak{p}^{N^{\text{new}}}}(A)$, in order to get a scalar-sum-product set \hat{A} such that $\pi_{\mathfrak{p}^{N^{\text{new}}}}(\hat{A})$ contains $\pi_{\mathfrak{p}^{\lceil \varepsilon_2^{m-1} N \rceil}}(\mathfrak{p}^{\lceil \varepsilon_2^m N \rceil} \mathcal{O})$. Now using the sum-product set $\langle A \rangle_{3\lceil 1/\varepsilon_1 \rceil}$ we shift \hat{A} at most $1/\varepsilon_2$ many times and add them in order to fill out the entire \mathcal{O} -segment $\pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon_2^m \rceil} \mathcal{O})$ *without introducing a new scaling parameter*. This is crucial as later we need to get rid of the used scalars; and each time we reduce the number of used scalars, it comes with a *cost* on the thickness of \mathcal{O} -segment. So we need to start with a *thick* enough \mathcal{O} -segment at the beginning of the process.

Step 6. *A multi-scaled version of the Bourgain-Katz-Tao argument and proof of Theorem 4* (see [BKT04, Proofs of Lemma 4.2 and Theorem 4.3]). Roughly the following steps were employed in [BKT04] to prove a bounded generation result in a finite field \mathfrak{f} (see [EM03] where a similar approach is used to prove Erdős-Volkmann's ring conjecture):

- (a) (Scalar-Sum bounded generation) There is a linear function $l : \mathfrak{f}^k \rightarrow \mathfrak{f}, l(x_1, \dots, x_k) := \sum_{i=1}^k \alpha_i x_i$ such that $l(A^k) = \mathfrak{f}$ (here the number k of the needed scalars depends on $\log |A| / \log |\mathfrak{f}|$);
- (b) (Reducing the number of involved scalars) We have either **(injectivity)** l is injective on $A \times \dots \times A$ or **(reduction)** there is a linear function $l' : \mathfrak{f}^{k-1} \rightarrow \mathfrak{f}$ with one less variable such that $l'(\langle A \rangle_2^{k-1}) = \mathfrak{f}$;
- (c) (Analyzing the injectivity case) If a linear function $l : \mathfrak{f}^k \rightarrow \mathfrak{f}$ is injective on A^k and $l(A^k) = \mathfrak{f}$, then A is a subfield of \mathfrak{f} .

(Here A is a subset of \mathfrak{f} with cardinality at least $|\mathfrak{f}|^\varepsilon$ and $0, 1 \in A$. See Lemma 43 for the details and the precise statement.) For the purposes of this note, we need a *multi-scaled* version of these steps. That means we will be needing the injectivity of a linear map on a *large neighborhood* of a given set in order to be able to deduce existence of some algebraic structure on it. In the algebraic language the difficulty arises as we have lots of nilpotent elements in $\pi_{\mathfrak{p}^N}(\mathcal{O})$, but in a field \mathfrak{f} any non-zero element is invertible. Here we briefly explain how we overcome this difficulty.

Choice of the set of digits $\Omega \cup \{0\}$. So far we have been working with an arbitrary set Ω of representatives in \mathcal{O} of the non-zero elements of the residue field; and the needed scalars for the scalar-sum-product bounded generation were picked from $\prod_C \Omega \cup \{1\}$. At this step in order to have a slightly neater version of the process, we assume that Ω is a subgroup of the group of units of \mathcal{O} ; using the Hensel lemma we know that there is a subgroup Ω of the group of units of \mathcal{O} such that $\pi_p : \Omega \rightarrow \mathfrak{f}^\times$ is a group isomorphism. So we can and will assume $\prod_C \Omega \cup \{1\} = \Omega$ for any positive integer C , and more importantly the inverse of an element of Ω is again in Ω .

The main dichotomy. To explain this part, we introduce the symbol $\text{BG}(A; \varepsilon, k, C)$; for a subset A of $\pi_{p^N}(\mathcal{O})$, a positive number ε , and positive integers k, C , we say $\text{BG}(A; \varepsilon, k, C)$ holds if

$$(15) \quad \pi_{p^N}(\mathfrak{p}^{\lceil \varepsilon N \rceil} \mathcal{O}) \subseteq \langle A \rangle_C + \pi_{p^N}(\alpha_1) \langle A \rangle_C + \cdots + \pi_{p^N}(\alpha_k) \langle A \rangle_C$$

for some $\alpha_1, \dots, \alpha_k \in \Omega$. So roughly $\text{BG}(A; \varepsilon, k, C)$ is a compact way of saying that an \mathcal{O} -segment, whose box dimension at scale $|p^N|$ is at least $1 - \varepsilon$, can be generated by A in C steps sum-product, and $k + 1$ steps scalar-sum with scalars in Ω . Let us observe that for $0 < \varepsilon_1 \ll \varepsilon_2 \ll 1$ and a positive integer m (under certain conditions on the set A), by Theorem 10, $\text{BG}(A; \varepsilon_2^m, k(\varepsilon_1), C(\varepsilon_1, m))$ holds. This will be serving us as the *initial seed* of a process similar to the explained Bourgain-Katz-Tao argument.

Assuming $\text{BG}(A; \varepsilon, k, C)$ holds, for any $\delta_0 > 0$, we prove (see Claim 1 in the proof of Lemma 42) that either

- (a) (δ_0 -**injectivity**) for any $\mathbf{x}, \mathbf{x}' \in \langle A \rangle_{2C}^{k+1}$, $l(\mathbf{x}) = l(\mathbf{x}_0)$ implies $\mathbf{x} - \mathbf{x}' \in \pi_{p^N}(\mathfrak{p}^{\lfloor \delta_0 N \rfloor})$, where $l(x_0, \dots, x_k) = x_0 + \pi_{p^N}(\alpha_1)x_1 + \cdots + \pi_{p^N}(\alpha_k)x_k$ and α_i 's satisfy (15), or
- (b) (**Reduction**) $\text{BG}(A; \varepsilon + \delta_0, k - 1, 8C)$ holds.

Analyzing the case where the reduction fails. Suppose $\delta_0 > \varepsilon$, $\text{BG}(A; \varepsilon, k, C)$ holds for the sequence $\alpha_1, \dots, \alpha_k$ of scales in Ω , and we have the δ_0 -injectivity for $\langle A \rangle_{2C}$ and $l(x_0, \dots, x_k) := x_0 + \pi_{p^N}(\alpha_1)x_1 + \cdots + \pi_{p^N}(\alpha_k)x_k$; then we prove (see Lemma 42) that

$$(16) \quad \pi_{p^{\lfloor \delta_0 N \rfloor}} \left(\langle A \rangle_C \cap \pi_{p^N}(\mathfrak{p}^{\lceil \varepsilon N \rceil} \mathcal{O}) \right)$$

is a subring of $\pi_{p^{\lfloor \delta_0 N \rfloor}}(\mathcal{O})$.

Gaining an algebraic structure without using scalars. We use the conclusion of Theorem 10 as the *initial seed* for using the main dichotomy. That means $\text{BG}(A; \varepsilon_2^m, k(\varepsilon_1), C(\varepsilon_1, m))$ holds for $0 < \varepsilon_1 \ll \varepsilon_2 \ll 1$ and a positive integer m (under certain assumptions on A). Then using the main dichotomy we reduce the number of needed scalars. But each time the main dichotomy is used, assuming the reduction occurs, the number of needed scalars is reduced in the cost of getting a *smaller* \mathcal{O} -segment. So the next time we should pick a larger scale δ_0 for using the main dichotomy. This shows how important it is to know that the number k of needed scalars only depends on ε_1 and it is *independent* of m . Now by choosing m large enough depending only on ε_1 and choosing the scalars carefully, we get (see the proof of Lemma 42) that

$$\pi_{p^{\lfloor t\varepsilon' N \rfloor}} \left(\langle A \rangle_{C'} \cap \pi_{p^N}(\mathfrak{p}^{\lceil \varepsilon' N \rceil} \mathcal{O}) \right)$$

is a ring for some $\varepsilon_2^m \leq \varepsilon' \leq \varepsilon_2$ and $C' := C'(\varepsilon_1)$ (where t is a given fixed integer).

Controlling the gap of indexes of non-zero grades, and finishing the proof of Theorem 4. Another application of the associated graded algebra and Mann's theorem as in Step 4 helps us finish the proof of Theorem 4.

Step 7. *Proof of Theorem 1.* The main shortcoming of Theorem 4 is on the assumption that there is an element $a \in A - A$ whose p -adic valuation is 1. In fact, the set A might be in a smaller field, or the ring R generated by A might behave as the ring of integers of different subfields in different scales.

Having a description of closed subrings of \mathcal{O} in hand, using a corollary of Theorem 10, we follow scheme of Bourgain's proof in [BG09, Section A.5]. It is worth repeating that Bourgain had assumed p is a fixed prime and K/\mathbb{Q}_p is not widely ramified; and both of these assumptions were utilized for understanding structure of certain subrings of \mathcal{O} .

Assuming the ring R generated by A is \mathcal{O} . Our starting point is where there is no complication on the ring R . The following is an immediate corollary of Theorem 10, which is a uniform version (in the sense that the implied constants do not depend on the field K) of [BG09, Corollary A.1].

Corollary 11. *For any $0 < \varepsilon_1 \ll \varepsilon_2 \ll 1$, $0 < \delta \ll_{\varepsilon_1} 1$, and positive integer $1 \ll_{\varepsilon_1} C$, and any finite extension K of \mathbb{Q}_p with large, depending on ε_1 , residue field \mathfrak{f} the following holds:*

Let \mathcal{O} be the ring of integers of K , and \mathfrak{p} be a uniformizing element of K . Suppose $A \subseteq \pi_{\mathfrak{p}^N}(\mathcal{O})$ such that

- (a) $|\pi_{\mathfrak{p}^i}(A)| \geq |\mathfrak{f}|^{i\varepsilon_1}$ for any $N\delta \leq i \leq N$.
- (b) $\pi_{\mathfrak{p}^{e'}}(A) = \pi_{\mathfrak{p}^{e'}}(\mathcal{O})$, where $e' = 1$ if K is an unramified extension, and $e' = 2$ otherwise.

Then

$$\pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon_2 N \rceil} \mathcal{O}) \subseteq \sum_{C_1} \prod_{C_2} A - \sum_{C_1} \prod_{C_2} A,$$

for some integers $C_1, C_2 \leq C$.

Assuming the first N grades of R with respect to powers of p have equal cardinality. In Proposition 44, we consider the case where $|\text{gr}_{0,p}(R; \mathcal{O})| = \cdots = |\text{gr}_{N-1,p}(R; \mathcal{O})|$. Using the description of closed subrings of \mathcal{O} , we deduce that $\pi_{\mathfrak{p}^{N-4}}(R) = \pi_{\mathfrak{p}^{N-4}}(\mathcal{O}_0)$ where \mathcal{O}_0 is the ring of integers of a closed subfield if N is large compared to the degree of the field extension K/\mathbb{Q}_p . Then we use Corollary 11 to get the desired bounded generation. For small N , Bourgain-Katz-Tao's sum-product result for finite fields is used.

Bourgain's technique of detecting mutations. In Lemma 45, under the assumption that the box dimension of A is at least ε at the scale $|\mathfrak{p}^i|$ for any integer i in $[0, N-1]$, we prove a bounded generation result. It is clear that the cardinality of the grades $\text{gr}_{i,p}(R; \mathcal{O})$ of the ring R generated by A with respect to powers of p is a non-decreasing sequence; and it has at most $[K : \mathbb{Q}_p]$ many jumps. So at least one of the ranges where equality of grades occurs is *large*. And one would hope to zoom in this portion and use the previous step. However there are two important issues: (1) we need to rescale a subset of A in order to zoom in to the equal grade portion; and this changes the ring; (2) it is not clear why we can get any element at the particular grade where the mutation occurred only in a *bounded* number of steps. For these reasons, we recursively define a sequence of quadruples (A_i, R_i, n_i, C_i) where R_i is the ring generated by A_i ; the first n_i grades of R_i have the same cardinality; in C_i steps A_i generates a *large* ideal of R_i at the scale $|p^{n_i}|$; $\pi_{\mathfrak{p}}(A_i) = \pi_{\mathfrak{p}}(R_i)$; and $|\pi_{\mathfrak{p}}(R_i)|$ is getting larger (see Step 0 of the proof of Lemma 45). Having this sequence, we get the needed bounded generation result.

Finishing proof of Theorem 1. For small residue fields, we use Bourgain's result. We notice that Bourgain had assumed that p is a fixed *large* prime; but in the proof the largeness of p is only used to ensure the extension K/\mathbb{Q}_p is not widely ramified. This, in turn, is only used to understand closed subrings of \mathcal{O} . So results of Section 2 of this note automatically extends Bourgain's result to any fixed prime p . For large residue fields, another application of the regularization technique gives us a set where we can use the previous step and deduce the desired result.

1.5. Notation. In this note, K is a finite extension of \mathbb{Q}_p , \mathcal{O} is its ring of integers, $\mathfrak{p}|p$ is a uniformizing element, $\mathfrak{f} := \mathcal{O}/\langle \mathfrak{p} \rangle$ is its residue field, and e is the ramification index of K over \mathbb{Q}_p , i.e. $\langle p \rangle = \langle \mathfrak{p}^e \rangle$. For any ring R and $a \in R$, $\pi_a : R \rightarrow R/\langle a \rangle$ is the canonical quotient map $\pi_a(x) := x + \langle a \rangle$.

We use the usual Vinogradov notation: $x \gg y$ means that there is a universal positive constant c such that $x \geq cy$, and $x \gg_{z_1, z_2} y$ means that there is a positive function $c(z_1, z_2)$ of z such that $x \geq c(z_1, z_2)y$.

For a subset A of a ring R and a positive integer C , let $\sum_C A := \{\sum_{i=1}^C a_i \mid a_i \in A\}$, $\prod_C A := \{\prod_{i=1}^C a_i \mid a_i \in A\}$, and $\langle A \rangle_C := \{\sum_{i=1}^C \prod_{j=1}^C a_{ij} - \sum_{i=1}^C \prod_{j=1}^C a'_{ij} \mid a_{ij}, a'_{ij} \in A\}$; and $\langle A \rangle$ is the subring generated by A .

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2. STRUCTURE OF SUBRINGS

In this section we describe structure of subrings of the ring \mathcal{O} of integers of a finite extension K of \mathbb{Q}_p (see Theorem 7). To do so first graded subrings of the ring of polynomials over a finite field are described (see Proposition 12). Along the way a result for semigroups of non-negative integers is proved that is of independent interest (see Proposition 13). It should be said that Theorem 7 will be used only towards the end of the article in the proof of Theorem 1. But the technique of assigning a graded set to a subset of \mathcal{O} in order to understand the shape of this set in various scales is well illustrated in this section. One can take Theorem 7 as a blackbox and skip this section; but the basic properties proved and notation introduced in Lemma 16 and Corollary 17 should be reviewed.

2.1. Structure of graded subrings of the ring of polynomials over a finite field. The main goal of this section is to prove the following proposition.

Proposition 12. *Let \mathfrak{f} be a finite field, and $S := \bigoplus_{i=0}^{\infty} S_i t^i$ be a graded subring of the ring of polynomials $\mathfrak{f}[t]$ over the field \mathfrak{f} ; that means S_i is an additive subgroup of \mathfrak{f} and $S_i S_j \subseteq S_{i+j}$, for any i, j ; in particular S_0 is a subfield of \mathfrak{f} . Suppose*

$$(17) \quad [\mathfrak{f} : S_0] := d \quad \text{and} \quad S_m \neq 0.$$

For any integer $C \geq 3$ and any integer $F \gg_{C,m,d} 1$, there are integers r, a , and b , an element $\lambda \in \mathfrak{f}$, and a subfield \mathfrak{f}_0 of \mathfrak{f} such that $r|m$, $S_0 \subseteq \mathfrak{f}_0$,

$$(18) \quad b - a \gg_{C,d,m} F, \text{ and } b \geq Ca,$$

$$(19) \quad \pi_{t^{bm}}(S \cap t^{am}\mathfrak{f}[t]) = \pi_{t^{bm}}(t^{am}\mathfrak{f}_0[\lambda t^r])$$

where $\pi_{t^k} : \mathfrak{f}[t] \rightarrow \mathfrak{f}[t]/t^k\mathfrak{f}[t]$ is the canonical quotient map and $\mathfrak{f}_0[\lambda t^r] := \bigoplus_{i=0}^{\infty} \mathfrak{f}_0(\lambda t^r)^i$.

Proposition 12 has an immediate implication for numerical semigroups that is of independent interest.

Proposition 13. *Let J be a subsemigroup of non-negative integers. Suppose the greatest common divisor of elements of J is one (such a semigroup is called a numerical semigroup). Let $m := \min J \setminus \{0\}$ (it is called the multiplicity of J), and let $F := \max \mathbb{Z} \setminus J$ (it is called the Frobenius number of J). Suppose $C \geq 3$ and $F \gg_{C,m} 1$. Then there are integers a, b , and d , such that $d|m$, $d > 1$,*

$$(20) \quad b - a \gg_{C,m} F, \text{ and } b \geq Ca,$$

$$(21) \quad J \cap [a, b) = d\mathbb{Z} \cap [a, b).$$

Proof. Let $S := \bigoplus_{j \in J} \mathfrak{f}(t^j)$. Since J is a subsemigroup of non-negative integers, S is a graded subring of $\mathfrak{f}[t]$. Since $S_0 = \mathfrak{f}$, by Proposition 12, there are integers $d := r$, a' , and b' such that $d|m$

$$(22) \quad b' - a' \gg_{C,m} F, b' \geq Ca', \text{ and } \pi_{t^{b'm}}(S \cap t^{a'm} \mathfrak{f}[t]) = \pi_{t^{b'm}}(t^{a'm} \mathfrak{f}[t^d]).$$

Since $\pi_{t^{b'm}}(S \cap t^{a'm} \mathfrak{f}[t]) = \pi_{t^{b'm}}(\bigoplus_{j \in J \cap [a'm, b'm)} \mathfrak{f}(t^j))$, by (22) we deduce that

$$J \cap [a'm, b'm) = d\mathbb{Z} \cap [a'm, b'm);$$

and the claim follows. \square

The following picture shows us a numerical semigroup J with the multiplicity $m(J) = 6$ and the Frobenius number $F(J) = 133$. We can see that the set of non-negative integers can be covered with at most $m(J)$ windows with different *patterns*. And one of the *large* windows is going to have a *regular strip* pattern.

0	1	2	3	4	5
6	7	8	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23
24	25	26	27	28	29
30	31	32	33	34	35
36	37	38	39	40	41
42	43	44	45	46	47
48	49	50	51	52	53
54	55	56	57	58	59
60	61	62	63	64	65
66	67	68	69	70	71
72	73	74	75	76	77
78	79	80	81	82	83
84	85	86	87	88	89
90	91	92	93	94	95
96	97	98	99	100	101
102	103	104	105	106	107
108	109	110	111	112	113
114	115	116	117	118	119
120	121	122	123	124	125
126	127	128	129	130	131
132	133	134	135	136	137
138	139	140	141	142	143

Large window where exactly multiples of $d = 2$ can be seen.

To prove Proposition 12, we start with the following combinatorial lemma.

Lemma 14. *Suppose $\{d_i\}_{i=0}^\infty$ is a sequence of non-negative integers with the following properties.*

- (A1) *For a positive integer d we have $0 \leq d_i \leq d$ for any i .*
- (A2) *If $d_i \neq 0$, then for any non-negative integer j we have $d_j \leq d_{i+j}$.*
- (A3) *For some positive integer m , d_m is not zero; and d_0 is not zero.*

Then for any integers $C \geq 2$ and $F \geq C^{md+1}$ there are integers a, b , and x_0, \dots, x_{m-1} such that

- (C1) *(m -periodic) For any $i \in [am, bm)$, $d_i = x_{r_i}$ where r_i is the remainder of i divided by m .*
- (C2) *(Length of periodicity) $b \geq Ca$, and $b \geq C^{-md-1}F$.*

Remark 15. It is worth mentioning that, if $d = 1$, then the assumptions (A1) and (A2) are equivalent to saying that the map $i \mapsto d_i$ is the characteristic function of a subsemigroup of non-negative integers.

Proof of Lemma 14. Since $d_m \neq 0$ (see (A3)), we have that $i \mapsto d_{mi+j}$ is increasing for any non-negative integer j . And so for non-negative integers j and l the set $\{i \in \mathbb{Z}^{\geq 0} \mid d_{im+j} = l\}$ is a segment of integers. Since $0 \leq d_{mi+j} \leq d$, for any j there is a partition $P_j := \{I_{0,j}, \dots, I_{d,j}\}$ of the set of non-negative integers such that for any $i \in I_{l,j}$ we have $d_{mi+j} = l$ (some of $I_{l,j}$'s might be empty). Collecting all the end points of the intervals $I_{l,j} \cap [0, F]$ (for $0 \leq l \leq d$ and $0 \leq j \leq m-1$) we get integers $0 =: g_0 \leq g_1 \leq \dots \leq g_{md} \leq g_{md+1} := F$ such that for any index j and integer $i \in [g_j m, g_{j+1} m)$ the number d_i only depends on the remainder of i divided by m .

Claim 1. There is an integer $j \in [0, md]$ such that $(g(j+1) + 1) \geq C(g(j) + 1)$.

Proof of Claim 1. Suppose to the contrary that for any $j \in [1, md]$ we have $(g(j+1) + 1) < C(g(j) + 1)$; then inductively we get that $(g(md+1) + 1) < C^j(g(md+1-j) + 1)$ for any integer $j \in [1, md+1]$. And for $j = md+1$, we get that $F + 1 < C^{md+1}$; and this gives us a contradiction.

Claim 2. There is an integer $j \in [1, md]$ such that $(g(j+1) + 1) \geq C(g(j) + 1)$ and $g(j+1) \geq C^{-md-1}F$.

Proof of Claim 2. Let j_0 be the largest integer in $[0, md]$ such that $(g(j_0 + 1) + 1) \geq C(g(j_0) + 1)$ (using Claim 1 we know that such an integer exists). Since j_0 is the largest integer with this property, we have

$$(g(i + 1) + 1) < C(g(i) + 1), \quad \text{for any integer } i \in [j_0 + 1, md].$$

Hence $g(md + 1) + 1 < C^{md-j_0}(g(j_0 + 1) + 1) < C^{md}(g(j_0 + 1) + 1)$; and this implies

$$C^{-md-1}F < C^{-md}F + C^{-md} - 1 < g(j_0 + 1).$$

Let $b := g(j_0 + 1)$ and $a := g(j_0)$. Claim 1 and Claim 2 imply that a and b satisfy the conclusion (C2) in the statement of Lemma 14. For an integer $j \in [0, m)$, let $x_j := d_{am+j}$. Since for any integer $i \in [am, bm)$ the number d_i only depends on the remainder r_i of i divided by m , we get that $d_i = x_{r_i}$ for any integer $i \in [am, bm)$; and the claim follows. \square

Proof of Proposition 12. Since $S = \bigoplus_{i=0}^{\infty} S_i t^i$ is a graded subring of the ring of polynomials $\mathbb{f}[t]$ over a finite field, we have that S_0 is a subfield of \mathbb{f} and S_i is a S_0 -vector space for any non-negative integer i . Let $d_i := \dim_{S_0} S_i$. Next we check that the assumptions (A1), (A2), and (A3) of Lemma 14 hold for the sequence $\{d_i\}_{i=0}^{\infty}$. As S_i 's are S_0 -subspaces of \mathbb{f} , we get that $\dim_{S_0} S_i \leq \dim_{S_0} \mathbb{f}$; and so the assumption (A1) of Lemma 14 holds for the sequence $\{d_i\}_{i=0}^{\infty}$. Since for non-negative integers i, j we have $S_i S_j \subseteq S_{i+j}$, we get that the assumption (A2) of Lemma 14 holds for the sequence $\{d_i\}_{i=0}^{\infty}$. And because of (17), the assumption (A3) of Lemma 14 holds for the sequence $\{d_i\}_{i=0}^{\infty}$. So by Lemma 14, there are integers $a, b, x_0, \dots, x_{m-1}$ that satisfy the conclusions (C1) and (C2) of Lemma 14.

Let $\lambda \in S_m \setminus \{0\}$ and $\bar{S}_i := \lambda^{-\lfloor i/m \rfloor} S_i$. Then $1 \in \bar{S}_m$; and so $\bar{S}_i \subseteq \bar{S}_m \bar{S}_i$. On the other hand, $S_m S_i \subseteq S_{i+m}$ implies that $\bar{S}_m \bar{S}_i \subseteq \bar{S}_{i+m}$. For any integer $i \in [ma, m(b-1))$, we have that $\dim_{S_0} \bar{S}_{i+m} = d_{i+m} = d_i = \dim_{S_0} \bar{S}_i$. Altogether we get that

$$(23) \quad \bar{S}_i = \bar{S}_{i+m}, \quad \text{for any integer } i \in [am, (b-1)m).$$

Let $B_i := \bar{S}_{am+i}$ for any integer $i \in [0, m)$. And so by (23) we have that for any integer $i \in [am, bm)$, $\bar{S}_i = B_{r_i}$ where r_i is the remainder of i divided by m . We will identify the set of integers in the interval $[0, m)$ with $\mathbb{Z}/m\mathbb{Z}$, and in the rest of the proof $\pi_m(i) := i + m\mathbb{Z}$ will be used instead of r_i ; that means we have $B_{\pi_m(i)} = \bar{S}_i$ for any integer $i \in [am, bm)$.

Claim. Let $I := \{\bar{i} \in \mathbb{Z}/m\mathbb{Z} \mid B_{\bar{i}} \neq 0\}$. Then I is a subgroup of $\mathbb{Z}/m\mathbb{Z}$.

Proof of Claim. Suppose i, j are two integers in $[0, m)$ and $\pi_m(i), \pi_m(j) \in I$. Then $B_{\pi_m(i)}$ and $B_{\pi_m(j)}$ are non-zero S_0 -vector spaces. Since $(2a+1)m < Cam \leq bm$, we have

$$(24) \quad S_{am+i} = \lambda^{\lfloor i/m \rfloor} B_{\pi_m(i)}, \quad S_{am+j} = \lambda^{\lfloor j/m \rfloor} B_{\pi_m(j)}, \quad \text{and } S_{2am+i+j} = \lambda^{\lfloor i+j/m \rfloor} B_{\pi_m(i+j)};$$

and so the grading equation $S_{am+i} S_{am+j} \subseteq S_{2am+i+j}$ implies that

$$(25) \quad 0 \neq B_{\pi_m(i)} B_{\pi_m(j)} \subseteq \lambda^{(\lfloor i+j/m \rfloor - \lfloor i/m \rfloor - \lfloor j/m \rfloor)} B_{\pi_m(i+j)} = \lambda^{\lfloor i+j/m \rfloor} B_{\pi_m(i+j)};$$

in particular $\pi_m(i+j) = \pi_m(i) + \pi_m(j) \in I$. And the claim follows.

Suppose r is the positive integer in $[0, m)$ such that $I = r\mathbb{Z}/m\mathbb{Z}$; in particular $r|m$. Let $s := m/r$. Suppose $\eta \in B_r \setminus \{0\}$, and let $\bar{B}_i := \eta^{-i} B_{\pi_m(ri)}$ for any integer i in $[0, s)$. So (25) implies that $\bar{B}_0 \cdot \bar{B}_i \eta^i \subseteq \bar{B}_i \eta^i$ for any integer i in $[0, s)$. Therefore we get

$$(26) \quad \bar{B}_i = \bar{B}_0 \cdot \bar{B}_i,$$

for any integer i in $[0, s)$; in particular \bar{B}_0 is a subfield of \mathbb{f} , and \bar{B}_i 's are \bar{B}_0 -vector spaces. Another application of (25) implies that $\bar{B}_1 \eta \cdot \bar{B}_i \eta^i \subseteq \bar{B}_{i+1} \eta^{i+1}$ for any integer i in $[0, s-1)$. Therefore we get

$$(27) \quad \bar{B}_1 \cdot \bar{B}_i \subseteq \bar{B}_{i+1},$$

for any integer i in $[0, s-1)$. By (26), (27), and the fact that $1 \in \bar{B}_1$, we deduce that

$$(28) \quad \bar{B}_0 \subseteq \bar{B}_1 \subseteq \dots \subseteq \bar{B}_{s-1}.$$

Next we again use (25) to deduce that $\overline{B}_1\eta \cdot \overline{B}_{s-1}\eta^{s-1} \subseteq \lambda\overline{B}_0$. Therefore we have

$$(29) \quad \lambda^{-1}\eta^s\overline{B}_1\overline{B}_{s-1} \subseteq \overline{B}_0.$$

By (28), (29), the fact that $1 \in \overline{B}_1$, and comparing the dimensions, we get that there is a subfield \mathfrak{f}_0 of \mathfrak{f} such that

$$(30) \quad \overline{B}_0 = \overline{B}_1 = \cdots = \overline{B}_{s-1} = \mathfrak{f}_0, \quad \text{and} \quad \lambda\mathfrak{f}_0^\times = \eta^s\mathfrak{f}_0^\times.$$

Since $\overline{B}_0 = \lambda^{-m}S_{am} \supseteq S_0$, we get that \mathfrak{f}_0 is a field extension of S_0 . By (24), $\overline{B}_i := \eta^{-i}B_{\pi_m(r_i)}$, (30), $\overline{B}_i := \eta^{-i}B_{\pi_m(r_i)}$, and $m = rs$, we have

$$(31) \quad \begin{aligned} \bigoplus_{i=am}^{bm-1} S_i t^i &= \bigoplus_{i=a}^{b-1} \left(\overline{B}_0 \oplus \overline{B}_1 \eta t^r \oplus \cdots \oplus \overline{B}_{s-1} \eta^{s-1} t^{r(s-1)} \right) (\lambda t^m)^i \\ &= \bigoplus_{i=as}^{bs-1} \mathfrak{f}_0 (\eta t^r)^i. \end{aligned}$$

By (31), we get (19); and since a and b satisfy the conclusions (C1) and (C2) of Lemma 14, we get (18). And the claim follows. \square

2.2. Structure of subrings of the ring \mathcal{O} of integers of a finite extension K of \mathbb{Q}_p . The main goal of this section is to prove Theorem 7; but before we get to that, let us define certain graded algebras which are crucial throughout this note. For any $\eta \in \mathfrak{p}\mathcal{O}$, we get a filtration $\{\eta^i\mathcal{O}\}_{i=0}^\infty$ of \mathcal{O} ; and then we can define a corresponding graded algebra: let $\text{gr}_\eta(\mathcal{O}) := \bigoplus_{i=0}^\infty \text{gr}_{i,\eta}(\mathcal{O})$ where $\text{gr}_{i,\eta} := \pi_{\eta^{i+1}}(\eta^i\mathcal{O})$. For any $x \in \mathcal{O}$, there is a unique non-negative integer i such that $x \in \eta^i\mathcal{O} \setminus \eta^{i+1}\mathcal{O}$; we denote such non-negative integer by $\text{deg}_\eta(x)$, and it is clear that $\text{deg}_\eta(x) = \lfloor v_{\mathfrak{p}}(x)/v_{\mathfrak{p}}(\eta) \rfloor$. For any $x \in \mathcal{O}$ and $\eta \in \mathfrak{p}\mathcal{O}$, we let $l_\eta(x) := \pi_{\mathfrak{p}^{\text{deg}_\eta(x)+1}}(x)$ and $\tilde{l}_\eta(x) := \pi_\eta(\eta^{-\text{deg}_\eta(x)}x) \in \pi_\eta(\mathcal{O})$ and call $\tilde{l}_\eta(x)$ the η -leading term of x . The following is a useful lemma that justifies the use of the above terminology for $\tilde{l}_\eta(x)$.

Lemma 16. *In the above setting, let $\phi_\eta : \text{gr}_\eta(\mathcal{O}) \rightarrow \pi_\eta(\mathcal{O})[t]$ be the graded map induced by*

$$\phi_\eta(l_\eta(x)) := \tilde{l}_\eta(x)t^{\text{deg}_\eta(x)},$$

where $\pi_\eta(\mathcal{O})[t]$ is the ring of polynomials over $\pi_\eta(\mathcal{O})$. Then ϕ_η is a graded ring isomorphism.

Proof. Suppose $\bar{x} \in \text{gr}_{i,\eta}(\mathcal{O})$; then $\bar{x} = \pi_{\eta^{i+1}}(x)$ for some $x \in \eta^i\mathcal{O}$. And we have $\phi_\eta(\bar{x}) = \pi_\eta(\eta^{-i}x)t^i$. Therefore for $\bar{x}, \bar{x}' \in \text{gr}_{i,\eta}(\mathcal{O})$, there are $x, x' \in \eta^i\mathcal{O}$ such that $\pi_{\eta^i}(x) = \bar{x}$ and $\pi_{\eta^i}(x') = \bar{x}'$; and

$$\phi_\eta(\bar{x} + \bar{x}') = \phi_\eta(\pi_{\eta^{i+1}}(x + x')) = \pi_\eta(\eta^{-i}(x + x'))t^i = \pi_\eta(\eta^{-i}x)t^i + \pi_\eta(\eta^{-i}x')t^i = \phi_\eta(\bar{x}) + \phi_\eta(\bar{x}').$$

Since \mathcal{O} is an integral domain, $\pi_{\eta^{i+1}}(x) \mapsto \pi_\eta(\eta^{-i}x)$ is a bijection from $\pi_{\eta^{i+1}}(\eta^i\mathcal{O})$ to $\pi_\eta(\mathcal{O})$. Hence ϕ_η is an additive group isomorphism from $\text{gr}_\eta(\mathcal{O})$ to $\pi_\eta(\mathcal{O})[t]$.

For $\bar{x} \in \text{gr}_{i,\eta}(\mathcal{O})$ and $\bar{x}' \in \text{gr}_{j,\eta}(\mathcal{O})$, we have that $\bar{x} = \pi_{\eta^{i+1}}(x)$ and $\bar{x}' = \pi_{\eta^{j+1}}(x')$ for some $x \in \eta^i\mathcal{O}$ and $x' \in \eta^j\mathcal{O}$; and $\phi_\eta(\bar{x}) = \pi_\eta(\eta^{-i}x)t^i$ and $\phi_\eta(\bar{x}') = \pi_\eta(\eta^{-j}x')t^j$. Then based on the graded structure of $\text{gr}_\eta(\mathcal{O})$, we have $\overline{xx'} = \pi_{\eta^{i+j+1}}(xx')$; and so $\phi_\eta(\overline{xx'}) = \pi_\eta(xx')t^{i+j} = \phi_\eta(\bar{x})\phi_\eta(\bar{x}')$. And the claim follows. \square

Corollary 17. *Suppose $x \in \mathcal{O} \setminus \mathfrak{p}\mathcal{O}$; then for any non-negative integers i and k multiplication by $x\eta^i$ induces a bijection $x_k + \eta^{k+1}\mathcal{O} \mapsto x_k x \eta^i + \eta^{k+i+1}\mathcal{O}$ from $\text{gr}_{k,\eta}(\mathcal{O})$ to $\text{gr}_{k+i,\eta}(\mathcal{O})$.*

Proof. We notice that x is a unit in \mathcal{O} ; and so $\pi_\eta(x)$ is a unit in $\pi_\eta(\mathcal{O})$; and so multiplication by $\pi_\eta(x)$ induces a bijection from $\pi_\eta(\mathcal{O})$ to itself. And by Lemma 16 the claim follows. \square

To any subset X of \mathcal{O} we associate a graded subset $\text{gr}_\eta(X; \mathcal{O})$ of $\text{gr}_\eta(\mathcal{O})$. For a non-negative integer i , we let $\text{gr}_{i,\eta}(X; \mathcal{O}) := \pi_{\eta^{i+1}}(X \cap \eta^i\mathcal{O})$; and we define $\text{gr}_\eta(X; \mathcal{O}) := \bigoplus_{i=0}^\infty \text{gr}_{i,\eta}(X; \mathcal{O})$.

Lemma 18. *In the above setting, suppose R is a subring of \mathcal{O} ; then $\text{gr}_\eta(R; \mathcal{O})$ is a subring of $\text{gr}_\eta(\mathcal{O})$.*

Proof. It is clear. \square

Proof of Theorem 7. By Lemma 18, we have that $\text{gr}_{\mathfrak{p}}(R; \mathcal{O})$ is a subring of $\text{gr}_{\mathfrak{p}}(\mathcal{O})$; and by Lemma 16 there is a graded algebra isomorphism $\phi : \text{gr}_{\mathfrak{p}}(\mathcal{O}) \rightarrow \mathfrak{f}[t]$ where \mathfrak{f} is the residue field of \mathcal{O} . Since 1 is in R , we have that $\text{gr}_{0,\mathfrak{p}}(R; \mathcal{O}) := \pi_{\mathfrak{p}}(R)$ is a subfield of \mathfrak{f} and $p \in R$. The latter implies that $\text{gr}_{e,\mathfrak{p}}(R; \mathcal{O}) \neq 0$ where $e := v_{\mathfrak{p}}(p)$ is the ramification index of K over \mathbb{Q}_p . It is well-known that the degree $[K : \mathbb{Q}_p]$ of the field extension K of \mathbb{Q}_p is equal to the product of its ramification index e and its residue degree $[\mathfrak{f} : \mathfrak{f}_p]$. Hence by Proposition 12 there are a subfield \mathfrak{f}_0 of \mathfrak{f} which is an extension of $\text{gr}_{0,\mathfrak{p}}(R; \mathcal{O})$, $\lambda \in \mathfrak{f}$, and positive integers r, a, b such that $r|e$ and

$$(32) \quad b - a \gg_{C,[K:\mathbb{Q}_p]} F, \quad b \geq Ca, \quad \text{and} \quad \bigoplus_{ae \leq i < be} \phi(\text{gr}_{i,\mathfrak{p}}(R; \mathcal{O})) = \bigoplus_{ae/r \leq j < be/r} \mathfrak{f}_0(\lambda t^r)^j.$$

An immediate consequence of (32) is the existence of certain elements in R which help us pass to smaller scales *within* R . By (32), for any integer j in $[1, e/r)$, there is $\widehat{\xi}_j \in R$ such that $v_{\mathfrak{p}}(\widehat{\xi}_j) = ea + rj$. By letting $\widehat{\xi}_0 := p^a$, we get the same claim for $j = 0$ as well. For any integer i in $[a, b)$ and integer j in $[0, e/r)$, let $\widehat{\xi}_{i,j} := p^{i-a} \widehat{\xi}_j$. Hence for any such integers i and j , we have

$$(33) \quad \widehat{\xi}_{i,j} \in R \quad \text{and} \quad v_{\mathfrak{p}}(\widehat{\xi}_{i,j}) = ei + rj.$$

For any integer k in $[0, (b-a)e/r)$, let $\xi_k := p^{-a} \widehat{\xi}_{\bar{q}, \bar{r}}$ where \bar{q} is the quotient of $rk + ea$ divided by e and $r\bar{r}$ is the remainder of $rk + ea$ divided by e . Therefore for any such k, \bar{q} , and \bar{r} by (33) we have

$$(34) \quad p^a \xi_k \in R \quad \text{and} \quad v_{\mathfrak{p}}(\xi_k) = -ea + v_{\mathfrak{p}}(\widehat{\xi}_{\bar{q}, \bar{r}}) = -ea + (e\bar{q} + r\bar{r}) = rk.$$

Another consequences of (32) is the fact that $\text{gr}_{i,\mathfrak{p}}(R; \mathcal{O})$ is zero if $r \nmid i$ and $i < be$; and so for any non-negative integer $i < be$ we have

$$(35) \quad R \cap \mathfrak{p}^i \mathcal{O} = R \cap \mathfrak{p}^{r \lceil i/r \rceil} \mathcal{O}.$$

And the last immediate consequence of (32) (see Corollary 17) that we mention here is that for any integer i in $[ae, (b-1)e)$

$$(36) \quad x + \mathfrak{p}^{i+1} \mathcal{O} \mapsto px + \mathfrak{p}^{e+i+1} \mathcal{O} \text{ is a bijection from } \text{gr}_{i,\mathfrak{p}}(R; \mathcal{O}) \text{ to } \text{gr}_{e+i,\mathfrak{p}}(R; \mathcal{O}).$$

By (32) we also deduce that there is a function $\tilde{s} : \mathfrak{f}_0 \rightarrow R \cap p^a \mathcal{O}$ such that $\pi_{\mathfrak{p}}(p^{-a} \tilde{s}(x)) = x$ for any $x \in \mathfrak{f}_0$.

Claim 1. Let $s : \mathfrak{f}_0 \rightarrow \mathcal{O}, s(x) := p^{-a} \tilde{s}(x)$ and r be as in (32). Then $\pi_{\mathfrak{p}^r} \circ s : \mathfrak{f}_0 \rightarrow \pi_{\mathfrak{p}^r}(\mathcal{O})$ is a ring embedding.

Proof of Claim 1. For any $x \in \mathfrak{f}_0$, we have $\pi_{\mathfrak{p}}(s(x)) = x$. Hence for any $x_1, x_2 \in \mathfrak{f}_0$, we have $s(x_1 + x_2) - s(x_1) - s(x_2) \in \mathfrak{p}\mathcal{O}$; this implies that

$$(37) \quad \tilde{s}(x_1 + x_2) - \tilde{s}(x_1) - \tilde{s}(x_2) \in R \cap \mathfrak{p}^{ea+1} \mathcal{O}.$$

Hence by (35) we have

$$(38) \quad \tilde{s}(x_1 + x_2) - \tilde{s}(x_1) - \tilde{s}(x_2) \in R \cap \mathfrak{p}^{ea+r} \mathcal{O}.$$

And so $\pi_{\mathfrak{p}^r}(s(x_1 + x_2)) = \pi_{\mathfrak{p}^r}(s(x_1)) + \pi_{\mathfrak{p}^r}(s(x_2))$.

We also have $\tilde{s}(x_1) \tilde{s}(x_2) \in R \cap p^{2a} \mathcal{O}$. By (36), there is $x' \in R \cap p^a \mathcal{O}$ such that

$$(39) \quad p^a x' - \tilde{s}(x_1) \tilde{s}(x_2) \in R \cap \mathfrak{p}^{2ae+1} \mathcal{O}.$$

We also have $\pi_{\mathfrak{p}}(p^{-a} \tilde{s}(x_1 x_2)) = x_1 x_2 = \pi_{\mathfrak{p}}(p^{-a} \tilde{s}(x_1)) \pi_{\mathfrak{p}}(p^{-a} \tilde{s}(x_2)) = \pi_{\mathfrak{p}}(p^{-2a} \tilde{s}(x_1) \tilde{s}(x_2))$. And so by (39) we get $\pi_{\mathfrak{p}}(p^{-a} x') = \pi_{\mathfrak{p}}(p^{-a} \tilde{s}(x_1 x_2))$. Hence we get

$$(40) \quad p^a \tilde{s}(x_1 x_2) - \tilde{s}(x_1) \tilde{s}(x_2) \in R \cap \mathfrak{p}^{2ea+1} \mathcal{O}.$$

By (40) and (35) we have

$$(41) \quad p^a \tilde{s}(x_1 x_2) - \tilde{s}(x_1) \tilde{s}(x_2) \in R \cap \mathfrak{p}^{2ae+r} \mathcal{O}.$$

Hence $p^{-a}\tilde{s}(x_1x_2) - (p^{-a}\tilde{s}(x_1))(p^{-a}\tilde{s}(x_2)) \in \mathfrak{p}^r\mathcal{O}$; and this implies that

$$(42) \quad \pi_{\mathfrak{p}^r}(s(x_1x_2)) = \pi_{\mathfrak{p}^r}(s(x_1))\pi_{\mathfrak{p}^r}(s(x_2));$$

and the claim follows. QED.

By Hensel's lemma we know that there is a group embedding $\theta : \mathfrak{f}^\times \rightarrow \mathcal{O}^\times$ such that for any $x \in \mathfrak{f}$ we have $x = \pi_{\mathfrak{p}}(\theta(x))$. Next we show that a good approximation of a multiple of $\theta(\mathfrak{f}_0^\times)$ can be found in R . This will help us to get the *unramified part* of K_0 .

Claim 2. Suppose a, b are given as in (32) for a constant C which is more than $3[K : \mathbb{Q}_p]$. Then there is $\hat{s} : \mathfrak{f}_0^\times \rightarrow R \cap p^{2a}\mathcal{O}$ such that $\hat{s}(x) - p^{2a}\theta(x) \in p^{b-2a([\mathfrak{f}_0 : \mathfrak{f}_p]-1)}\mathcal{O}$ for any $x \in \mathfrak{f}_0^\times$.

Proof of Claim 2. Let us fix $x_0 \in \mathfrak{f}_0^\times$. We notice that $\theta(x_0)$ is a zero of $t^{|\mathfrak{f}_0|-1} - 1$; and so its (monic) minimal polynomial $\Phi(t)$ over \mathbb{Q}_p is actually in $\mathbb{Z}_p[t]$ and $\pi_{\mathfrak{p}}(\Phi(t))$ does not have multiple zeros in \mathfrak{f} . And so by the Hensel's lemma, if $\Phi(y) \in \mathfrak{p}^l\mathcal{O}$ for some $y \in \mathcal{O}$ and positive integer l and $\pi_{\mathfrak{p}}(y) = x_0$, then $y - \theta(x_0) \in \mathfrak{p}^l\mathcal{O}$. Moreover we notice that $d := \deg \Phi$ is at most the residue index $d_0 := [\mathfrak{f} : \mathfrak{f}_p]$ of K over \mathbb{Q}_p . Next for integers k in $[0, (b - 2ad_0)e/r)$ we inductively find $y_k \in \mathcal{O}$ such that

$$(43) \quad \pi_{\mathfrak{p}}(y_k) = x_0, \quad y_k - y_{k-1} \in \mathfrak{p}^{rk}\mathcal{O}, \quad p^{2a}y_k \in R, \quad \text{and } \Phi(y_k) \in \mathfrak{p}^{r(k+1)}\mathcal{O},$$

where r is as in (32). The argument is similar to the proof of Hensel's lemma but we have to be careful that every step of estimation is done within the ring R and only in certain scales we have a control on R . We write the Taylor expansion of $\Phi(t)$ at y_k ,

$$(44) \quad \Phi(y_k + t) = \Phi(y_k) + \sum_{i=1}^d \frac{\Phi^{(i)}(y_k)}{i!} t^i;$$

and we point out that, since the coefficients of Φ are in \mathbb{Z}_p , we have that $\frac{\Phi^{(i)}(t)}{i!}$ is again in $\mathbb{Z}_p[t]$. Therefore, for any integer i in $[1, d]$, $p^{2a(d-i)}\frac{\Phi^{(i)}(y_k)}{i!}$ is in the \mathbb{Z}_p -algebra generated by $p^{2a}y_k$; and so by the induction hypothesis

$$(45) \quad p^{2a(d-i)}\frac{\Phi^{(i)}(y_k)}{i!} \in R. \quad \text{And in addition } y_k \in \mathcal{O} \text{ implies that } \frac{\Phi^{(i)}(y_k)}{i!} \in \mathcal{O}.$$

As we mentioned earlier, $\Phi(t)$ is a divisor of $t^{|\mathfrak{f}_0|-1} - 1$; since $t^{|\mathfrak{f}_0|-1} - 1$ does not have multiple zeros in \mathfrak{f} , $\Phi(t)$ does not have multiple zeros in \mathfrak{f} . Hence $\pi_{\mathfrak{p}}(\Phi(y_k)) = 0$ implies that

$$(46) \quad \pi_{\mathfrak{p}}(\Phi'(y_k)) \neq 0.$$

By (45) and (33), we have

$$p^{2ad}\Phi'(y_k)\xi_{k+1}s(\mathfrak{f}_0) = (p^{2a(d-1)}\Phi'(y_k))(p^a\xi_{k+1})(p^as(\mathfrak{f}_0)) \subseteq R \cap \mathfrak{p}^{2ade+r(k+1)}\mathcal{O};$$

and by (46), we have that

$$|\pi_{\mathfrak{p}^{2ade+r(k+1)+1}}(p^{2ad}\Phi'(y_k)\xi_{k+1}s(\mathfrak{f}_0))| = |\mathfrak{f}_0|;$$

since, by (32), $|\text{gr}_{2ade+r(k+1), \mathfrak{p}}(R; \mathcal{O})| = |\mathfrak{f}_0|$ (notice that $k+1 < (b - 2ad_0)e/r$), we deduce that

$$(47) \quad \text{gr}_{2ade+r(k+1), \mathfrak{p}}(R; \mathcal{O}) = \pi_{\mathfrak{p}^{2ade+r(k+1)+1}}(p^{2ad}\Phi'(y_k)\xi_{k+1}s(\mathfrak{f}_0)).$$

By (45) and (47), there is $x_{k+1} \in \mathfrak{f}_0$ such that

$$p^{2ad}\Phi(y_k) - p^{2ad}\Phi'(y_k)\xi_{k+1}s(x_{k+1}) \in \mathfrak{p}^{2ade+r(k+1)+1}\mathcal{O},$$

which implies

$$(48) \quad \Phi(y_k) + \Phi'(y_k)(\xi_{k+1}s(x_{k+1})) \in \mathfrak{p}^{r(k+1)+1}\mathcal{O}.$$

We also notice that by (34) and the induction hypothesis (see the third condition in (43)) we have

$$(49) \quad p^{2a}(y_k + \xi_{k+1}s(x_{k+1})) = p^{2a}y_k + (p^a\xi_{k+1})(p^as(x_{k+1})) \in R.$$

By (44), (45), (48), and (49) we have

$$(50) \quad \Phi(y_k + \xi_{k+1}s(x_{k+1})) = \Phi(y_k) + \Phi'(y_k)(\xi_{k+1}s(x_{k+1})) + \sum_{i=2}^d \frac{\Phi^{(i)}(y_k)}{i!} (\xi_{k+1}s(x_{k+1}))^i \in \mathfrak{p}^{r(k+1)+1}\mathcal{O},$$

and

$$(51) \quad p^{2ad}\Phi(y_k + \xi_{k+1}s(x_{k+1})) = p^{2ad}\Phi(y_k) + \sum_{i=1}^d \left(p^{2a(d-i)} \frac{\Phi^{(i)}(y_k)}{i!} \right) (p^a \xi_{k+1})^i (p^a s(x_{k+1}))^i \in R.$$

By (50) and (51), we have $p^{2ad}\Phi(y_k + \xi_{k+1}s(x_{k+1})) \in R \cap \mathfrak{p}^{2ead+r(k+1)+1}\mathcal{O}$; and so by (35) and having $k+1 < (b-2ad_0)e/r$ we deduce that $p^{2ad}\Phi_d(y_k + \xi_{k+1}s(x_{k+1})) \in R \cap \mathfrak{p}^{2ead+r(k+2)}\mathcal{O}$. Hence we get

$$(52) \quad \Phi(y_k + \xi_{k+1}s(x_{k+1})) \in \mathfrak{p}^{r(k+2)}\mathcal{O}.$$

Therefore by (34), (49), and (52) we get that $y_{k+1} := y_k + \xi_{k+1}s(x_{k+1})$ satisfies all the conditions mentioned in (43).

We notice that $\pi_{\mathfrak{p}}(y_k) = x_0$ and $\Phi(y_k) \in \mathfrak{p}^{r(k+1)}\mathcal{O}$ imply that

$$(53) \quad y_k - \theta(x_0) \in \mathfrak{p}^{r(k+1)}\mathcal{O}.$$

Let $k_0 := (b-2ad_0)e/r - 1$ and $\widehat{s}(x_0) := p^{2a}y_{k_0}$. Then by (43) and (53) we have

$$\widehat{s}(x_0) \in R, \text{ and } \widehat{s}(x_0) - p^{2a}\theta(x_0) \in \mathfrak{p}^{2ae+r(k_0+1)}\mathcal{O} = p^{2a+(b-2ad_0)}\mathcal{O};$$

and the claim follows. QED.

Claim 2 implies that $\pi_{\mathfrak{p}^{b-2a(d_0-1)}}(p^{2a}\theta(\mathfrak{f}_0^\times)) \subseteq \pi_{\mathfrak{p}^{b-2a(d_0-1)}}(R)$. On the other hand, as $\theta(\mathfrak{f}_0^\times)$ is a subgroup of the group \mathcal{O}^\times of units of \mathcal{O} , we have that

$$\mathbb{Z}_p[\theta(\mathfrak{f}_0^\times)] = \{ \sum_{x \in \mathfrak{f}_0^\times} c_x \theta(x) \mid c_x \in \mathbb{Z}_p \};$$

and so we get

$$(54) \quad \pi_{\mathfrak{p}^{b-2a(d_0-1)}}(p^{2a}\mathbb{Z}_p[\theta(\mathfrak{f}_0^\times)]) \subseteq \pi_{\mathfrak{p}^{b-2a(d_0-1)}}(R).$$

Claim 3. Suppose a, b, r are given as in (32) for a constant C which is more than $3[K : \mathbb{Q}_p]$. And as before let $\widehat{\xi}_1 \in R \cap \mathfrak{p}^{ae+r}\mathcal{O} \setminus \mathfrak{p}^{ae+r+1}\mathcal{O}$. Let $b' := b - 2a([\mathfrak{f}_0 : \mathfrak{f}_p] - 1)$. Then

$$\pi_{\mathfrak{p}^{b'}}(R \cap p^{4ar'}\mathcal{O}) = \pi_{\mathfrak{p}^{b'}}(p^{ar'}(\mathcal{O}_{0,u} + \mathcal{O}_{0,u}\widehat{\xi}_1 + \cdots + \mathcal{O}_{0,u}\widehat{\xi}_1^{r'-1}) \cap p^{4ar'}\mathcal{O}),$$

where $\mathcal{O}_{0,u} := \mathbb{Z}_p[\theta(\mathfrak{f}_0^\times)]$ and $r' := e/r$.

Proof of Claim 3. By (54) we have $\pi_{\mathfrak{p}^{b'}}(p^{2a}\mathcal{O}_{0,u}) \subseteq \pi_{\mathfrak{p}^{b'}}(R)$. As $\pi_{\mathfrak{p}^{b'}}(R)$ is a ring and $\pi_{\mathfrak{p}^{b'}}(\widehat{\xi}_1) \in \pi_{\mathfrak{p}^{b'}}(R)$, we have

$$(55) \quad \pi_{\mathfrak{p}^{b'}}(p^{2a}(\mathcal{O}_{0,u} + \mathcal{O}_{0,u}\widehat{\xi}_1 + \cdots + \mathcal{O}_{0,u}\widehat{\xi}_1^{r'-1})) \subseteq \pi_{\mathfrak{p}^{b'}}(R).$$

Let $\widetilde{\xi}_{i,j} := p^{j+a(r'-i)}\widehat{\xi}_1^i$ for integers i in $[0, r']$ and j in $[0, b' - ar']$; then $v_{\mathfrak{p}}(\widetilde{\xi}_{i,j}) = ej + ea(r' - i) + eai + ri = ear' + ej + ri$. Hence for any integer k in $[ar'^2, b'r']$ there is $\widetilde{\xi}_k \in \bigcup_{j=0}^{r'-1} \mathbb{Z}_p \widetilde{\xi}_1^j$ such that $v_{\mathfrak{p}}(\widetilde{\xi}_k) = rk$.

Hence by Corollary 17 and (32) we have that

$$(56) \quad x + \mathfrak{p}^{l+1}\mathcal{O} \mapsto \widetilde{\xi}_k x + \mathfrak{p}^{l+rk+1}\mathcal{O} \text{ is a bijection from } \text{gr}_{l,\mathfrak{p}}(R; \mathcal{O}) \text{ to } \text{gr}_{l+rk,\mathfrak{p}}(R; \mathcal{O})$$

if k is an integer in $[ar'^2, b'r']$ and l is an integer in $[ae, be - rk)$.

Let us fix $y_0 \in R \cap p^{4ar'}\mathcal{O}$. Inductively we prove that for any integer m in $[4ar'^2, b'r']$ we have

$$\pi_{\mathfrak{p}^{rm}}(y_0) \in \pi_{\mathfrak{p}^{rm}}(p^{ar'}(\mathcal{O}_{0,u} + \mathcal{O}_{0,u}\widehat{\xi}_1 + \cdots + \mathcal{O}_{0,u}\widehat{\xi}_1^{r'-1})).$$

The case of $m = 4ar'^2$ is clear as $\mathfrak{p}^{r(4ar'^2)}\mathcal{O} = p^{4ar'}\mathcal{O}$.

By the induction hypothesis, we have that there is

$$(57) \quad v_m \in p^{ar'}(\mathcal{O}_{0,u} + \mathcal{O}_{0,u}\widehat{\xi}_1 + \cdots + \mathcal{O}_{0,u}\widehat{\xi}_1^{r'-1}) \text{ such that } y_0 - v_m \in \mathfrak{p}^{rm}\mathcal{O}.$$

Since $y_0 \in p^{4ar'}\mathcal{O}$, $y_0 - v_m \in \mathfrak{p}^{rm}\mathcal{O} \subseteq p^{4ar'}\mathcal{O}$ (as $m \geq 4ar'^2$), we deduce that $v_m \in p^{4ar'}\mathcal{O}$; in particular if $r' = 1$, then $v_m \in p^{ar'}\mathcal{O}_{0,u} \cap p^{4ar'}\mathcal{O} = p^{4ar'}\mathcal{O}_{0,u}$ (notice that by [Neu99, Chapter II, Proposition 7.12] p is a uniformizing element of $\mathcal{O}_{0,u}$; see Claim 4, Subclaim (a) for more included details). Therefore in any case $v_m \in p^{2a}(\mathcal{O}_{0,u} + \mathcal{O}_{0,u}\widehat{\xi}_1 + \cdots + \mathcal{O}_{0,u}\widehat{\xi}_1^{r'-1})$; and so by (55), there are $z_m \in R$ and $e_m \in \mathcal{O}$ such that $v_m = z_m + p^{b'}e_m$. And so by (57)

$$(58) \quad y_0 - z_m \in R \cap \mathfrak{p}^{rm}\mathcal{O}.$$

On the other hand applying (56) for the parameters $l = 2ae$ and $k = m - 2ar' \geq 4ar'^2 - 2ar' \geq ar'^2$ we have that

$$y_0 - z_m - \widetilde{\xi}_{m-2ar'}\widehat{s}(x_m) \in \mathfrak{p}^{r(m+1)}\mathcal{O} \cap R;$$

and so by (35)

$$(59) \quad y_0 - z_m - \widetilde{\xi}_{m-2ar'}\widehat{s}(x_m) \in \mathfrak{p}^{r(m+1)}\mathcal{O} \cap R.$$

On the other hand, by Claim 2 and (57),

$$(60) \quad \begin{aligned} \pi_{p^{b'}}(z_m + \widetilde{\xi}_{m-2ar'}\widehat{s}(x_m)) &= \pi_{p^{b'}}(v_m) + \pi_{p^{b'}}(\widetilde{\xi}_{m-2ar'})\pi_{p^{b'}}(\widehat{s}(x_m)) \\ &= \pi_{p^{b'}}(v_m) + \pi_{p^{b'}}(\widetilde{\xi}_{m-2ar'})\pi_{p^{b'}}(p^{2a}\theta(x_m)) \\ &= \pi_{p^{b'}}(v_m) + \pi_{p^{b'}}(\widetilde{\xi}_m)\pi_{p^{b'}}(\theta(x_m)) \\ &\in \pi_{p^{b'}}(p^{ar'}(\mathcal{O}_{0,u} + \mathcal{O}_{0,u}\widehat{\xi}_1 + \cdots + \mathcal{O}_{0,u}\widehat{\xi}_1^{r'-1})); \end{aligned}$$

in the last assertion we are using

$$\begin{aligned} v_{\mathfrak{p}}(\widetilde{\xi}_{m-2ar'}) - v_{\mathfrak{p}}(\widehat{\xi}_1^{r'-1}) &\geq (4ar'^2 - 2ar')r - (r' - 1)(ae + r) \\ &> e((3r' - 1)a - 1) \geq ear' = v_{\mathfrak{p}}(p^{ar'}). \end{aligned}$$

By (60) there is

$$(61) \quad v_{m+1} \in p^{ar'}(\mathcal{O}_{0,u} + \mathcal{O}_{0,u}\widehat{\xi}_1 + \cdots + \mathcal{O}_{0,u}\widehat{\xi}_1^{r'-1}) \text{ such that } z_m + \widetilde{\xi}_{m-2ar'}\widehat{s}(x_m) - v_{m+1} \in p^{b'}\mathcal{O}.$$

By (61) and (59) we have that there is

$$(62) \quad v_{m+1} \in p^{ar'}(\mathcal{O}_{0,u} + \mathcal{O}_{0,u}\widehat{\xi}_1 + \cdots + \mathcal{O}_{0,u}\widehat{\xi}_1^{r'-1}) \text{ such that } y_0 - v_{m+1} \in \mathfrak{p}^{r(m+1)}\mathcal{O};$$

and by (55), (62), and the remark made for $r' = 1$ case, the claim follows. QED.

Claim 4. Suppose a, b, r are as in (32) for a constant $C \geq 12[K : \mathbb{Q}_p]$. Suppose $\widehat{\xi}_1 \in R \cap \mathfrak{p}^{ae+r}\mathcal{O} \setminus \mathfrak{p}^{ae+r+1}\mathcal{O}$. Then there is $\xi \in \mathcal{O}$ such that

$$\xi - \widehat{\xi}_1 \in p^{\lfloor b/2 \rfloor}\mathcal{O}, \text{ and } \pi_{p^{\lfloor b/2 \rfloor}}(R \cap p^{4ar'}\mathcal{O}) = \pi_{p^{\lfloor b/2 \rfloor}}(p^{ar'}\mathcal{O}_{0,u}[\xi] \cap p^{4ar'}\mathcal{O}),$$

where as before $\mathcal{O}_{0,u} = \mathbb{Z}_p[\theta(\mathfrak{f}_0^\times)]$ and $r' := e/r$. Moreover $\mathbb{Q}_p[\theta(\mathfrak{f}_0^\times)][\xi]$ is a totally ramified extension of $\mathbb{Q}_p[\theta(\mathfrak{f}_0^\times)]$ and $[\mathbb{Q}_p[\theta(\mathfrak{f}_0^\times)][\xi] : \mathbb{Q}_p[\theta(\mathfrak{f}_0^\times)]] = r'$.

Proof of Claim 4. We know that $v_{\mathfrak{p}}(\xi^{r'}) = v_{\mathfrak{p}}(p^{3ar'}\widehat{\xi}_1^{r'}) = 4aer' + e > v_{\mathfrak{p}}(p^{4ar'})$. And so by Claim 3, there are $c_0, \dots, c_{r'-1} \in \mathcal{O}_{0,u}$ such that

$$(63) \quad p^{3ar'}\widehat{\xi}_1^{r'} + c_{r'-1}\widehat{\xi}_1^{r'-1} + \cdots + c_1\widehat{\xi}_1 + c_0 \in p^{b'}\mathcal{O},$$

where $b' := b - 2(\lfloor \mathfrak{f}_p : \mathfrak{f}_p \rfloor - 1)a$. Before we continue our analysis, we prove the following well-known subclaim. (Since its proof is short, we include it for the convenience of reader; see [Neu99, Chapter II, Proposition 7.2] for a more general statement.)

Subclaim (a). Let $K_{0,u} := \mathbb{Q}_p[\theta(\mathfrak{f}_0^\times)]$. Then $K_{0,u}$ is an unramified extension of \mathbb{Q}_p , its ring of integers is $\mathcal{O}_{0,u} := \mathbb{Z}_p[\theta(\mathfrak{f}_0^\times)]$, and its residue field is \mathfrak{f}_0 ; in particular for any $c \in K_{0,u}^\times$ we have $v_{\mathfrak{p}}(c) \in e\mathbb{Z}$.

Proof of Subclaim (a). Let $\mathcal{O}'_{0,u}$ be the ring of integers of $K_{0,u}$, and \mathfrak{f}'_0 be its residue field. Then $\mathcal{O}_{0,u} := \mathbb{Z}_p[\theta(\mathfrak{f}'_0)] \subseteq \mathcal{O}'_{0,u}$. And so the $\mathfrak{f}_0 \subseteq \mathfrak{f}'_0$. Let β be a generator of \mathfrak{f}'_0 . Since $\theta(\beta)$ is integral over \mathbb{Z}_p , by Hensel's lemma we have $[\mathbb{Q}_p[\theta(\beta)] : \mathbb{Q}_p] = [\mathfrak{f}_p[\beta] : \mathfrak{f}_p]$; this means $[K_{0,u} : \mathbb{Q}_p] = [\mathfrak{f}_0 : \mathfrak{f}_p]$. But we know that the degree $[K_{0,u} : \mathbb{Q}_p]$ of the field extension $K_{0,u}$ of \mathbb{Q}_p is equal to the product of its ramification index and its residue degree. So we conclude that $\mathfrak{f}'_0 = \mathfrak{f}_0$ and $K_{0,u}$ is an unramified extension of \mathbb{Q}_p . Since $\mathcal{O}_{0,u}$ is a complete subring of $\mathcal{O}'_{0,u}$ and $\pi_p(\mathcal{O}_{0,u}) = \pi_p(\mathcal{O}'_{0,u})$, we deduce that $\mathcal{O}'_{0,u} = \mathcal{O}_{0,u}$. And the subclaim (a) follows.

Let $c_{r'} := p^{3ar'}$ and $f(t) := \sum_{i=0}^{r'} c_i t^i \in \mathcal{O}_{0,u}[t]$. Next we will find an upper bound for the p-adic valuation $v_p(f'(\widehat{\xi}_1))$ of the value of the derivative $f'(t)$ of $f(t)$ at $t = \widehat{\xi}_1$.

Subclaim (b). In the above setting $v_p(f'(\widehat{\xi}_1)) \leq 4er'a$.

Proof of Subclaim (b). We have $f'(\widehat{\xi}_1) = \sum_{i=1}^{r'} i c_i \widehat{\xi}_1^{i-1}$. And by Subclaim (a) we have

$$(64) \quad v_p(i c_i \widehat{\xi}_1^{i-1}) \equiv r(i-1) \pmod{e}$$

for any integer i in $[1, r']$. Since $0 \leq r(i-1) < e$ for $i \in [1, r']$, by (64) we have

$$(65) \quad v_p(i c_i \widehat{\xi}_1^{i-1}) \neq v_p(j c_j \widehat{\xi}_1^{j-1})$$

for distinct integers i and j in $[1, r']$. By (65) we deduce that

$$v_p(f'(\widehat{\xi}_1)) = \min_{1 \leq i \leq r'} v_p(i c_i \widehat{\xi}_1^{i-1}) \leq v_p(r' p^{3ar'} \widehat{\xi}_1^{r'-1}) \leq 4er'a$$

(to get the last assertion we are assuming $a \geq 3$); and the subclaim (b) follows.

By (63) and Subclaim (b), we have that

$$(66) \quad v_p(f(\widehat{\xi}_1)) - 2v_p(f'(\widehat{\xi}_1)) \geq e(b' - 4r'a) = e(b - 2([\mathfrak{f}_0 : \mathfrak{f}_p] - 1 + 2r')a) \geq e(1 - 6[K : \mathbb{Q}_p]/C)b \geq eb/2.$$

Hence by [Lan94, Chapter II, Section 2, Proposition 2] and (66), there is $\xi \in \mathcal{O}$ such that

$$(67) \quad f(\xi) = 0, \text{ and } v_p(\xi - \widehat{\xi}_1) \geq v_p(f(\widehat{\xi}_1)/(f'(\widehat{\xi}_1))^2) \geq eb/2.$$

Let $K_0 := K_{0,u}[\xi]$. Since $f(\xi) = 0$ and $f(t) \in K_{0,u}[t]$, we have $[K_0 : K_{0,u}] \leq \deg f = r'$. By the second part of (67) we can deduce that $v_p(\xi) = v_p(\widehat{\xi}_1) = ae + r$; and so $r \in v_p(K_0^\times)$. This and Subclaim (a) imply that the ramification index of the field extension K_0 over $K_{0,u}$ is at least $[r\mathbb{Z} : e\mathbb{Z}] = r'$. Since the index $[K_0 : K_{0,u}]$ of the field extension K_0 over $K_{0,u}$ is equal to the product of the ramification index and the residue index of this field extension, by the above discussion we deduce that

$$(68) \quad [K_0 : K_{0,u}] = r', \text{ } K_0/K_{0,u} \text{ is a totally ramified extension, and } f(t) \text{ is irreducible in } K_{0,u}[t].$$

Since ξ is integral over $\mathcal{O}_{0,u}$, $\mathcal{O}_{0,u}$ is integrally closed, and the degree of ξ over $K_{0,u}$ is r' , we have that

$$(69) \quad \mathcal{O}_{0,u}[\xi] = \mathcal{O}_{0,u} + \mathcal{O}_{0,u}\xi + \cdots + \mathcal{O}_{0,u}\xi^{r'-1}.$$

Hence we have

$$\pi_{p^{\lfloor b/2 \rfloor}}(R \cap p^{4ar'} \mathcal{O}) = \pi_{p^{\lfloor b/2 \rfloor}}(p^{ar'}(\mathcal{O}_{0,u} + \mathcal{O}_{0,u}\widehat{\xi}_1 + \cdots + \mathcal{O}_{0,u}\widehat{\xi}_1^{r'-1}) \cap p^{4ar'} \mathcal{O}) \quad (\text{by Claim 3})$$

$$= \pi_{p^{\lfloor b/2 \rfloor}}(p^{ar'}(\mathcal{O}_{0,u} + \mathcal{O}_{0,u}\xi + \cdots + \mathcal{O}_{0,u}\xi^{r'-1}) \cap p^{4ar'} \mathcal{O}). \quad (\text{by (67)})$$

$$= \pi_{p^{\lfloor b/2 \rfloor}}(p^{ar'} \mathcal{O}_{0,u}[\xi] \cap p^{4ar'} \mathcal{O}) \quad (\text{by (69)})$$

QED.

Claim 5. Suppose a, b, r are as in (32) for a constant $C \geq 12[K : \mathbb{Q}_p]$. Suppose ξ is as in Claim 4, and let $K_{0,u} := \mathbb{Q}_p[\theta(\mathfrak{f}'_0)]$ and $K_0 := K_{0,u}[\xi]$. Let \mathcal{O}_0 be the ring of integers of K_0 . Then

$$\pi_{p^{\lfloor b/2 \rfloor}}(R \cap p^{4ar'} \mathcal{O}) = \pi_{p^{\lfloor b/2 \rfloor}}(\mathcal{O}_0 \cap p^{4ar'} \mathcal{O}).$$

Proof of Claim 5. Suppose $x \in \mathcal{O}_0 \cap p^{4ar'} \mathcal{O}$. Then there are $c'_i \in K_{0,u}$ such that $x = \sum_{i=0}^{r'-1} c'_i \xi^i$. By Subclaim (a) (of Claim 4), we have that $v_{\mathfrak{p}}(K_{0,u}^\times) = e\mathbb{Z}$. And, by Claim 4, $v_{\mathfrak{p}}(\xi) = ae + r$. Hence we have that

$$(70) \quad v_{\mathfrak{p}}(c'_i \xi^i) \equiv ri \pmod{e}.$$

By (70) and the fact that $0 \leq ri < e$ for $i \in [0, r' - 1]$, we deduce that $v_{\mathfrak{p}}(c'_i \xi^i)$'s are pairwise distinct; and therefore

$$(71) \quad 5er'a \leq v_{\mathfrak{p}}(x) = v_{\mathfrak{p}}\left(\sum_{i=0}^{r'-1} c'_i \xi^i\right) = \min_{0 \leq i \leq r'-1} \{v_{\mathfrak{p}}(c'_i) + i(ae + r)\}.$$

Hence for any integer i in $[0, r' - 1]$ we have

$$v_{\mathfrak{p}}(c'_i) \in [4er'a - eia - ir, \infty) \cap e\mathbb{Z} = [4er'a - eia, \infty) \cap e\mathbb{Z}$$

which implies $c''_i := c'_i/p^{3ar'} \in \mathcal{O}_{0,u}$. Hence by (69) we have

$$(72) \quad x = p^{3ar'} \sum_{i=0}^{r'-1} c''_i \xi^i \in p^{3ar'} \mathcal{O}_{0,u}[\xi].$$

Therefore by (72) and Claim 4, we have $\pi_{p^{\lfloor b/2 \rfloor}}(x) \in \pi_{p^{\lfloor b/2 \rfloor}}(p^{ar'} \mathcal{O}_{0,u}[\xi] \cap p^{4ar'} \mathcal{O}) = \pi_{p^{\lfloor b/2 \rfloor}}(R \cap p^{4ar'} \mathcal{O})$. And so

$$(73) \quad \pi_{p^{\lfloor b/2 \rfloor}}(\mathcal{O}_0 \cap p^{4ar'} \mathcal{O}) \subseteq \pi_{p^{\lfloor b/2 \rfloor}}(R \cap p^{4ar'} \mathcal{O}).$$

On the other hand, by Claim 4 we have

$$(74) \quad \pi_{p^{\lfloor b/2 \rfloor}}(R \cap p^{4ar'} \mathcal{O}) = \pi_{p^{\lfloor b/2 \rfloor}}(p^{ar'} \mathcal{O}_{0,u}[\xi] \cap p^{4ar'} \mathcal{O}) \subseteq \pi_{p^{\lfloor b/2 \rfloor}}(\mathcal{O}_0 \cap p^{4ar'} \mathcal{O});$$

and by (73) and (74) claim follows. QED.

Claim 6. Suppose R is a subring of \mathcal{O} , and for positive integers a and b we have $a \leq b$ and $\pi_{p^b}(R \cap p^a \mathcal{O}) = \pi_{p^b}(R \cap p^a \mathcal{O}_0)$ where \mathcal{O}_0 is the ring of integers of a subfield K_0 of K . Then $\pi_{p^{b-a}}(R) \subseteq \pi_{p^{b-a}}(\mathcal{O}_0)$.

Proof of Claim 6. For any $x \in R$, there is $x' \in \mathcal{O}_0$ such that $p^a x - x' \in p^b \mathcal{O}$. And so $x - p^{-a} x' \in p^{b-a} \mathcal{O}$, which implies $p^{-a} x' \in K_0 \cap \mathcal{O} = \mathcal{O}_0$. Therefore $\pi_{p^{b-a}}(x) \in \pi_{p^{b-a}}(\mathcal{O}_0)$; and the claim follows. QED.

Claim 5 and Claim 6 imply the assertion of Theorem 7. \square

2.3. Subrings with the same graded structure as the ring \mathcal{O}_0 of integers of a subfield. The main goal of this section is to show knowing $\text{gr}_p(R; \mathcal{O}) = \text{gr}_p(\mathcal{O}_0; \mathcal{O})$ and a bit more information is enough to deduce that $R = \mathcal{O}_0$.

Proposition 19. *Suppose R is a subring of the ring \mathcal{O} of integers of a finite extension K of \mathbb{Q}_p . Suppose N is a positive integer which is at least $6[K : \mathbb{Q}_p]$. Suppose \mathcal{O}_0 is the ring of integers of a closed subfield K_0 of K . If $\pi_{p^{6[K:\mathbb{Q}_p]}}(R) = \pi_{p^{6[K:\mathbb{Q}_p]}}(\mathcal{O}_0)$ and $\text{gr}_{i,p}(R; \mathcal{O}) = \text{gr}_{i,p}(\mathcal{O}_0; \mathcal{O})$ for any integer i in $[0, N - 1]$, then $\pi_{p^{N-4}}(R) = \pi_{p^{N-4}}(\mathcal{O}_0)$.*

Proof. Step 1. (Getting the unramified part) First we notice that (32) holds for $a = 0$ and $b = N$. Let \mathfrak{f}_0 and \mathfrak{f} be the residue fields of \mathcal{O}_0 and \mathcal{O} , respectively. Let \mathfrak{p} be a uniformizing element of \mathcal{O} . As before, let $\theta : \mathfrak{f}^\times \rightarrow \mathcal{O}^\times$ be the group embedding such that $\pi_{\mathfrak{p}}(\theta(x)) = x$ for any $x \in \mathfrak{f}^\times$. So by Claim 2 in the proof of Theorem 7, we have that there is $\widehat{s} : \mathfrak{f}_0^\times \rightarrow R$ such that $\widehat{s}(x) - \theta(x) \in p^N \mathcal{O}$ for any $x \in \mathfrak{f}_0^\times$ (we extend the domain of \widehat{s} to \mathfrak{f} by setting $\widehat{s}(0) := 0$). This implies that $\pi_{p^N}(\theta(\mathfrak{f}_0^\times)) \subseteq \pi_{p^N}(R)$, and so

$$(75) \quad \pi_{p^N}(\mathbb{Z}_p[\theta(\mathfrak{f}_0^\times)]) \subseteq \pi_{p^N}(R).$$

Notice that $K_{0,u} := \mathbb{Q}_p[\theta(\mathfrak{f}_0^\times)]$ is a subfield of K_0 , $K_{0,u}$ is an unramified extension of \mathbb{Q}_p , and K_0 is a totally ramified extension of $K_{0,u}$. And the ring $\mathcal{O}_{0,u}$ of integers of $K_{0,u}$ is $\mathbb{Z}_p[\theta(\mathfrak{f}_0^\times)]$.

Step 2. (Describing elements of R) Suppose \mathfrak{p}_0 be a uniformizing element of \mathcal{O}_0 . Let $e_0 := v_{\mathfrak{p}}(\mathfrak{p}_0)$ and $d_0 := e/e_0$. Since $\pi_{p^{6[K:\mathbb{Q}_p]}}(R) = \pi_{p^{6[K:\mathbb{Q}_p]}}(\mathcal{O}_0)$, there is $\xi_0 \in R$ such that

$$(76) \quad \xi_0 - \mathfrak{p}_0 \in p^{6[K:\mathbb{Q}_p]}\mathcal{O}.$$

And so we have

$$(77) \quad \begin{aligned} \pi_p(\widehat{s}(\mathfrak{f}_0) + \widehat{s}(\mathfrak{f}_0)\xi_0 + \cdots + \widehat{s}(\mathfrak{f}_0)\xi_0^{d_0-1}) &= \mathfrak{f}_0 + \mathfrak{f}_0\pi_p(\mathfrak{p}_0) + \cdots + \mathfrak{f}_0\pi_p(\mathfrak{p}_0^{d_0-1}) \\ &= \pi_p(\theta(\mathfrak{f}_0) + \theta(\mathfrak{f}_0)\mathfrak{p}_0 + \cdots + \theta(\mathfrak{f}_0)\mathfrak{p}_0^{d_0-1}) \\ &= \pi_p(\mathcal{O}_0) = \pi_p(R). \end{aligned}$$

On the other hand, $|\mathrm{gr}_{i,p}(R; \mathcal{O})| = |\mathrm{gr}_{i,p}(\mathcal{O}_0; \mathcal{O})| = |\pi_p(\mathcal{O}_0)|$ for any integer i in $[0, N)$. So by Corollary 17 and (77), we deduce that

$$(78) \quad \mathrm{gr}_{i,p}(R; \mathcal{O}) = \pi_{p^{i+1}}(\widehat{s}(\mathfrak{f}_0)p^i + \widehat{s}(\mathfrak{f}_0)p^i\xi_0 + \cdots + \widehat{s}(\mathfrak{f}_0)p^i\xi_0^{d_0-1}).$$

Since $\widehat{s}(\mathfrak{f}_0) \subseteq R$, $\xi_0 \in R$, and $p \in R$, by (78) we get that

$$(79) \quad \pi_{p^N}(R) = \pi_{p^N} \left(\sum_{i=1}^N \sum_{j=0}^{d_0-1} \widehat{s}(\mathfrak{f}_0)p^i\xi_0^j \right).$$

Step 3. (Finding a zero of a degree d_0 polynomial close to ξ_0) Since $\xi_0^{d_0}$ is in R , by (79) there is a monic polynomial $f_0(x) := \sum_{i=0}^{d_0} c_i x^i \in \mathcal{O}_{0,u}[x]$ of degree d_0 such that

$$(80) \quad f_0(\xi_0) \in p^N \mathcal{O}.$$

As $v_{\mathfrak{p}}(\xi_0) = v_{\mathfrak{p}}(\mathfrak{p}_0) = e_0$ and $v_{\mathfrak{p}}(\mathcal{O}_{0,u} \setminus \{0\}) \subseteq e\mathbb{Z}$, we get that for any integer i in $[1, d_0]$

$$v_{\mathfrak{p}}(ic_i\xi_0^{i-1}) \equiv (i-1)e_0 \pmod{e};$$

and so $v_{\mathfrak{p}}(ic_i\xi_0^{i-1})$ are distinct integers. Hence

$$(81) \quad v_{\mathfrak{p}}(f'_0(\xi_0)) = \min_{1 \leq i \leq d_0} v_{\mathfrak{p}}(ic_i\xi_0^{i-1}) \leq v_{\mathfrak{p}}(d_0\xi_0^{d_0-1}) \leq v_{\mathfrak{p}}(d_0) + (d_0-1)e_0 < 2e.$$

By [Lan94, Chapter II, Section 2, Proposition 2], (80), and (81), there is a zero $\mathfrak{p}'_0 \in \mathcal{O}$ of $f_0(x)$ such that

$$(82) \quad v_{\mathfrak{p}}(\xi_0 - \mathfrak{p}'_0) \geq v_{\mathfrak{p}}(f_0(\xi_0)) - 2v_{\mathfrak{p}}(f'_0(\xi_0)) > (N-4)e.$$

Step 4. (Showing that \mathfrak{p}'_0 is in K_0) By (76), (80), and $f_0(x) \in \mathcal{O}_{0,u}[x]$, we deduce that

$$(83) \quad v_{\mathfrak{p}}(f_0(\mathfrak{p}_0)) \geq 6[K:\mathbb{Q}_p]e.$$

By (76), (81), and $f_0(x) \in \mathcal{O}_{0,u}[x]$, we deduce that

$$(84) \quad v_{\mathfrak{p}}(f'_0(\mathfrak{p}_0)) < 2e.$$

Hence again by [Lan94, Chapter II, Section 2, Proposition 2], (83), and (84), there is a zero $\mathfrak{p}''_0 \in \mathcal{O}$ of $f_0(x)$ such that

$$(85) \quad v_{\mathfrak{p}}(\mathfrak{p}_0 - \mathfrak{p}''_0) \geq v_{\mathfrak{p}}(f_0(\mathfrak{p}_0)) - 2v_{\mathfrak{p}}(f'_0(\mathfrak{p}_0)) > (6[K:\mathbb{Q}_p] - 4)e.$$

By (82) and (85), we have

$$(86) \quad v_{\mathfrak{p}}(\mathfrak{p}'_0 - \mathfrak{p}''_0) \geq (6[K:\mathbb{Q}_p] - 4)e.$$

Hence, if $\mathfrak{p}'_0 \neq \mathfrak{p}''_0$, we get

$$(6[k:\mathbb{Q}_p] - 4)e \leq v_{\mathfrak{p}}(\mathfrak{p}'_0 - \mathfrak{p}''_0) \leq v_{\mathfrak{p}}(f'_0(\mathfrak{p}_0)) < 2e,$$

which is a contradiction. Hence $\mathfrak{p}'_0 = \mathfrak{p}''_0$. Let L be the splitting field of $f_0(x)$ over $K_{0,u}$, and let $|\cdot|$ be the unique extension of $|\cdot|_{\mathfrak{p}}$ to L . Then for any root β of $f_0(x)$ in L that is not equal to \mathfrak{p}'_0 , by (84), (85), and $\mathfrak{p}'_0 = \mathfrak{p}''_0$ we have

$$(87) \quad |\beta - \mathfrak{p}'_0| \geq |f'_0(\mathfrak{p}'_0)| > |\mathfrak{f}|^{-2e} \geq |\mathfrak{f}|^{-(6[K:\mathbb{Q}_p]-4)e} > |\mathfrak{p}_0 - \mathfrak{p}'_0|.$$

Hence by Krasner's lemma (for instance see [Lan94, Chapter II, Section 2, Proposition 3]) we have

$$\mathfrak{p}'_0 \in K_{0,u}[\mathfrak{p}_0] = K_0, \text{ and } v_{\mathfrak{p}}(\mathfrak{p}_0 - \mathfrak{p}'_0) > (6[K : \mathbb{Q}_p] - 4)e.$$

Step 5. (Arguing why $\mathcal{O}_0 = \mathcal{O}_{0,u}[\mathfrak{p}'_0]$) By Step 4, we know that $v_{\mathfrak{p}}(\mathfrak{p}_0) = v_{\mathfrak{p}}(\mathfrak{p}'_0)$ and $\mathfrak{p}'_0 \in K_0$. Hence \mathfrak{p}'_0 is a uniformizing element of K_0 . As K_0 is a purely ramified extension of $K_{0,u}$, we have that $\mathcal{O}_0 = \mathcal{O}_{0,u}[\mathfrak{p}'_0]$.

Step 6. (Finishing proof) By (79) (see Step 1), (82) (see Step 3), and Step 5, we have

$$\pi_{p^{N-4}}(R) = \pi_{p^{N-4}} \left(\sum_{i=1}^N \sum_{j=0}^{d_0-1} \widehat{s}(\mathfrak{f}_0) p^i \mathfrak{p}'_0^j \right) = \pi_{p^{N-4}}(\mathcal{O}_0).$$

□

3. SCALAR-SUM-PRODUCT PHENOMENA.

In this section, using conditional (Shannon) entropy we study *scalar-sum-product properties* of ring of integers \mathcal{O} of a finite extension K of \mathbb{Q}_p .

3.1. Scalar-Sum inequality for regular sets. The main goal of this section is to prove Proposition 20.

Proposition 20 (Scalar-Sum inequality for regular sets). *Let K be a finite extension of \mathbb{Q}_p , \mathcal{O} be its ring of integers, and \mathfrak{f} be its residue field. Let $\Omega \subseteq \mathcal{O}$ be such that $\pi_{\mathfrak{p}}$ induces a bijection between Ω and \mathfrak{f}^{\times} .*

Let A and B be (m_0, \dots, m_{N-1}) -regular and (l_0, \dots, l_{N-1}) -regular¹ subsets of $\pi_{\mathfrak{p}^N}(\mathcal{O})$, respectively. Then

$$\max_{\omega \in \Omega} |A + \pi_{\mathfrak{p}^N}(\omega)B| \geq \prod_{i=0}^{N-1} \max \left(1, \left(\frac{1}{m_i l_i} + \frac{1}{|\mathfrak{f}|} \right)^{-1} \right).$$

Let us fix a subset $\Omega \subseteq \mathcal{O}^{\times}$ such that $\pi_{\mathfrak{p}}$ induces a bijection between Ω and \mathfrak{f}^{\times} . As it was mentioned in Section 1.4, for any element $X \in \mathcal{O}$, there are unique $D_{i,\Omega}(X) \in \Omega \cup \{0\}$ such that

$$X = D_{0,\Omega}(X) + \mathfrak{p}D_{1,\Omega}(X) + \mathfrak{p}^2D_{2,\Omega}(X) + \dots;$$

and we call $D_{i,\Omega}(X)$ the i -th \mathfrak{p} -adic digit with respect to Ω . We fix Ω at the beginning of each given proof and write $D_i(X)$ instead of $D_{i,\Omega}(X)$. Again as explained in Section 1.4, we can and will talk about the i -th \mathfrak{p} -adic digit of an element X of $\pi_{\mathfrak{p}^N}(\mathcal{O})$ for any integer i in $[0, N-1]$; and we have

$$X = D_{0,\Omega}(X) + \mathfrak{p}D_{1,\Omega}(X) + \dots + \mathfrak{p}^{N-1}D_{N-1,\Omega}(X) + \mathfrak{p}^N \mathcal{O}.$$

Definition 21. *A subset A of $\pi_{\mathfrak{p}^N}(\mathcal{O})$ is called an $(m_0, m_1, \dots, m_{N-1})$ -regular subset if for any $0 \leq n \leq N-1$ and $\bar{x} := \mathfrak{p}^n \mathcal{O} + x$ we have that either $\bar{x} \not\subseteq \pi_{\mathfrak{p}^n}(A)$ or*

$$|\pi_{\mathfrak{p}^{n+1}}(A) \cap \pi_{\mathfrak{p}^{n+1}}(\mathfrak{p}^n \mathcal{O} + x)| = m_n.$$

The following Lemma gives us a good way of thinking about regular subsets.

Lemma 22. *Let A be an (m_0, \dots, m_{N-1}) -regular subset of $\pi_{\mathfrak{p}^N}(\mathcal{O})$. Let X be a random variable with respect to the probability counting measure on A . Then*

- (a) $\pi_{\mathfrak{p}^k}(X)$ is a random variable with respect to the probability counting measure on $\pi_{\mathfrak{p}^k}(A)$; and
- (b) for any $a \in A$ and any $0 \leq k \leq N-1$, the conditional probability measure

$$P(D_k(X) | D_0(X) = D_0(a), \dots, D_{k-1}(X) = D_{k-1}(a))$$

is a probability counting measure on a set of size m_k .

Proof. Both of the above claims are easy consequences of the fact that A is a regular set. □

¹For the definition of a regular set, see Definition 21.

As explained in Section 1.4 (Step 2), to prove Proposition 20, we work with random variables X and Y that are distributed according to the probability counting measures on the sets A and B , respectively. And we use basic properties of (Shannon) entropy and conditional entropy. Here we recall their definitions and basic properties.

Definition 23. Let X be a random variable on a finite set \mathcal{X} .

(a) The (Shannon) entropy $H(X)$ of X is

$$H(X) := \sum_{x \in \mathcal{X}} -\mathbb{P}(X = x) \log \mathbb{P}(X = x),$$

where $\mathbb{P}(X = x)$ is the probability of having $X = x$.

(b) Suppose Y is another random variable on \mathcal{X} . Then the entropy of X conditioned to Y is

$$H(X|Y) := \sum_{y \in \mathcal{X}} \mathbb{P}(Y = y) H(X|Y = y) = - \sum_{y \in \mathcal{X}} \mathbb{P}(Y = y) \sum_{x \in \mathcal{X}} \mathbb{P}(X = x|Y = y) \log \mathbb{P}(X = x|Y = y),$$

where $X|Y = y$ is the random variable X conditioned to the random variable Y taking a certain value y , and $\mathbb{P}(X = x|Y = y)$ is the probability of having $X = x$ conditioned to $Y = y$.

Here are some of the basic properties of entropy that will be used in this note.

Lemma 24. Suppose \mathcal{X} is a finite set, and X and Y are random variables with values in \mathcal{X} . Then

(a) $H(X, Y) = H(X) + H(Y|X)$.

(b) $H(X) \geq H(X|Y)$.

(c) $H(f(X)|X) = 0$ where f is a function; and so $H(Y|X, f(X)) = H(Y|X)$.

(d) Let $H_2(X) := -\log \sum_{x \in \mathcal{X}} \mathbb{P}(X = x)^2$; this is called the Rényi entropy. Let $H_0(X) := \log |\mathcal{X}|$, where $|\mathcal{X}|$ is the size of the support of X . Then

$$H_2(X) \leq H(X) \leq H_0(X).$$

(e) $H(X|f(Y)) \geq H(X|Y)$ where f is a function.

Proof. These are all well-known facts; for instance for parts (a)-(d) see [CT06, Theorem 2.4.1, Theorem 2.5.1, Theorem 2.6.4, Lemma 2.10.1, Problem 2.1]. Part (e) is a consequence of parts (b) and (c):

$$H(X|f(Y)) \geq H(X|f(Y), Y) = H(X|Y).$$

□

Lemma 25. Suppose \mathcal{O} is the ring of integers of a finite extension K of \mathbb{Q}_p . Let X and Y be two random variables with values in $\pi_{\mathfrak{p}^N}(\mathcal{O})$. Suppose Ω is a subset of \mathcal{O} such that $\pi_{\mathfrak{p}}$ induces a bijection from Ω to the set \mathfrak{f}^\times of non-zero elements of the residue field \mathfrak{f} . Let α be a random variable with respect to the probability counting measure on Ω ; and $\bar{\alpha} := \pi_{\mathfrak{p}^N}(\alpha)$. Then for any integer m in $[0, N - 1]$ we have

$$\begin{aligned} H(D_m(X + \bar{\alpha}Y)|\alpha, D_0(X + \bar{\alpha}Y), \dots, D_{m-1}(X + \bar{\alpha}Y)) \\ \geq H(\pi_{\mathfrak{p}}(D_m(X)) + \pi_{\mathfrak{p}}(\alpha)\pi_{\mathfrak{p}}(D_m(Y))|\pi_{\mathfrak{p}}(\alpha), D_0(X), \dots, D_{m-1}(X), D_0(Y), \dots, D_{m-1}(Y)), \end{aligned}$$

where $D_i(Z) := D_{i,\Omega}(Z)$ is the i -th \mathfrak{p} -adic digit of Z with respect to $\Omega \cup \{0\}$.

Proof. For a fixed α , let $\sigma_{m,\alpha} : ((\Omega \cup \{0\}) \times (\Omega \cup \{0\}))^m \rightarrow \Omega \cup \{0\}$ be the following carry over function:

$$(88) \quad \text{for any } a_0, \dots, a_{m-1}, b_0, \dots, b_{m-1} \in \Omega \cup \{0\}, \sigma_{m,\alpha}(\{(a_i, b_i)\}_{i=0}^{m-1}) := D_m\left(\sum_{i=0}^{m-1} a_i \mathfrak{p}^i + \alpha \sum_{i=0}^{m-1} b_i \mathfrak{p}^i\right).$$

Let $X_{m-1} := \sum_{i=0}^{m-1} D_i(X) \mathfrak{p}^i$ and $Y_{m-1} := \sum_{i=0}^{m-1} D_i(Y) \mathfrak{p}^i$. Then

$$\begin{aligned} X + \bar{\alpha}Y &\equiv (X_{m-1} + D_m(X)\mathfrak{p}^m) + \bar{\alpha}(Y_{m-1} + D_m(Y)\mathfrak{p}^m) \equiv (X_{m-1} + \bar{\alpha}Y_{m-1}) + (D_m(X) + \bar{\alpha}D_m(Y))\mathfrak{p}^m \\ &\equiv \sum_{i=0}^{m-1} D_i(X_{m-1} + \bar{\alpha}Y_{m-1})\mathfrak{p}^i + (\sigma_{m,\alpha}(\{(D_i(X), D_i(Y))\}_{i=0}^{m-1}) + D_m(X) + \bar{\alpha}D_m(Y))\mathfrak{p}^m \pmod{\mathfrak{p}^{m+1}}. \end{aligned}$$

Therefore $D_m(X + \bar{\alpha}Y)$ is uniquely determined by

$$\pi_{\mathfrak{p}}(\sigma_{m,\alpha}(\{(D_i(X), D_i(Y))\}_{i=0}^{m-1}) + D_m(X) + \bar{\alpha}D_m(Y));$$

and vice versa. Hence we have

$$\begin{aligned} H_m &:= H(D_m(X + \bar{\alpha}Y)|\alpha, D_0(X + \bar{\alpha}Y), \dots, D_{m-1}(X + \bar{\alpha}Y)) \\ &= H(\pi_{\mathfrak{p}}(\sigma_{m,\alpha}(\{(D_i(X), D_i(Y))\}_{i=0}^{m-1}) + D_m(X) + \bar{\alpha}D_m(Y))|\alpha, D_0(X + \bar{\alpha}Y), \dots, D_{m-1}(X + \bar{\alpha}Y)) \end{aligned}$$

So by Lemma 24, part (e), and the fact that the first $m - 1$ \mathfrak{p} -adic digits of $X + \bar{\alpha}Y$ are determined by $\{(D_i(X), D_i(Y))\}_{i=0}^{m-1}$ and α , we have

$$H_m \geq H(\pi_{\mathfrak{p}}(\sigma_{m,\alpha}(\{(D_i(X), D_i(Y))\}_{i=0}^{m-1}) + D_m(X) + \bar{\alpha}D_m(Y))|\alpha, \{(D_i(X), D_i(Y))\}_{i=0}^{m-1}).$$

And since for a given α and $\{(D_i(X), D_i(Y))\}_{i=0}^{m-1}$, $\pi_{\mathfrak{p}}(\sigma_{m,\alpha}(\{(D_i(X), D_i(Y))\}_{i=0}^{m-1}) + D_m(X) + \bar{\alpha}D_m(Y))$ is uniquely determined by $\pi_{\mathfrak{p}}(D_m(X) + \bar{\alpha}D_m(Y))$ and vice versa, we have

$$H_m \geq H(\pi_{\mathfrak{p}}(D_m(X) + \bar{\alpha}D_m(Y))|\alpha, \{(D_i(X), D_i(Y))\}_{i=0}^{m-1}).$$

And since $\pi_{\mathfrak{p}}$ induces a bijection between $\Omega \cup \{0\}$ and \mathfrak{f} , the claim follows. \square

Corollary 26. *Suppose \mathcal{O} is the ring of integers of a finite extension K of \mathbb{Q}_p . Let X and Y be two random variables with values in $\pi_{\mathfrak{p}^N}(\mathcal{O})$. Suppose Ω is a subset of \mathcal{O} such that $\pi_{\mathfrak{p}}$ induces a bijection from Ω to the set \mathfrak{f}^\times of non-zero elements of the residue field \mathfrak{f} . Let α be a random variable with respect to the probability counting measure on Ω ; and $\bar{\alpha} := \pi_{\mathfrak{p}^N}(\alpha)$. Then*

$$H(X + \bar{\alpha}Y|\alpha) \geq \sum_{m=0}^{N-1} \bar{H}_m,$$

where $\bar{H}_m := H(\pi_{\mathfrak{p}}(D_m(X)) + \pi_{\mathfrak{p}}(\alpha)\pi_{\mathfrak{p}}(D_m(Y))|\pi_{\mathfrak{p}}(\alpha), \{(D_i(X), D_i(Y))\}_{i=0}^{m-1})$.

Proof. Since $X + \bar{\alpha}Y$ is uniquely determined by its first $N - 1$ \mathfrak{p} -adic digits and vice versa, we have

$$H(X + \bar{\alpha}Y|\alpha) = H(\{D_i(X + \bar{\alpha}Y)\}_{i=0}^{N-1}|\alpha).$$

And so by Lemma 24, part (a), we have

$$(89) \quad H(X + \bar{\alpha}Y|\alpha) = \sum_{m=0}^{N-1} H_m,$$

where $H_m := H(D_m(X + \bar{\alpha}Y)|\alpha, \{D_i(X + \bar{\alpha}Y)\}_{i=0}^{m-1})$. Therefore by (89) and Lemma 25 we get

$$H(X + \bar{\alpha}Y|\alpha) \geq H(\pi_{\mathfrak{p}}(D_m(X)) + \pi_{\mathfrak{p}}(\alpha)\pi_{\mathfrak{p}}(D_m(Y))|\pi_{\mathfrak{p}}(\alpha), \{(D_i(X), D_i(Y))\}_{i=0}^{m-1});$$

and the claim follows. \square

Based on Corollary 26, we see the need of having a lower bound for the entropy of a linear combination of two random variables with values in the residue field \mathfrak{f} . As it was pointed out in Section 1.4 (see Question 8), at this point we do not know the answer to this question for arbitrary random variables. The following proposition partially answers this question.

Proposition 27. *Let \bar{A} and \bar{B} be two non-empty subsets of a finite field \mathfrak{f} . Suppose \bar{X} and \bar{Y} are two random variables with values in \mathfrak{f} with respect to the probability counting measures on \bar{A} and \bar{B} , respectively. Suppose $\bar{\alpha}$ is a uniform random variable with values in \mathfrak{f}^\times . Then*

$$H(\bar{X} + \bar{\alpha}\bar{Y}|\bar{\alpha}) \geq -\log\left(\frac{1}{|\bar{A}||\bar{B}|} + \frac{1}{|\mathfrak{f}|}\right) \geq \min\{H(\bar{X}) + H(\bar{Y}), \log|\mathfrak{f}|\} - \log 2.$$

To prove Proposition 27, we start with a Lemma that helps us control an average of the Rényi entropies $H_2(\bar{X} + c\bar{Y})$ as c varies in \mathfrak{f}^\times .

Lemma 28. *Let \bar{A}, \bar{B} be two non-empty subsets of a finite field \mathfrak{f} . Let $\mathcal{P}_{\bar{A}}$ and $\mathcal{P}_{\bar{B}}$ be the probability counting measures on \bar{A} and \bar{B} , respectively. For two functions $f, g : \mathfrak{f} \rightarrow \mathbb{C}$, let $f * g(x) := \sum_{y \in \mathfrak{f}} f(y)g(x-y)$ be the convolution of f and g . Then*

$$\frac{1}{|\mathfrak{f}^\times|} \sum_{c \in \mathfrak{f}^\times} \|\mathcal{P}_{\bar{A}} * c\mathcal{P}_{\bar{B}}\|_2^2 \leq \min\left(1, \frac{1}{|\bar{A}||\bar{B}|} + \frac{1}{|\mathfrak{f}|}\right),$$

where $c\mathcal{P}_{\bar{B}}$ is the push-forward of $\mathcal{P}_{\bar{B}}$ under the multiplication by c .

Proof. Let us recall that for any two subsets U and V of \mathfrak{f} , the additive energy of U and V is

$$E(U, V) = |\{(x_1, y_1, x_2, y_2) \in U \times V \times U \times V \mid x_1 + y_1 = x_2 + y_2\}|,$$

and we have

$$E(X, Y) = \|\mathbf{1}_X * \mathbf{1}_Y\|_2^2.$$

where $\mathbf{1}_X$ is the characteristic function of the set X . Hence we get

$$\begin{aligned} \frac{1}{|\mathfrak{f}^\times|} \sum_{\alpha \in \mathfrak{f}^\times} \|\mathcal{P}_{\bar{A}} * \alpha\mathcal{P}_{\bar{B}}\|_2^2 &= \frac{1}{|\mathfrak{f}^\times||\bar{A}|^2|\bar{B}|^2} \sum_{\alpha \in \mathfrak{f}^\times} \|\mathbf{1}_{\bar{A}} * \mathbf{1}_{\alpha\bar{B}}\|_2^2 = \frac{1}{|\mathfrak{f}^\times||\bar{A}|^2|\bar{B}|^2} \sum_{\alpha \in \mathfrak{f}^\times} E(\bar{A}, \alpha\bar{B}) \\ &= \frac{1}{|\mathfrak{f}^\times||\bar{A}|^2|\bar{B}|^2} \sum_{\alpha \in \mathfrak{f}^\times} |\{(a_1, b_1, a_2, b_2) \in \bar{A} \times \bar{B} \times \bar{A} \times \bar{B} \mid a_1 + \alpha b_1 = a_2 + \alpha b_2\}| \\ &= \frac{1}{|\mathfrak{f}^\times||\bar{A}|^2|\bar{B}|^2} \sum_{\alpha \in \mathfrak{f}^\times} |\{(a_1, b_1, a_2, b_2) \in \bar{A} \times \bar{B} \times \bar{A} \times \bar{B} \mid a_1 = a_2, a_1 + \alpha b_1 = a_2 + \alpha b_2\}| \\ &\quad + \frac{1}{|\mathfrak{f}^\times||\bar{A}|^2|\bar{B}|^2} \sum_{\alpha \in \mathfrak{f}^\times} |\{(a_1, b_1, a_2, b_2) \in \bar{A} \times \bar{B} \times \bar{A} \times \bar{B} \mid a_1 \neq a_2, a_1 + \alpha b_1 = a_2 + \alpha b_2\}| \\ &= \frac{1}{|\bar{A}||\bar{B}|} \\ &\quad + \frac{1}{|\mathfrak{f}^\times||\bar{A}|^2|\bar{B}|^2} \sum_{\alpha \in \mathfrak{f}^\times} |\{(a_1, b_1, a_2, b_2) \in \bar{A} \times \bar{B} \times \bar{A} \times \bar{B} \mid a_1 \neq a_2, b_1 \neq b_2, \alpha = \frac{a_1 - a_2}{b_2 - b_1}\}| \\ &= \frac{1}{|\bar{A}||\bar{B}|} + \frac{|(\bar{A}^2 \setminus \Delta(\bar{A})) \times (\bar{B}^2 \setminus \Delta(\bar{B}))|}{|\mathfrak{f}^\times||\bar{A}|^2|\bar{B}|^2}, \text{ where } \Delta(X) := \{(x, x) \mid x \in X\}, \\ &= \frac{1}{|\bar{A}||\bar{B}|} + \frac{(|\bar{A}| - 1)(|\bar{B}| - 1)}{|\mathfrak{f}^\times||\bar{A}||\bar{B}|} \leq \min\left(1, \frac{1}{|\bar{A}||\bar{B}|} + \frac{1}{|\mathfrak{f}|}\right). \end{aligned}$$

□

Proof of Proposition 27. Step 1. (Entropy \geq Rényi entropy) Using Lemma 24, part (d), we have

$$(90) \quad H(\bar{X} + \bar{\alpha}\bar{Y}|\bar{\alpha}) = \frac{1}{|\mathfrak{f}^\times|} \sum_{c \in \mathfrak{f}^\times} H(\bar{X} + c\bar{Y}) \geq \frac{1}{|\mathfrak{f}^\times|} \sum_{c \in \mathfrak{f}^\times} H_2(\bar{X} + c\bar{Y}) = -\frac{1}{|\mathfrak{f}^\times|} \sum_{c \in \mathfrak{f}^\times} \log \|\mathcal{P}_{\bar{A}} * c\mathcal{P}_{\bar{B}}\|_2^2.$$

Step 2. (Convexity of $-\log$) By the convexity of $-\log$ function and Jensen's inequality, we have

$$(91) \quad -\log \left(\frac{1}{|\mathfrak{f}^\times|} \sum_{c \in \mathfrak{f}^\times} \|\mathcal{P}_{\bar{A}} * c\mathcal{P}_{\bar{B}}\|_2^2 \right) \leq -\frac{1}{|\mathfrak{f}^\times|} \sum_{c \in \mathfrak{f}^\times} \log \|\mathcal{P}_{\bar{A}} * c\mathcal{P}_{\bar{B}}\|_2^2.$$

Step 3. (Finishing the proof) By Lemma 28, (90), and (91), we get that

$$H(\bar{X} + \bar{\alpha}\bar{Y}|\bar{\alpha}) \geq -\log \left(\frac{1}{|\bar{A}||\bar{B}|} + \frac{1}{|\mathfrak{f}|} \right).$$

And since $H(\bar{X}) = \log |\bar{A}|$ and $H(\bar{Y}) = \log |\bar{B}|$, claim follows. \square

Proof of Proposition 20. For a random variable Z with values in a set \mathcal{Z} and a function $f : \mathcal{Z} \rightarrow \mathbb{C}$, let $\mathbb{E}_Z(f)$ be the expectation of the random variable $f(Z)$; for instance for a function $f : \Omega \rightarrow \mathbb{C}$, we have $\mathbb{E}_\alpha(f) = \frac{1}{|\mathfrak{f}^\times|} \sum_{c \in \mathfrak{f}^\times} f(c)$.

Let X and Y be uniform random variables on the sets A and B , respectively. Suppose Ω is a subset of \mathcal{O} such that $\pi_{\mathfrak{p}}$ induces a bijection from Ω to \mathfrak{f}^\times .

Step 1. (From cardinality to entropy) Using Lemma 24, part (d), we have

$$(92) \quad \max_{c \in \Omega} \log |A + cB| \geq \mathbb{E}_\alpha(\log |A + \alpha B|) \geq \mathbb{E}_\alpha(H(X + \alpha Y)) = H(X + \alpha Y|\alpha).$$

Step 2. (Entropy to relative entropies of digits) By Corollary 26, we have

$$(93) \quad H(X + \bar{\alpha}Y|\alpha) \geq \sum_{m=0}^{N-1} \bar{H}_m,$$

where $\bar{H}_m := H(\pi_{\mathfrak{p}}(D_m(X)) + \pi_{\mathfrak{p}}(\alpha)\pi_{\mathfrak{p}}(D_m(Y))|\pi_{\mathfrak{p}}(\alpha), \{(D_i(X), D_i(Y))\}_{i=0}^{m-1})$.

Step 3. (Regularity and bound for relative entropies) By Lemma 22, for any $a \in A$, $b \in B$, and integer k in $[0, N-1]$, the conditional probability measures

$$\mathbb{P}(\pi_{\mathfrak{p}}(D_k(X))|\{D_i(X)\}_{i=0}^{k-1} = \{D_i(a)\}_{i=0}^{k-1}) \text{ and } \mathbb{P}(\pi_{\mathfrak{p}}(D_k(Y))|\{D_i(Y)\}_{i=0}^{k-1} = \{D_i(b)\}_{i=0}^{k-1})$$

are probability counting measures on sets of size m_k and l_k , respectively. And so by Proposition 27 we have

$$(94) \quad \bar{H}_k \geq \max \left\{ 0, -\log \left(\frac{1}{m_k l_k} + \frac{1}{|\mathfrak{f}|} \right) \right\}.$$

Step 4. (Finishing the proof) By (92), (93), and (94), we get

$$\max_{c \in \Omega} \log |A + cB| \geq \sum_{k=0}^{N-1} \max \left\{ 0, -\log \left(\frac{1}{m_k l_k} + \frac{1}{|\mathfrak{f}|} \right) \right\} = \log \left(\prod_{k=0}^{N-1} \max \left\{ 1, \left(\frac{1}{m_k l_k} + \frac{1}{|\mathfrak{f}|} \right)^{-1} \right\} \right);$$

and the claim follows. \square

3.2. Scalar-Sum-Product expansion for regular sets. The following is the main result of this section.

Proposition 29 (Scalar-Sum-Product expansion for regular sets). *For any positive number ε , positive integer d , $0 < \delta \ll \varepsilon^4$, and any finite extension K of \mathbb{Q}_p with large, depending on ε , residue field \mathfrak{f} the following holds:*

Let \mathcal{O} be the ring of integers of K , and \mathfrak{p} be a uniformizing element of K . Let $\Omega \subseteq \mathcal{O}$, and suppose $\pi_{\mathfrak{p}}$ induces a bijection between $\Omega \subseteq \mathcal{O}$ and \mathfrak{f}^\times . Suppose $A \subseteq \pi_{\mathfrak{p}^N}(\mathcal{O})$ satisfies the following properties:

- (a) A is an (m_0, \dots, m_{N-1}) -regular subset; and m_0, m_1 are more than 1.
- (b) $|A| \leq |\mathfrak{f}|^{N(1-\varepsilon)}$.
- (c) $|\pi_{\mathfrak{p}^l}(A)| \geq |\mathfrak{f}|^{l\varepsilon}$ for any $N\delta \leq l \leq N$.

Then

$$\max_{\alpha \in \Omega, a \in A-A} |A + \pi_{\mathfrak{p}^N}(\alpha)aA| \geq |A||\mathfrak{f}|^{N\delta}.$$

We prove Proposition 29 by contradiction. For the rest of this section, A and m_i 's satisfy all the conditions of Proposition 29. Moreover we assume to the contrary that $|A + \pi_{\mathfrak{p}^N}(\alpha)aA| < |A||\mathfrak{f}|^{N\delta}$ (for a small enough δ to be determined later) for any $\alpha \in \Omega$ and $a \in A - A$, and let

$$(95) \quad x_i := \frac{\log m_i}{\log |\mathfrak{f}|}$$

for any $0 \leq i \leq N - 1$.

Lemma 30. *Let x_i 's be as in (95). Then*

$$(96) \quad x_0, x_1 \neq 0 \text{ and } 0 \leq x_i \leq 1 \text{ for any } 0 \leq i \leq N - 1,$$

$$(97) \quad \sum_{i=0}^{N-1} x_i \leq N(1 - \varepsilon),$$

$$(98) \quad \sum_{i=0}^{l-1} x_i \geq l\varepsilon, \text{ for any } N\delta \leq l \leq N,$$

$$(99) \quad \sum_{i=0}^{N-1-k} \min(x_i, 1 - x_{i+k}) \leq N \left(\delta + \frac{\log 2}{\log |\mathfrak{f}|} \right) \text{ if } x_k \neq 0.$$

Proof. Since $1 \leq m_i \leq |\mathfrak{f}|$, $|A| = \prod_{i=0}^{N-1} m_i \leq |\mathfrak{f}|^{N(1-\varepsilon)}$, and $|\pi_{\mathfrak{p}^l}(A)| = \prod_{i=0}^{l-1} m_i \geq |\mathfrak{f}|^{l\varepsilon}$ for $N\delta \leq l \leq N$, one can see that (96), (97), and (98) hold.

Suppose $x_k \neq 0$. So there is $a \in A - A$ such that $a \in \pi_{\mathfrak{p}^N}(\mathfrak{p}^k \mathcal{O}) \setminus \pi_{\mathfrak{p}^N}(\mathfrak{p}^{k+1} \mathcal{O})$. Hence aA is an $(1, \dots, 1, m_0, \dots, m_{N-1-k})$ -regular subset of $\pi_{\mathfrak{p}^N}(\mathcal{O})$. Let $m_{-i} = 1$ for any $i \in \mathbb{Z}^+$. Therefore by Proposition 20 we have

$$\begin{aligned} \max_{\alpha \in \Omega} |A + \pi_{\mathfrak{p}^N}(\alpha)aA| &\geq \prod_{i=0}^{N-1} \max \left(1, \left(\frac{1}{m_i m_{i-k}} + \frac{1}{|\mathfrak{f}|} \right)^{-1} \right) \\ &\geq \left(\prod_{i=0}^{N-1} m_i \right) \prod_{i=0}^{N-1} \left(\frac{1}{m_{i-k}} + \frac{m_i}{|\mathfrak{f}|} \right)^{-1} \\ &\geq |A| \prod_{i=0}^{N-1} \frac{\min(m_{i-k}, |\mathfrak{f}|/m_i)}{2}. \end{aligned}$$

Thus by the contrary assumption we have

$$N\delta \log |\mathfrak{f}| \geq -N \log 2 + \sum_{i=0}^{N-1} \min(\log m_{i-k}, \log |\mathfrak{f}| - \log m_i).$$

And so

$$N \left(\delta + \frac{\log 2}{\log |\mathfrak{f}|} \right) \geq \sum_{i=0}^{N-1-k} \min(x_i, 1 - x_{i+k}).$$

□

Now we follow Lindenstrauss-Varjú's treatment [LV] to prove that, if $|\mathfrak{f}| \gg_\varepsilon 1$ and $0 < \delta \ll \varepsilon^4$, then there are no real numbers x_0, \dots, x_{N-1} that satisfy properties mentioned in Lemma 30. This is based on Mann's theorem on Schnirelmann density of subsets of non-negative integers.

Definition 31. The Schnirelmann density $\sigma(X)$ of a non-empty subset X of non-negative integers is

$$\sigma(X) := \inf_{n \in \mathbb{Z}^+} \frac{|X \cap [1, n]|}{n}.$$

Theorem 32 (Mann's Theorem). *Let X, Y be two non-empty subsets of non-negative integers. Suppose X and Y contain 0. Then either $X + Y = \mathbb{Z}^{\geq 0}$ or $\sigma(X + Y) \geq \sigma(X) + \sigma(Y)$.*

Before we proceed with the proof of Proposition 29, let us recall the definition of the i -th grade $\text{gr}_{i, \mathfrak{p}}(A)$ of A with respect to powers of \mathfrak{p} . For any non-negative integer i , we let $\text{gr}_{i, \mathfrak{p}}(A) := \pi_{\mathfrak{p}^{i+1}}(\tilde{A} \cap \mathfrak{p}^i \mathcal{O}) \subseteq \mathfrak{p}^i \mathcal{O} / \mathfrak{p}^{i+1} \mathcal{O}$ where $\tilde{A} := \pi_{\mathfrak{p}^N}^{-1}(A)$. Then for $\{x_k\}_{k=0}^{N-1}$ as in Equation (95), we have $\text{gr}_{k, \mathfrak{p}}(A - A) \neq 0$ if and only if $x_k \neq 0$. Let

$$(100) \quad J := \{k \in [0, N] \mid x_k \neq 0\}.$$

Lemma 33 (Lindenstrauss-Varjú [LV]). *Let x_i 's be real numbers that satisfy conditions (96) and (98) of Lemma 30. Then we have*

$$(N \lceil 1/\varepsilon \rceil \delta, N) \cap \mathbb{Z} \subseteq \sum_{3 \lceil 1/\varepsilon \rceil} J.$$

Proof. By (96) and (98), we have that

$$(101) \quad |(J + 1) \cap [1, l]| \geq \sum_{i=0}^{l-1} x_i \geq l\varepsilon$$

for any $l \in [N\delta, N)$. Let k_0 be the largest integer such that $|(J + 1) \cap [1, k_0]| < k_0\varepsilon$. Hence, by (101), we have $k_0 < \delta N$. For any $k_0 < l \leq N$ we deduce that

$$(102) \quad |(J + 1) \cap [k_0 + 1, l]| = |(J + 1) \cap [1, l]| - |(J + 1) \cap [1, k_0]| \geq (l - k_0)\varepsilon;$$

in particular applying (102) for $l = k_0 + 1$ we get that $k_0 \in J$. Next, we let

$$X := \{j - k_0 + 1 \mid j \in J, j \geq k_0 - 1\} \cup \{k \in \mathbb{Z} \mid k \geq N - k_0\}.$$

By (102) we have $|(J - k_0 + 1) \cap [1, l - k_0]| \geq (l - k_0)\varepsilon$ for any $k_0 < l \leq N$. Hence we have that the Schnirelmann density $\sigma(X)$ of X is at least ε . Therefore by Mann's theorem (Theorem 32) we have

$$\mathbb{Z}^{\geq 0} = \sum_{\lceil 1/\varepsilon \rceil} (X \cup \{0\}).$$

So for any integer $\lceil 1/\varepsilon \rceil \delta N < m < N$ there are $t \leq \lceil 1/\varepsilon \rceil$ elements of X that add up to $m - \lceil 1/\varepsilon \rceil k_0$. Since $m - \lceil 1/\varepsilon \rceil k_0 < N - k_0$, there are $j_1, \dots, j_t \in J \cap [k_0, \infty)$ such that

$$(j_1 - k_0 + 1) + \dots + (j_t - k_0 + 1) = m - \lceil 1/\varepsilon \rceil k_0.$$

Thus we have

$$m = j_1 + \dots + j_t + (\lceil 1/\varepsilon \rceil - t)k_0 + t \in \sum_{3 \lceil 1/\varepsilon \rceil} J,$$

as $k_0, 0, 1 \in J$. □

For $\{x_k\}_{k=0}^{N-1}$ as in (95), let

$$(103) \quad L := \{k \in [0, N] \mid x_k \geq 1/2\}.$$

In particular, $L \subseteq J$. So L consists of indexes i , where the i -th grade $\text{gr}_{i, \mathfrak{p}}(A - A)$ of $A - A$ is *large*. Now the idea is that the contrary assumption implies when we shift L by an element of J we cannot get *lots of new* indexes; the almost invariance of L under the shifts by elements of J leads us to a contradiction. Let $D_L(k)$ be the number of new elements that are gained after a k -shift of L ; that means

$$(104) \quad D_L(k) := |(L + k \cap [0, N]) \setminus L|.$$

It is useful to notice that $D_L(k) = |(L \cap [0, N - k]) \setminus (L - k)|$. Here is the main property of the sets J, L , and the function $D_L(k)$.

Lemma 34. *Suppose $k \in J$. Then*

$$D_L(k) \leq 2N \left(\delta + \frac{\log 2}{\log |\mathfrak{f}|} \right),$$

where $D_L(k)$ is as in (104).

Proof. Suppose $i \in (L \cap [0, N - k]) \setminus (L - k)$. Then

$$(105) \quad \min(x_i, 1 - x_{i+k}) \geq 1/2.$$

On the other hand, since $k \in J$, $x_k \neq 0$. Therefore by (99) we have

$$\begin{aligned} \frac{D_L(k)}{2} &= \frac{|(L \cap [0, N - k]) \setminus (L - k)|}{2} \leq \sum_{i \in (L \cap [0, N - k]) \setminus (L - k)} \min(x_i, 1 - x_{i+k}) \\ &\leq \sum_{i=0}^{N-k-1} \min(x_i, 1 - x_{i+k}) \leq N \left(\delta + \frac{\log 2}{\log |\mathfrak{f}|} \right). \end{aligned}$$

□

Lemma 35. [LV] *For any pair of positive integers $k_1, k_2 < N$ we have $D_L(k_1 + k_2) \leq D_L(k_1) + D_L(k_2)$.*

Proof. For any three sets A, B , and C we have $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$. Therefore

$$\begin{aligned} D_L(k_1 + k_2) &= |(L \cap [0, N - 1 - k_1 - k_2]) \setminus (L - k_1 - k_2)| \\ &\leq |(L \cap [0, N - 1 - k_1 - k_2]) \setminus ((L - k_1) \cap [0, N - 1 - k_1 - k_2])| \\ &\quad + |((L - k_1) \cap [0, N - 1 - k_1 - k_2]) \setminus (L - k_1 - k_2)| \\ &\leq D_L(k_1) + D_L(k_2). \end{aligned}$$

□

Lemma 36. [LV] *For some universal implied constants we have*

$$\frac{1}{N} \sum_{k=0}^{N-1} D_L(k) \geq N\varepsilon^3/16$$

if $0 < \varepsilon \ll 1$, $\delta \ll \varepsilon^2$ and $1 \ll_\varepsilon |\mathfrak{f}|$.

Proof. By (99) we have

$$(106) \quad N \left(\delta + \frac{\log 2}{\log |\mathfrak{f}|} \right) \geq \sum_{i=0}^{N-1} \min(x_i, 1 - x_i) \geq \sum_{i \in [0, N] \setminus L} x_i.$$

By (98), for any integer $l \in [N\delta, N]$, we have

$$l\varepsilon \leq \sum_{i=0}^{l-1} x_i \leq \sum_{i \in [0, N] \setminus L} x_i + |L \cap [0, l]|.$$

Hence by (106) we have

$$|L \cap [0, l]| \geq l \left(\varepsilon - \frac{N}{l} \left(\delta + \frac{\log 2}{\log |\mathfrak{f}|} \right) \right).$$

Suppose $\max(\delta, \frac{\log 2}{\log |\mathfrak{f}|}) < \varepsilon^2/16$. Then we have

$$(107) \quad |L \cap [0, N\varepsilon/4]| \geq N\varepsilon^2/8.$$

For any $i \in (N\varepsilon/4, N)$ and any $k \in L \cap [0, N\varepsilon/4)$ we have $i \in (L \cap [0, N\varepsilon/4)) + (i - k)$. Hence for any integer $i \in (N\varepsilon/4, N)$ we have

$$\sum_{0 \leq j < \frac{N(1-\varepsilon)}{4}} \mathbf{1}_{(L \cap [0, \frac{N\varepsilon}{4}) + j) \setminus L}(i) \geq |L \cap [0, \frac{N\varepsilon}{4})| \mathbf{1}_{[0, N) \setminus L}(i),$$

where $\mathbf{1}_Y$ is the characteristic function of a set Y . By adding over i in the above range we get

$$(108) \quad \sum_{j=0}^{N(1-\varepsilon/4)} D_L(j) \geq |L \cap [0, N\varepsilon/4)| \cdot |(N\varepsilon/4, N) \setminus L|.$$

By (99) we have

$$N \left(\delta + \frac{\log 2}{\log |\mathfrak{f}|} \right) \geq \sum_{i=0}^{N-1} \min(x_i, 1 - x_i) \geq \sum_{i \in L} (1 - x_i) \geq |L| - \sum_{i=0}^{N-1} x_i \geq |L| - (1 - \varepsilon)N$$

Therefore, by our assumption $\max(\delta, \frac{\log 2}{\log |\mathfrak{f}|}) < \varepsilon^2/16$, we have

$$(109) \quad |L| \leq N(1 - \varepsilon + \varepsilon^2/8).$$

Hence by (107), (108), and (109) we have

$$\sum_{j=0}^{N-1} D_L(j) \geq (N\varepsilon^2/8) \cdot N((1 - \varepsilon/4) - (1 - \varepsilon + \varepsilon^2/8)) \geq N^2\varepsilon^3/16.$$

□

Corollary 37. *For some integer $j_0 \in [N\varepsilon^3/32, N)$, we have $D_L(j_0) \geq N\varepsilon^3/32$ if $0 < \varepsilon \ll 1$, $0 < \delta \ll \varepsilon^2$ and $1 \ll_\varepsilon |\mathfrak{f}|$.*

Proof. By Lemma 36 we have

$$\sum_{j \in [N\varepsilon^3/32, N-1]} D_L(j) \geq \sum_{i=0}^{N-1} D_L(j) - (N\varepsilon^3/32)(N) \geq N^2\varepsilon^3/16 - N^2\varepsilon^3/32 = N^2\varepsilon^3/32.$$

And so for some $j_0 \in [N\varepsilon^3/32, N)$ we have $D_L(j_0) \geq N\varepsilon^3/32$. □

Proof of Proposition 29. Suppose $\max(\delta, \frac{\log 2}{\log |\mathfrak{f}|}) < \varepsilon^4/512$, and for some A the assertion of Proposition 29 does not hold. Then we consider J and L as above. Hence by Corollary 37 we have

$$D_L(j_0) \geq N\varepsilon^3/32$$

for some integer $j_0 \in [N\varepsilon^3/32, N)$. On the other hand, by Lemma 33, since $j_0 \geq N\varepsilon^3/32 > N[1/\varepsilon]\delta$, for some integer t in $[1, 3[1/\varepsilon]]$ there are t many elements b_1, \dots, b_t of J such that

$$j_0 = b_1 + \dots + b_t.$$

Hence by Lemma 35,

$$(110) \quad D_L(b_i) > N\varepsilon^4/100$$

for some i .

On the other hand, by Lemma 34, we have that for any $b \in J$

$$D_L(b) \leq 2N \left(\delta + \frac{\log 2}{\log |\mathfrak{f}|} \right) < N\varepsilon^4/128,$$

which contradicts (110). □

3.3. Proof of Theorem 9: Scalar-Sum-Product expansion. As in [BG09] (also see [Bou08] or [SG17, Section 2.3]), we start by a regularization process. The p-adic filtration $\{\pi_{\mathfrak{p}^N}(\mathfrak{p}^i \mathcal{O})\}_{i=1}^N$ induces an $|\mathfrak{f}|$ -regular rooted tree structure (with N -levels) on $\pi_{\mathfrak{p}^N}(\mathcal{O})$. So by a similar argument as the above mentioned articles we get the following large regular subset of A .

Lemma 38. *Let $0 < \delta < \varepsilon < 1$ and $|\mathfrak{f}| \gg_{\varepsilon, \delta} 1$. Then for $0 < \delta' \leq \varepsilon \delta / 4$ the following holds: Let $A \subseteq \pi_{\mathfrak{p}^N}(\mathcal{O})$. Suppose that A satisfies the following properties:*

- (a) $|\pi_{\mathfrak{p}^i}(A)| \geq |\mathfrak{f}|^{i\varepsilon}$ for any $N\delta' \leq i \leq N$,
- (b) $|A + A| \leq |A||\mathfrak{f}|^{N\delta'}$.

Then there is $A' \subseteq A$ such that

- (a) A' is (m_0, \dots, m_{N-1}) -regular.
- (b) $|A'| \geq |A| / (2 \log |\mathfrak{f}|)^N$.
- (c) $|\pi_{\mathfrak{p}^i}(A')| \geq |\mathfrak{f}|^{i\varepsilon/2}$ for $N\delta \leq i \leq N$.

Proof. By [SG17, Section 2.3], there is a subset $A' \subseteq A$ such that $|A'| \geq |A| / (2 \log |\mathfrak{f}|)^N$ and A' is an (m_0, \dots, m_{N-1}) -regular set. Let $\bar{n} := \max\{i \in [0, N-1] \mid |\pi_{\mathfrak{p}^i}(A')| < |\mathfrak{f}|^{i\varepsilon/2}\}$. To show that A' satisfies the above three conditions, it is enough to show that, if $0 < \delta' \leq \varepsilon \delta / 4$ and $|\mathfrak{f}| \gg_{\varepsilon, \delta} 1$, then $\bar{n} < N\delta$.

Suppose to the contrary that $\bar{n} \geq N\delta > N\delta'$. Then by the assumption $|\pi_{\mathfrak{p}^{\bar{n}}}(A)| \geq |\mathfrak{f}|^{\bar{n}\varepsilon}$. On the other hand, there is a subset $A'' \subseteq A'$ such that $|\pi_{\mathfrak{p}^{\bar{n}}}(A'')| = 1$ and

$$|A''| = \frac{|A'|}{|\pi_{\mathfrak{p}^{\bar{n}}}(A')|} > \frac{|A|}{(2 \log |\mathfrak{f}|)^N |\mathfrak{f}|^{\bar{n}\varepsilon/2}}.$$

Therefore we have

$$|A||\mathfrak{f}|^{N\delta'} \geq |A + A| \geq |A''| |\pi_{\mathfrak{p}^{\bar{n}}}(A)| \geq \frac{|A||\mathfrak{f}|^{\bar{n}\varepsilon}}{(2 \log |\mathfrak{f}|)^N |\mathfrak{f}|^{\bar{n}\varepsilon/2}},$$

which implies that

$$(111) \quad (2 \log |\mathfrak{f}|)^N \geq |\mathfrak{f}|^{\bar{n}\varepsilon/2 - N\delta'} \geq |\mathfrak{f}|^{N(\delta\varepsilon/2 - \delta')} \geq |\mathfrak{f}|^{N(\delta\varepsilon/4)}.$$

For $|\mathfrak{f}| \gg_{\varepsilon, \delta} 1$ (so that $2 \log |\mathfrak{f}| < |\mathfrak{f}|^{\delta\varepsilon/8}$), (111) implies that $\varepsilon/8 \geq \varepsilon/4$, which is a contradiction. \square

Proof of Theorem 9. Let $\delta_r(\varepsilon/2)$ (r stands for regular) be such that $0 < \delta_r(\varepsilon/2) \ll (\varepsilon/2)^4$ where the implied constant is given by Proposition 29. Suppose $|\mathfrak{f}| \gg_{\varepsilon} 1$, where the implied constant is given by Lemma 38 for $\varepsilon/2$ and $\delta_r(\varepsilon/2)$. Now let $\delta' \ll (\varepsilon/2)\delta_r(\varepsilon/2)$ be given by Lemma 38.² We claim δ' satisfies the desired conditions.

By the choice of δ' and Lemma 38, there is $A' \subseteq A$ such that

- (a) $|A'|$ is an (m_0, \dots, m_{N-1}) -regular subset.
- (b) $|\pi_{\mathfrak{p}^i}(A')| \geq |\mathfrak{f}|^{i\varepsilon/2}$ for $N\delta_r(\varepsilon/2) \leq i \leq N$,
- (c) $|A'| \geq |A| / (2 \log |\mathfrak{f}|)^N$.

Next we modify A' a bit, if necessary, to make sure that m_0 and m_1 are at least 2.

If $m_0 = 1$ and $m_1 > 1$, then $A' + \{a_{01}, a_{02}\}$ is a $(2, m_1, \dots, m_{N-1})$ -regular subset of $A + A$.

If $m_0 = m_1 = 1$, then $A' + \{a_{11}, a_{12}\} + \{a_{01}, a_{12}\}$ is a $(2, 2, m_2, \dots, m_{N-1})$ -regular subset of $A + A + A$.

If $m_0 > 1$ and $m_1 = 1$, then

- (a) there is a subset X_0 of A such that $|X_0| = |\pi_{\mathfrak{p}}(X_0)| = |\pi_{\mathfrak{p}}(A)|$,

²To avoid further confusion with the δ used in Lemma 38, we are using δ' , here. This is, in fact, supposed to be the claimed δ in Theorem 9.

(b) there is a $(1, 1, m_2, \dots, m_{N-1})$ -regular subset A'_0 of A'

Then $A'_0 + \{a_{11}, a_{12}\} + X_0$ is a regular $(|\pi_{\mathfrak{p}}(A)|, 2, m_2, \dots, m_{N-1})$ -regular subset of $A + A + A$.

So in all the cases we get an (m_0, \dots, m_{N-1}) -regular subset A' of $A + A + A$ such that

- (a) $m_0, m_1 > 1$.
- (b) $|\pi_{\mathfrak{p}^i}(A')| \geq |\mathfrak{f}|^{i\epsilon/2}$ for $N\delta_r(\epsilon/2) \leq i \leq N$.
- (c) $|A'| \geq |A|/(2 \log |\mathfrak{f}|)^N$.

If $|\langle A \rangle_6| \geq |A||\mathfrak{f}|^{N\delta}$ (for small enough δ to be determined later), we are done. So suppose this does not hold. In particular, $|A + A + A| \leq |A||\mathfrak{f}|^{N\delta}$. Hence $|A + A + A| \leq |\mathfrak{f}|^{N(1-\epsilon+\delta)}$. So assuming $\delta < \epsilon/2$, we have that $|A'| \leq |\mathfrak{f}|^{N(1-\epsilon/2)}$. Hence A' satisfies all the conditions of Proposition 29. Therefore we have

$$(112) \quad \max_{\omega \in \Omega, x \in A' - A'} |A' + \pi_{\mathfrak{p}^N}(\omega)x A'| \geq |A'| |\mathfrak{f}|^{N\delta_r(\epsilon/2)}.$$

Since at least one of a_{01}, a_{02} is a unit, we have that

$$|\langle A \rangle_6 + \pi_{\mathfrak{p}^N}(\omega)\langle A \rangle_6| \geq |A' + \pi_{\mathfrak{p}^N}(\omega)x A'|.$$

Therefore we have

$$\max_{\omega \in \Omega} |\langle A \rangle_6 + \pi_{\mathfrak{p}^N}(\omega)\langle A \rangle_6| \geq |A'| |\mathfrak{f}|^{N\delta_r(\epsilon/2)} \geq |A| \left(\frac{|\mathfrak{f}|^{\delta_r(\epsilon/2)}}{2 \log |\mathfrak{f}|} \right)^N.$$

Suppose $|\mathfrak{f}| \gg_{\epsilon} 1$ so that $|\mathfrak{f}|^{\delta_r(\epsilon/2)/2} \geq 2 \log |\mathfrak{f}|$. Hence we get

$$\max_{\omega \in \Omega} |\langle A \rangle_6 + \pi_{\mathfrak{p}^N}(\omega)\langle A \rangle_6| \geq |A| |\mathfrak{f}|^{N\delta_r(\epsilon/2)/2} \geq |A| |\mathfrak{f}|^{N\delta'}.$$

□

3.4. Proof of Theorem 10: a scalar-sum-product set contains a large congruence set.

Proposition 39. *For any $0 < \epsilon_1 \ll \epsilon_2 \ll 1$, $0 < \delta \ll \epsilon_1^5$, and positive integer $1 \ll_{\epsilon_1} C$, and any finite extension K of $\mathbb{Q}_{\mathfrak{p}}$ with large, depending on ϵ_1 , residue field \mathfrak{f} the following holds:*

Let \mathcal{O} be the ring of integers of K , and \mathfrak{p} be a uniformizing element of K . Let $\Omega \subseteq \mathcal{O}$, and suppose $\pi_{\mathfrak{p}}$ induces a bijection between $\Omega \subseteq \mathcal{O}$ and \mathfrak{f}^{\times} . Suppose $A \subseteq \pi_{\mathfrak{p}^N}(\mathcal{O})$ such that

- (a) $|\pi_{\mathfrak{p}^i}(A)| \geq |\mathfrak{f}|^{i\epsilon_1}$ for any $N\delta \leq i \leq N$.
- (b) *there are $a_{01}, a_{02}, a_{11}, a_{12} \in A$ such that $a_{i1} - a_{i2} \in \pi_{\mathfrak{p}^N}(\mathfrak{p}^i \mathcal{O} \setminus \mathfrak{p}^{i+1} \mathcal{O})$.*

Then

$$\pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \epsilon_2 N \rceil} \mathcal{O}) \subseteq \langle A \rangle_C + \pi_{\mathfrak{p}^N}(\omega_1)\langle A \rangle_C + \dots + \pi_{\mathfrak{p}^N}(\omega_C)\langle A \rangle_C,$$

for some $\omega_i \in \prod_C(\Omega \cup \{1\})$.

Proof of Theorem 10 modulo Proposition 39. Let $\delta \ll \epsilon_1^{5m+1}$. Hence by Proposition 39 applied to the set $\pi_{\mathfrak{p}^{N_m}}(A)$, where $N_m := \lfloor \epsilon_2^{m-1} N \rfloor$, we get that for a positive integer $1 \ll_{\epsilon_1} k$ there are $\omega_i \in \prod_k(\Omega \cup \{1\})$ such that

$$(113) \quad \begin{aligned} \pi_{\mathfrak{p}^{N_m}}(\mathfrak{p}^{\lceil \epsilon_2^m N \rceil} \mathcal{O}) &\subseteq \pi_{\mathfrak{p}^{N_m}}(\mathfrak{p}^{\epsilon_2 N_m} \mathcal{O}) \\ &\subseteq \langle \pi_{\mathfrak{p}^{N_m}}(A) \rangle_k + \pi_{\mathfrak{p}^{N_m}}(\omega_1)\langle \pi_{\mathfrak{p}^{N_m}}(A) \rangle_k + \dots + \pi_{\mathfrak{p}^{N_m}}(\omega_k)\langle \pi_{\mathfrak{p}^{N_m}}(A) \rangle_k \\ &\subseteq \pi_{\mathfrak{p}^{N_m}}(\langle A \rangle_k + \pi_{\mathfrak{p}^N}(\omega_1)\langle A \rangle_k + \dots + \pi_{\mathfrak{p}^N}(\omega_k)\langle A \rangle_k), \end{aligned}$$

Next we find lots of elements with controlled scales in a sum-product set of A . First we notice that by enlarging $k \gg_{\epsilon_1} 1$, if necessary, we can and will assume that $|\langle A \rangle_k + \langle A \rangle_k| \leq |\langle A \rangle_k| |\mathfrak{f}|^{N\delta}$. Hence by Lemma 38 and an argument similar to proof of Theorem 9 there is an (m_0, \dots, m_{N-1}) -regular subset $A' \subseteq \langle A \rangle_{3k}$ such

that $m_0, m_1 > 1$ and $|\pi_{\mathfrak{p}^i}(A')| \geq |\mathfrak{f}|^{i\varepsilon_1/2}$ for any $N(4\delta/\varepsilon_1) \leq i \leq N$. By Lemma 33 for the numbers $x_i := \log m_i / \log |\mathfrak{f}|$ we have

$$(114) \quad (N(8\delta/\varepsilon_1^2), N) \cap \mathbb{Z} \subseteq \sum_{6 \lceil 1/\varepsilon_1 \rceil} v_{\mathfrak{p}}(\langle A \rangle_{6k}) \subseteq v_{\mathfrak{p}}(\langle A \rangle_{36 \lceil 1/\varepsilon_1 \rceil}).$$

For any $1 \leq j \leq \varepsilon_2^{-m}$ and $k' \gg_{\varepsilon_1} 1$, since $N(8\delta/\varepsilon_1^2) \leq j\varepsilon_2^m N \leq N$, by (114), there is $y_j \in \langle A \rangle_{k'}$ such that

$$y_j \in \mathfrak{p}^{\lfloor j\varepsilon_2^m N \rfloor} \pi_{\mathfrak{p}^N}(\mathcal{O}) \setminus \mathfrak{p}^{\lfloor j\varepsilon_2^m N \rfloor + 1} \pi_{\mathfrak{p}^N}(\mathcal{O}).$$

Hence by (113) we have

$$\begin{aligned} \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon_2^m N \rceil} \mathcal{O}) &\subseteq \langle A \rangle_{k'} + \pi_{\mathfrak{p}^N}(\omega_1) \langle A \rangle_{k'} + \cdots + \pi_{\mathfrak{p}^N}(\omega_k) \langle A \rangle_{k'} \\ &\quad + y_1 \langle A \rangle_{k'} + \pi_{\mathfrak{p}^N}(\omega_1) \langle A \rangle_{k'} + \cdots + \pi_{\mathfrak{p}^N}(\omega_k) \langle A \rangle_{k'} \\ &\quad + \cdots \\ &\quad + y_{\lceil \varepsilon_2^{-m} \rceil} (\langle A \rangle_{k'} + \pi_{\mathfrak{p}^N}(\omega_1) \langle A \rangle_{k'} + \cdots + \pi_{\mathfrak{p}^N}(\omega_k) \langle A \rangle_{k'}). \end{aligned}$$

And, since A contains a unit and $\varepsilon_1 \leq \varepsilon_2$, we have that for a positive integer $1 \ll_{m, \varepsilon_1} C$

$$\pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon_2^m N \rceil} \mathcal{O}) \subseteq \langle A \rangle_C + \pi_{\mathfrak{p}^N}(\omega_1) \langle A \rangle_C + \cdots + \pi_{\mathfrak{p}^N}(\omega_k) \langle A \rangle_C.$$

□

To prove Proposition 39, let us start with a direct corollary of [BG09, Lemma A.1].

Lemma 40. *Let K be a finite extension of \mathbb{Q}_p , \mathcal{O} be the ring of integers of K , and \mathfrak{p} be a uniformizing element of K . Suppose $B \subseteq \pi_{\mathfrak{p}^N}(\mathcal{O})$ such that for any $1 \leq k \leq N$,*

$$\max_{\xi} |\{x \in B \mid \pi_{\mathfrak{p}^k}(x) = \xi\}| < |\mathfrak{f}|^{-(3/4)k} |B|.$$

Then $\langle B \rangle_{200} = \pi_{\mathfrak{p}^N}(\mathcal{O})$.

Proof. It is a consequence of [BG09, Lemma A.1] as it is observed in [BG09, Proof of Corollary A.1]. □

Next following [BG09, Proof of Corollary A.1] we show how Lemma 40 helps us to deal with (extremely) large sets.

Lemma 41. *For any $0 < \varepsilon \ll 1$, $0 < \delta \ll \varepsilon$, and any finite extension K of \mathbb{Q}_p the following holds:*

Let \mathcal{O} be the ring of integers of K , \mathfrak{p} be a uniformizing element of K , and \mathfrak{f} be the residue field. Suppose $A \subseteq \pi_{\mathfrak{p}^N}(\mathcal{O})$ such that $|A| \geq |\mathfrak{f}|^{N(1-\delta)}$. Then

$$\pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon N \rceil} \mathcal{O}) \subseteq \langle A \rangle_{200}.$$

Proof. ³ Let

$$n_0 := \max\{k \mid \max_{\xi} |\{x \in A \mid \pi_{\mathfrak{p}^k}(x) = \xi\}| > |\mathfrak{f}|^{-(3/4)k} |A|\}.$$

Hence for small enough δ (to be determined later) we have

$$|\mathfrak{f}|^{N-n_0} > |\mathfrak{f}|^{-(3/4)n_0} |A| \geq |\mathfrak{f}|^{-(3/4)n_0} |\mathfrak{f}|^{N(1-\delta)}.$$

Therefore we have

$$(115) \quad n_0 < 4N\delta.$$

Let $\xi \in \pi_{\mathfrak{p}^{n_0}}(\mathcal{O})$ be such that $A' := \{x \in A \mid \pi_{\mathfrak{p}^{n_0}}(x) = \xi\}$ has at least $|\mathfrak{f}|^{-(3/4)n_0} |A|$ -many elements. And let

$$B := \pi_{\mathfrak{p}^{N-n_0}}(\{x \in \mathcal{O} \mid \pi_{\mathfrak{p}^N}(x_0 + \mathfrak{p}^{n_0}x) \in A'\}),$$

³This is identical to [BG09, Proof of Corollary A.1]. It is included for the convenience of the reader.

where $\pi_{\mathfrak{p}^{n_0}}(x_0) = \xi$. By Lemma 40, we have that

$$\langle B \rangle_{200} = \pi_{\mathfrak{p}^{N-n_0}}(\mathcal{O}).$$

Hence

$$\langle A \rangle_{200} \supseteq \pi_{\mathfrak{p}^N}(\mathfrak{p}^{200n_0}\mathcal{O}).$$

Now (115) gives us the claim. \square

Proof of Proposition 39. By Lemma 41, it is enough to prove that

$$(116) \quad |\langle A \rangle_C + \pi_{\mathfrak{p}^N}(\omega_1)\langle A \rangle_C + \cdots + \pi_{\mathfrak{p}^N}(\omega_C)\langle A \rangle_C| \geq |\mathfrak{f}|^{N(1-O(\varepsilon_2))},$$

for a positive integer $C \gg_{\varepsilon_1} 1$ and $\omega_i \in \prod_C(\Omega \cup \{1\})$. One can get (116) by applying Theorem 9 repeatedly and using the fact that $\varepsilon_1 \ll \varepsilon_2$. \square

Proof of Corollary 11. Since $\pi_{\mathfrak{p}^{e'}}(A) = \pi_{\mathfrak{p}^{e'}}(\mathcal{O})$, we have $\pi_{\mathfrak{p}}(A) = \mathfrak{f}$. Therefore there is a subset $\Omega \subseteq A$ such that $\pi_{\mathfrak{p}}$ induces a bijection between Ω and \mathfrak{f}^\times . If K is ramified over \mathbb{Q}_p , then $e' = 2$. So by the assumption, we can apply Theorem 10 to $\Omega \subseteq A$ and A , which implies the claim. Now suppose K is an unramified extension of \mathbb{Q}_p , and let $s : \mathfrak{f} \rightarrow \pi_{\mathfrak{p}^2}(A)$ be a section of $\pi_{\mathfrak{p}} : \pi_{\mathfrak{p}^2}(A) \rightarrow \mathfrak{f}$. Since K is an unramified extension of \mathbb{Q}_p , \mathfrak{f} cannot be embedded into $\pi_{\mathfrak{p}^2}(\mathcal{O})$ as an additive group. Hence there are $x_1, x_2 \in \mathfrak{f}$ such that $s(x_1) + s(x_2) - s(x_1 + x_2) \neq 0$. Therefore this time we can apply Theorem 10 to $\Omega \subseteq A$ and $\langle A \rangle_2$ and get the claim. \square

4. GETTING A THICK \mathbb{Z}_p -SEGMENT IN A SUM-PRODUCT OF A LARGE SET.

In this section, first we get a *thick \mathbb{Z}_p -segment* in a *small scale* in a sum-product set where the implied constants are independent of local field K , but the caveat is that we assume $\text{gr}_{1,\mathfrak{p}}(\langle A \rangle; \mathcal{O}) \neq 0$. This is based on a multi-scale analog of the Bourgain-Katz-Tao argument and another application of Mann's theorem. It is worth pointing out that this result is not needed to prove Theorem 1. Next we prove Theorem 1, where we relax the condition on the set A , but assume that the degree of the field extension K/\mathbb{Q}_p is bounded.

4.1. Multi-scale version of the Bourgain-Katz-Tao argument. In this section we give a \mathfrak{p} -adic version of [BKT04, Theorem 4.3]. One surprising result is that the implied constants are independent of the choice of local field K .

Lemma 42. *For any positive integer t , positive numbers $0 < \varepsilon_1 \ll \varepsilon_2 \ll_t 1$, $0 < \delta \ll_{\varepsilon_1} 1$, any positive integer $C \gg_{\varepsilon_1} 1$, and any finite extension K of \mathbb{Q}_p with large, depending on ε_1 , residue field \mathfrak{f} the following holds: let \mathcal{O} be the ring of integers of K , and \mathfrak{p} be a uniformizing element of K . Suppose $A \subseteq \pi_{\mathfrak{p}^N}(\mathcal{O})$ such that*

- (a) $|\pi_{\mathfrak{p}^i}(A)| \geq |\mathfrak{f}|^{i\varepsilon_1}$ for any $N\delta \leq i \leq N$.
- (b) $0, 1 \in A$ and there are $a_1, a_2 \in A$ such that $a_1 - a_2 \in \pi_{\mathfrak{p}^N}(\mathfrak{p}\mathcal{O} \setminus \mathfrak{p}^2\mathcal{O})$.

Then either

$$\pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon_2 N \rceil} \mathcal{O}) \subseteq \langle A \rangle_C,$$

or

$$\pi_{\mathfrak{p}^{\lfloor t\varepsilon' N \rfloor}}(\langle A \rangle_C \cap \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon' N \rceil} \mathcal{O})) \text{ is a ring,}$$

for some ε' in $[\varepsilon_2^{m(\varepsilon_1)}, \varepsilon_2]$.

Proof. Let $m := m(\varepsilon_1)$ be a large integer (will be determined later). Let $\theta : \mathfrak{f}^\times \rightarrow \mathcal{O}^\times$ be the group homomorphism such that $\pi_{\mathfrak{p}}(\theta(x)) = x$ for any $x \in \mathfrak{f}^\times$ (recall that such group homomorphism exists by

Hensel's lemma). Let $\psi_N : \mathfrak{f} \rightarrow \pi_{\mathfrak{p}^N}(\mathcal{O}), \psi_N(0) = 0$, and $\psi_N(x) = \pi_{\mathfrak{p}^N}(\theta(x))$ for any $x \in \mathfrak{f}^\times$. Then by Theorem 10, if $0 < \delta \ll_{\varepsilon_1, m} 1$, we have that

$$(117) \quad \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon_2^m N \rceil} \mathcal{O}) \subseteq \langle A \rangle_{C_1} + \psi_N(\alpha_1) \langle A \rangle_{C_1} + \cdots + \psi_N(\alpha_k) \langle A \rangle_{C_1},$$

for some integers $k := k(\varepsilon_1)$ and $C_1 := C_1(\varepsilon_1, m)$, and $\alpha_i \in \mathfrak{f}^\times$.

Now we introduce a process through which the number k of the involved scalars will be reduced in the expense of enlarging C_1 and shrinking the size of the congruence subgroup, i.e. enlarging ε_2^m . Then we will analyze the case when this process halts before getting $k = 0$.

For simplicity we say $\text{BG}(A; \varepsilon, k, C)$ holds if for k elements $\alpha_i \in \mathfrak{f}^\times$ we have

$$(118) \quad \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon N \rceil} \mathcal{O}) \subseteq \langle A \rangle_C + \psi_N(\alpha_1) \langle A \rangle_C + \cdots + \psi_N(\alpha_k) \langle A \rangle_C.$$

Claim 1. *Suppose $0 < \delta_0 < 1$ and $\text{BG}(A; \varepsilon, k, C)$ holds. Then we have either (**reduction**)*

$$(119) \quad \text{BG}(A; \varepsilon + \delta_0, k - 1, 8C),$$

or (**δ_0 -injectivity**) for any $\mathbf{x}, \mathbf{x}' \in \langle A \rangle_{2C}^{k+1} := \langle A \rangle_{2C} \times \cdots \times \langle A \rangle_{2C}$ we have that

$$(120) \quad l(\mathbf{x}) = l(\mathbf{x}') \Rightarrow \mathbf{x} - \mathbf{x}' \in \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lfloor \delta_0 N \rfloor} \mathcal{O}),$$

where $l(x_0, \dots, x_k) := x_0 + \psi_N(\alpha_1)x_1 + \cdots + \psi_N(\alpha_k)x_k$ and $\alpha_i \in \mathfrak{f}^\times$ satisfy (118).

Proof of Claim 1. Suppose δ_0 -injectivity fails, i.e. there are $\mathbf{x}, \mathbf{x}' \in \langle A \rangle_{2C} \times \cdots \times \langle A \rangle_{2C}$ such that

- (a) $\mathbf{x} - \mathbf{x}' \notin \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lfloor \delta_0 N \rfloor} \mathcal{O})^{k+1}$, and
- (b) $l(\mathbf{x}) = l(\mathbf{x}')$.

Then, for some i_0 , $(x_{i_0} - x'_{i_0})\pi_{\mathfrak{p}^N}(\mathcal{O}) \supseteq \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lfloor \delta_0 N \rfloor} \mathcal{O})$. Without loss of generality let us assume that it happens for $i_0 = k$ (notice that, if $i_0 = 0$, we can multiply both sides by $\psi_N(\alpha_1^{-1})$ to make sure that one of the remaining coefficients is one). Hence we have

$$(121) \quad \begin{aligned} \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon N \rceil + \lfloor \delta_0 N \rfloor} \mathcal{O}) &\subseteq \psi_N(\alpha_0) \langle A \rangle_{2C}(x_k - x'_k) + \cdots + \psi_N(\alpha_k) \langle A \rangle_{2C}(x_k - x'_k) \\ (\text{since } l(\mathbf{x}) = l(\mathbf{x}'),) &\subseteq \psi_N(\alpha_0) \langle A \rangle_{4C} + \cdots + \psi_N(\alpha_{k-1}) \langle A \rangle_{4C} - \left(\sum_{i=0}^{k-1} \psi_N(\alpha_i)(x_i - x'_i) \right) \langle A \rangle_{2C} \\ &\subseteq \psi_N(\alpha_0) \langle A \rangle_{8C} + \cdots + \psi_N(\alpha_{k-1}) \langle A \rangle_{8C}, \end{aligned}$$

which means that $\text{BG}(A; \varepsilon + \delta_0, k - 1, 8C)$ holds. \square

Claim 2. *Suppose $\text{BG}(A; \varepsilon, k, C)$ holds and $\alpha_i \in \mathfrak{f}^\times$ satisfy (118). If $l(\mathbf{x}) := \sum_i \alpha_i x_i$ is δ_0 -injective on $\langle A \rangle_{2C}^{k+1}$ for some $\delta_0 > \varepsilon$ (see (120)), then*

$$\pi_{\mathfrak{p}^{\lfloor \delta_0 N \rfloor}} \left(\langle A \rangle_C \cap \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon N \rceil} \mathcal{O}) \right)$$

is closed under addition and multiplication.

Proof of Claim 2. Let $x, x' \in \langle A \rangle_C \cap \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon N \rceil} \mathcal{O})$. So there is $\mathbf{x}_1 \in \langle A \rangle_C \times \cdots \times \langle A \rangle_C$ such that $l(\mathbf{x}_1) = x + x' = l(x + x', 0, \dots, 0)$.

By assumption for any $\mathbf{x}_1, \mathbf{x}'_1 \in \langle A \rangle_{2C} \times \cdots \times \langle A \rangle_{2C}$ we have that

$$(122) \quad l(\mathbf{x}_1) = l(\mathbf{x}'_1) \Rightarrow \mathbf{x}_1 - \mathbf{x}'_1 \in \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lfloor \delta_0 N \rfloor} \mathcal{O})^{k+1}.$$

Hence we have

$$\mathbf{x}_1 \equiv (x + x', 0, \dots, 0) \pmod{\mathfrak{p}^{\lfloor \delta_0 N \rfloor}},$$

which implies that $\pi_{\mathfrak{p}^{\lfloor \delta_0 N \rfloor}}(A \cap \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon N \rceil} \mathcal{O}))$ is closed under addition.

Similarly there is $\mathbf{x}_2 \in \langle A \rangle_C \times \cdots \times \langle A \rangle_C$ such that $l(\mathbf{x}_2) = xx' = l(xx', 0, \dots, 0)$. As $xx' \in \langle A \rangle_{2C}$, again as above δ_0 -injectivity on $\langle A \rangle_{2C}^{k+1}$ implies that $\mathbf{x}_2 \equiv (xx', 0, \dots, 0) \pmod{\mathfrak{p}^{\lceil \delta_0 N \rceil}}$, which implies that $\pi_{\mathfrak{p}^{\lceil \delta_0 N \rceil}}(A \cap \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon N \rceil} \mathcal{O}))$ is closed under multiplication. \square

Having the above Claims, we inductively define three sequences $\{\varepsilon'_i\}, \{k'_i\}, \{C'_i\}$ of numbers:

$$\begin{aligned} \varepsilon'_0 &:= \varepsilon_2^m, & \varepsilon'_{i+1} &:= (t+1)\varepsilon'_i; \\ k'_0 &:= k, & k'_{i+1} &:= k_i - 1; \\ C'_0 &:= C_1, & C'_{i+1} &:= 8C'_i. \end{aligned}$$

First notice that for $\varepsilon_2 \ll_t 1$ and $k(\varepsilon_1) \leq m(\varepsilon_1)$ we have that

$$(123) \quad \varepsilon'_i \leq (t+1)^k \varepsilon_2^m \leq (t+1)^k (t+1)^{-m} \varepsilon_2 \leq \varepsilon_2, \quad C'_i \leq 8^k C_1 \ll_{\varepsilon_1} 1.$$

We know that $\text{BG}(A; \varepsilon'_0, k'_0, C'_0)$ holds. Suppose i_0 is the smallest non-negative integer such that

$$\text{BG}(A; \varepsilon'_{i_0+1}, k'_{i_0+1}, C'_{i_0+1})$$

does not hold. If $i_0 = k$, then $\text{BG}(A; \varepsilon_2, 0, 4^k C_1)$ holds. And we are done. Suppose $i_0 < k$. So, by Claim 1, $\text{BG}(A; \varepsilon'_{i_0}, k'_{i_0}, C'_{i_0})$ holds for some $\alpha_j \in \mathfrak{f}^\times$ and $l(\mathbf{x}) := \sum_j \alpha_j x_j$ is $t\varepsilon'_{i_0}$ -injective on $\langle A \rangle_{2C'_{i_0}}^{k'_{i_0}+1}$. Therefore, by Claim 2,

$$\pi_{\mathfrak{p}^{\lceil t\varepsilon'_{i_0} N \rceil}} \left(\langle A \rangle_{C'_{i_0}} \cap \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon'_{i_0} N \rceil} \mathcal{O}) \right)$$

is closed under addition and multiplication; and the claim follows. \square

In order to get a meaningful conclusion from Lemma 42, we have to show that $\langle A \rangle_C$ has an element with \mathfrak{p} -valuation roughly equal to $\lceil \varepsilon' N \rceil$ as otherwise $\pi_{\mathfrak{p}^{\lceil \varepsilon' N \rceil}}(\langle A \rangle_C \cap \pi_{\mathfrak{p}^N}(\mathfrak{p}^{\lceil \varepsilon' N \rceil} \mathcal{O}))$ can be a very small set. For that purpose, next we will observe that Lemma 33 gives us such a control.

Proof of Theorem 4. Suppose the implied constants are so that the given inequalities in Lemma 42 are satisfied for the parameters $\varepsilon_1, \varepsilon_2, t, \delta_0$, and C_0 . By changing the implied constants, we can further assume that $\delta_0 \leq \varepsilon_1^{m(\varepsilon_1)+1} \leq \varepsilon_1 \varepsilon_2^{m(\varepsilon_1)}$ and $C_0 \geq 6/\varepsilon_1$. Suppose $\delta \leq \varepsilon_1 \delta_0/4$ and $C \gg C_0/(\varepsilon_1^2 \delta_0)$. Notice that for some $C' \leq 4/(\varepsilon_1 \delta_0)$ we have that $|\langle A \rangle_{C'} + \langle A \rangle_{C'}| \leq |\langle A \rangle_{C'}| |\mathfrak{f}|^{N\varepsilon_1 \delta_0/4}$. Hence by Lemma 38 and a similar argument as in the proof of Theorem 10, there is an (m_0, \dots, m_{N-1}) -regular subset A' of $\langle A \rangle_{3C'}$ such that

- $|\pi_{\mathfrak{p}^i}(A')| \geq |\mathfrak{f}|^{i\varepsilon_1/2}$ for any $N\delta_0 \leq i \leq N$, and
- $m_0, m_1 > 1$

By Lemma 33 for the numbers $x_i := \log m_i / \log |\mathfrak{f}|$ we have

$$(124) \quad (N \lceil 2/\varepsilon_1 \rceil \delta_0, N) \cap \mathbb{Z} \subseteq \sum_{6 \lceil 1/\varepsilon_1 \rceil} v_{\mathfrak{p}}(\langle A \rangle_{6C'}) \subseteq v_{\mathfrak{p}}(\langle A \rangle_{36C' \lceil 1/\varepsilon_1 \rceil});$$

and the claim follows. \square

4.2. Proof of Theorem 1. Let us start with (a variation of) [BKT04, Theorem 4]. We include the proof for the convenience of the reader.

Lemma 43. *For any $0 < \varepsilon \ll 1$, positive integer $C \gg_{\varepsilon} 1$, and a finite field \mathfrak{f} the following holds:*

Suppose $B \subseteq \mathfrak{f}$, $|B| \geq |\mathfrak{f}|^{\varepsilon}$, and $0, 1 \in B$. Then $\langle B \rangle_C$ is a subfield of \mathfrak{f} .

Proof. By [BKT04, Lemma 4.1], there are $\alpha_1, \dots, \alpha_k \in \mathfrak{f}^\times$ such that $k \ll_{\varepsilon} 1$ and

$$\alpha_1 B + \cdots + \alpha_k B = \mathfrak{f}.$$

Claim 1: *Suppose $0, 1 \in X \subseteq \mathfrak{f}$ and $\alpha_i \in \mathfrak{f}^\times$ such that*

$$(125) \quad \mathfrak{f} = \alpha_1 X + \cdots + \alpha_k X.$$

Then either we have **(reduction)** $\mathfrak{f} = \sum_{i \neq i_0} \alpha_i \langle X \rangle_2$, for some i_0 , or **(injectivity)** for any $\mathbf{x}, \mathbf{x}' \in X^k$ $\sum_i \alpha_i x_i = \sum_i \alpha_i x'_i \Rightarrow \mathbf{x} = \mathbf{x}'$.

Proof of Claim. Suppose that the injectivity does not hold, i.e. there are $\mathbf{x} \neq \mathbf{x}' \in X^k$ such that

$$(126) \quad \sum_i \alpha_i x_i = \sum_i \alpha_i x'_i.$$

Without loss of generality we can assume that $x_k \neq x'_k$. Thus

$$\begin{aligned} \mathfrak{f} &= \alpha_1(x_k - x'_k)X + \cdots + \alpha_k(x_k - x'_k)X \\ (\text{by (126)}) \quad &\subseteq \alpha_1(X \cdot X - X \cdot X) + \cdots + \alpha_{k-1}(X \cdot X - X \cdot X) + \left(\sum_{i=1}^{k-1} \alpha_i(x_i - x'_i) \right) X \\ &\subseteq \alpha_1 \langle X \rangle_2 + \cdots + \alpha_{k-1} \langle X \rangle_2. \end{aligned}$$

□

Claim 2: Suppose $0, 1 \in X \subseteq \mathfrak{f}$ and $\alpha_i \in \mathfrak{f}^\times$ such that

$$\mathfrak{f} = \alpha_1 X + \cdots + \alpha_k X.$$

Suppose for any $\mathbf{x}, \mathbf{x}' \in \langle X \rangle_2^k$ we have

$$\sum_i \alpha_i x_i = \sum_i \alpha_i x'_i \Rightarrow \mathbf{x} = \mathbf{x}'.$$

Then X is a subfield of \mathfrak{f} .

Proof of Claim. It is enough to show $X \cdot X = X$ and $X + X = X$. For any $y, y' \in X$, there is $\mathbf{x} \in X^k$ such that

$$\alpha_1(y + y') = \sum_i \alpha_i x_i.$$

Hence $y + y' = x_1 \in X$. And so X is closed under addition. Similarly it is closed under multiplication. □

Now suppose $i_0 \leq k$ be the largest non-negative integer such that

$$\mathfrak{f} = \alpha'_1 \langle B \rangle_{4^{i_0}} + \cdots + \alpha'_{k-i_0} \langle B \rangle_{4^{i_0}},$$

for some $\alpha'_i \in \mathfrak{f}^\times$. If $i_0 = k$, we are done. If not, then by Claim 1 for $X = \langle \langle B \rangle_{4^{i_0}} \rangle_2$ we have that for any $\mathbf{x}, \mathbf{x}' \in \langle \langle B \rangle_{4^{i_0}} \rangle_2^{k-i_0}$ we have

$$\sum_i \alpha'_i x_i = \sum_i \alpha'_i x'_i \Rightarrow \mathbf{x} = \mathbf{x}'.$$

Hence, by Claim 2, $\langle B \rangle_{4^{i_0}}$ is a subfield of \mathfrak{f} . □

Proposition 44. For any positive integers $0 < \varepsilon_1 \ll \varepsilon_2 \leq 1/2$, $0 < \delta \ll_{\varepsilon_1} 1$, positive integers d and $1 \ll_{\varepsilon_1, d} C$, the following holds: suppose K is a field extension of \mathbb{Q}_p and $[K : \mathbb{Q}_p] \leq d$. Let \mathcal{O} be the ring of integers of K , \mathfrak{p} be a uniformizing element of K , and \mathfrak{f} be the residue field of K . Suppose $|\mathfrak{f}| \gg_{\varepsilon_1, d} 1$. Suppose A is a subset of \mathcal{O} which contains 0 and 1. Let R be the closure of the subring of \mathcal{O} that is generated by A . Suppose

- (C1) (Equality of grades) for any integer i in $[0, N-1]$, $|\text{gr}_{i, \mathfrak{p}}(R; \mathcal{O})| = |\text{gr}_{0, \mathfrak{p}}(R; \mathcal{O})|$, where $\text{gr}_{i, \mathfrak{p}}(R; \mathcal{O}) := \pi_{\mathfrak{p}^{i+1}}(R \cap \mathfrak{p}^i \mathcal{O})$.
- (C2) (Bound for the box dimension) for any integer i in $[N\delta, Ne]$, $|\pi_{\mathfrak{p}^i}(A)| \geq |\mathfrak{f}|^{\varepsilon_1}$, where e is the ramification index of K over \mathbb{Q}_p .
- (C3) (Bound for level \mathfrak{p}) $|\pi_{\mathfrak{p}}(A)| \geq |\mathfrak{f}|^{\varepsilon_1}$.

Then $\pi_{\mathfrak{p}^N}(\langle A \rangle_C) \supseteq \mathfrak{p}^{\lceil N\varepsilon_2 \rceil} \pi_{\mathfrak{p}^N}(R)$.

Proof. Step 1. (Describing R for large N) By Theorem 7, there is a positive integer N_0 depending only on $[K : \mathbb{Q}_p]$ such that, if $N \geq N_0$, then there are positive integers a and b , and a subfield K_0 of K such that $b - a \geq 6[K : \mathbb{Q}_p]$, $N \geq b$,

$$(127) \quad \pi_{p^b}(R) \subseteq \pi_{p^b}(\mathcal{O}_0), \text{ and } \pi_{p^b}(R \cap p^a \mathcal{O}) = \pi_{p^b}(\mathcal{O}_0 \cap p^a \mathcal{O}).$$

By Corollary 17 we have that $x + p\mathcal{O} \mapsto p^i x + p^{i+1} \mathcal{O}$ is an injection from $\text{gr}_{0,p}(R; \mathcal{O})$ to $\text{gr}_{i,p}(R; \mathcal{O})$. Since by our assumption $|\text{gr}_{i,p}(R; \mathcal{O})| = |\pi_p(R)|$ for any integer i in $[0, N-1]$, we deduce that $x + p\mathcal{O} \mapsto p^i x + p^{i+1} \mathcal{O}$ is a bijection from $\text{gr}_{0,p}(R; \mathcal{O})$ to $\text{gr}_{i,p}(R; \mathcal{O})$. By the equality of grades (condition C1) and (127) we deduce that

$$(128) \quad |\text{gr}_{i,p}(R; \mathcal{O})| = |\text{gr}_{a,p}(R; \mathcal{O})| = |\text{gr}_{a,p}(\mathcal{O}_0; \mathcal{O})| = |\text{gr}_{0,p}(\mathcal{O}_0; \mathcal{O})|$$

for any integer i in $[0, N-1]$. In particular, $|\pi_{p^j}(R \cap p^i \mathcal{O})| = |\pi_{p^{j-i}}(\mathcal{O}_0)|$ for any integers $i < j$ in $[0, N-1]$. By Corollary 17 we have that $x + p^{b-a} \mathcal{O} \mapsto p^a x + p^b \mathcal{O}$ is an injection from $\pi_{p^{b-a}}(R)$ to $\pi_{p^b}(R \cap p^a \mathcal{O})$; and as these sets have equal cardinality, we deduce that this map is a bijection. The same can be said for the ring \mathcal{O}_0 instead of R . Therefore by (127) and $b - a \geq 6[K : \mathbb{Q}_p]$, we have

$$(129) \quad \pi_{p^{6[K:\mathbb{Q}_p]}}(R) = \pi_{p^{6[K:\mathbb{Q}_p]}}(\mathcal{O}_0).$$

By (128), (129), and Proposition 19, we have

$$(130) \quad \pi_{p^{N-4}}(R) = \pi_{p^{N-4}}(\mathcal{O}_0).$$

Step 2. (Bounded generation of $\pi_{\mathfrak{p}\mathcal{O}(1)}(R)$) By condition (C3), we have $|\pi_{\mathfrak{p}}(A)| \geq |\mathfrak{f}|^{\varepsilon_1}$, and we also have that $0, 1 \in \pi_{\mathfrak{p}}(A)$; hence by Lemma 43 we get that

$$(131) \quad \pi_{\mathfrak{p}}(\langle A \rangle_{C_1}) = \pi_{\mathfrak{p}}(R) \text{ and } [\mathfrak{f} : \pi_{\mathfrak{p}}(R)] \leq 1/\varepsilon_1$$

where C_1 is an integer that only depends on ε_1 . By induction on i , we show that

$$(132) \quad \pi_{\mathfrak{p}^i}(\langle A \rangle_{C_i}) = \pi_{\mathfrak{p}^i}(R)$$

where C_i is an integer that depends only on i and ε_1 . If $\pi_{\mathfrak{p}^{i+1}}(\langle A \rangle_{C_i})$ is a ring, then $\pi_{\mathfrak{p}^{i+1}}(\langle A \rangle_{C_i}) = \pi_{\mathfrak{p}^{i+1}}(R)$; and we can set $C_{i+1} := C_i$. If not, $\pi_{\mathfrak{p}^{i+1}}(\langle A \rangle_{3C_i} \cap \mathfrak{p}^i \mathcal{O}) \neq 0$. By (131), $\pi_{\mathfrak{p}^{i+1}}(\mathfrak{p}^i \mathcal{O})$ is an $\pi_{\mathfrak{p}}(R)$ -vector space of dimension at most $1/\varepsilon_1$; and so we can deduce that the $\pi_{\mathfrak{p}}(R)$ -subspace spanned by $\pi_{\mathfrak{p}^{i+1}}(\langle A \rangle_{3C_i} \cap \mathfrak{p}^i \mathcal{O})$ is contained in $\pi_{\mathfrak{p}^{i+1}}(\langle A \rangle_{6\lceil 1/\varepsilon_1 \rceil C_i} \cap \mathfrak{p}^i \mathcal{O})$. Since length of any chain of $\pi_{\mathfrak{p}}(R)$ -subspaces of $\pi_{\mathfrak{p}^{i+1}}(\mathfrak{p}^i \mathcal{O})$ is at most $1/\varepsilon_1$, we deduce that $\pi_{\mathfrak{p}^{i+1}}(\langle A \rangle_{C_{i+1}}) = \pi_{\mathfrak{p}^{i+1}}(R)$ where $C_{i+1} := (6\lceil 1/\varepsilon_1 \rceil)^{\lceil 1/\varepsilon_1 \rceil} C_i$.

Step 3. (Finishing proof for large N) Suppose $N \geq N_0$ where N_0 is given in Step 1. Then by condition (C2), (130) (Step 1), (132) (Step 2), and Corollary 11, we have

$$(133) \quad \pi_{p^{N-4}}(p^{\lceil \varepsilon_2(N-4) \rceil} R) = \pi_{p^{N-4}}(p^{\lceil \varepsilon_2(N-4) \rceil} \mathcal{O}_0) \subseteq \pi_{p^{N-4}}(\langle A \rangle_C)$$

where C is any integer that is larger than a function of ε_1 . Next in (133), we need to change the level from p^{N-4} to p^N . Suppose $s : \pi_{p^{N-4}}(p^{\lceil \varepsilon_2(N-4) \rceil} R) \rightarrow \langle A \rangle_C$ is a section of $\pi_{p^{N-4}}$; that means

$$(134) \quad \text{for any } x \in \pi_{p^{N-4}}(p^{\lceil \varepsilon_2(N-4) \rceil} R) \text{ we have } \pi_{p^{N-4}}(s(x)) = x.$$

Let X be the image of s . Suppose $N > 6$; then $\lceil \varepsilon_2(N-4) \rceil < (N-4) - 1$. Since R is a finite rank free \mathbb{Z}_p -submodule of \mathcal{O} and $\lceil \varepsilon_2(N-4) \rceil < (N-4) - 1$, (134) implies that the \mathbb{Z} -span of X is dense in $p^{\lceil \varepsilon_2(N-4) \rceil} R$. Hence the group generated by $\pi_{p^{N-4}\mathfrak{p}}(X)$ is $\pi_{p^{N-4}\mathfrak{p}}(p^{\lceil \varepsilon_2(N-4) \rceil} R)$. On the other hand, by (134), we have that

$$v_{\mathfrak{p}}(s(x_1 + x_2) - s(x_1) - s(x_2)) \geq (N-4)e$$

for any $x_1, x_2 \in \pi_{p^{N-4}}(p^{\lceil \varepsilon_2(N-4) \rceil} R)$.

If $\pi_{p^{N-4}\mathfrak{p}} \circ s$ is a group homomorphism, then $\pi_{p^{N-4}\mathfrak{p}}(X) = \pi_{p^{N-4}\mathfrak{p}}(p^{\lceil \varepsilon_2(N-4) \rceil} R)$; in particular, we have

$$\begin{aligned} |\pi_{p^{N-4}}(p^{\lceil \varepsilon_2(N-4) \rceil} R)| &= |\pi_{p^{N-4}}(X)| = |X| \\ &\geq |\pi_{p^{N-4}\mathfrak{p}}(X)| = |\pi_{p^{N-4}\mathfrak{p}}(p^{\lceil \varepsilon_2(N-4) \rceil} R)| \\ &= |\pi_{p^{N-4}}(p^{\lceil \varepsilon_2(N-4) \rceil} R)| |\pi_{p^{N-4}\mathfrak{p}}(p^{\lceil \varepsilon_2(N-4) \rceil} R \cap p^{N-4} \mathcal{O})| \\ &\geq |\pi_{p^{N-4}}(p^{\lceil \varepsilon_2(N-4) \rceil} R)| |\pi_{\mathfrak{p}}(R)|, \end{aligned}$$

which is a contradiction. Therefore $\pi_{p^{N-4}\mathfrak{p}} \circ s$ is not a group homomorphism; and so there is $x' \in X - X - X$ such that $v_{\mathfrak{p}}(x') = (N-4)e$. Since $v_{\mathfrak{p}}(p^{-(N-4)}x') = 0$, we have that $|\pi_{\mathfrak{p}}(R)\pi_{\mathfrak{p}}(p^{-(N-4)}x')| = |\pi_{\mathfrak{p}}(R)|$; and so by (131) and Lemma 16, we have $\text{gr}_{N-4,p}(R; \mathcal{O}) \subseteq \pi_{p^{N-3}}(\langle A \rangle_{3C+C_1})$. Hence by (133) we deduce that

$$\pi_{p^{N-3}}(p^{\lceil \varepsilon_2(N-4) \rceil} R) \subseteq \pi_{p^{N-3}}(\langle A \rangle_{4C+C_1}).$$

Repeating this argument 3 more times, we get that

$$(135) \quad \pi_{p^N}(p^{\lceil \varepsilon_2(N-4) \rceil} R) \subseteq \pi_{p^N}(\langle A \rangle_{C'}),$$

where C' is any positive integer that is larger than a function of ε_1 .

Step 4. (Finishing proof for small N) If $N < \max(N_0, 7)$ where N_0 is given in Step 1, then by Step 2 we have $\pi_{p^N}(\langle A \rangle_C) = \pi_{p^N}(R)$ for any integer C that is larger than a function of N_0 \square

Proof of the next Lemma is an adaptation of the argument given in [BG09, Section A.5].

Lemma 45 (Bourgain). *For any positive integers $0 < \varepsilon \ll 1$, positive integers $d, 1 \ll_{d,\varepsilon} C$, and $1 \ll_{d,\varepsilon} N$, the following holds: suppose K is a field extension of \mathbb{Q}_p and $[K : \mathbb{Q}_p] \leq d$. Let \mathcal{O} be the ring of integers of K , \mathfrak{p} be a uniformizing element of K , and \mathfrak{f} be the residue field of K . Suppose $|\mathfrak{f}| \gg_{\varepsilon,d} 1$. Suppose A is a subset of \mathcal{O} which contains 0 and 1. Let R be the closure of the subring of \mathcal{O} that is generated by A . Suppose, for any integer i in $[1, Ne]$, $|\pi_{\mathfrak{p}^i}(A)| \geq |\mathfrak{f}|^{i\varepsilon}$, where e is the ramification index of K over \mathbb{Q}_p . Then there are positive integers m and n , $a \in \mathcal{O}$, and a closed subfield K_0 of K with ring of integers \mathcal{O}_0 such that*

$$\begin{aligned} n \ll_{d,\varepsilon} N \ll_{d,\varepsilon} n - m, & \quad (\text{Exponent conditions}) \\ \pi_{\mathfrak{p}^n}(\mathcal{O}_0 a) \subseteq \pi_{\mathfrak{p}^n}(\langle A \rangle_C), \quad v_{\mathfrak{p}}(a) = m, & \quad (\text{Bounded generation}) \\ |\pi_{\mathfrak{p}}(\mathcal{O}_0)| \geq |\pi_{\mathfrak{p}}(R)|. & \quad (\text{Box dimension control}) \end{aligned}$$

Proof. The key point is that we can detect in a *bounded* number of steps whether the grades of the ring generated by A are getting larger.

Let $\varepsilon_1 := \varepsilon$ and assume $0 < \varepsilon_1 \ll \varepsilon_2 \leq 1/4$ satisfy the inequality given in Proposition 44.

Step 0. (Setup) We will recursively define a sequence of quadruples (A_i, R_i, n_i, C_i) of subsets A_i of \mathcal{O} , subrings R_i of \mathcal{O} , and positive integers n_i and C_i with the following properties:

(P1) (Ring conditions) R_i is the closure of the ring generated by A_i ; and n_i is the largest integer in $[1, N]$ such that

$$|\text{gr}_{0,p}(R_i; \mathcal{O})| = |\text{gr}_{1,p}(R_i; \mathcal{O})| = \cdots = |\text{gr}_{n_i-1,p}(R_i; \mathcal{O})|.$$

(P2) (Set conditions)

(P2-a) $0, 1 \in A_i$.

(P2-b) For any integer j in $[1, Ne]$, $|\pi_{\mathfrak{p}^j}(A_i)| \geq |\mathfrak{f}|^{j\varepsilon_1}$.

(P2-c) $\pi_{\mathfrak{p}}(A_i) = \pi_{\mathfrak{p}}(R_i)$.

(P3) (Bounded generation) $\pi_{p^{n_i}}(\langle A_i \rangle_{C_i}) \supseteq \pi_{p^{n_i}}(p^{\lceil n_i \varepsilon_2 \rceil} R_i)$ and $C_i \ll_{d,\varepsilon_1} 1$.

(P4) (Connection between sets)

(P4-a) $\alpha_i A_i \subseteq \alpha_i A_{i+1} \subseteq \langle A_i \rangle_{C_i}$ for some $\alpha_i \in \mathcal{O}$ such that $v_{\mathfrak{p}}(\alpha_i) \ll_{d,\varepsilon} n_i$ and $\pi_{\mathfrak{p}}(p^{-v_{\mathfrak{p}}(\alpha_i)/e} \alpha_i) = 1$.

(P4-b) $|\pi_{\mathfrak{p}}(R_i)| < |\pi_{\mathfrak{p}}(R_{i+1})|$.

And we stop when $n_i \geq N\delta_0$ where δ_0 is a fixed positive number smaller than ε_1 and $1/d$.

Step 1. Since $|\pi_{\mathfrak{p}}(A)| \geq |\mathfrak{f}|^{\varepsilon_1}$, by Lemma 43 there is a positive integer C'_0 that is at most a function of ε_1 such that

$$(136) \quad \pi_{\mathfrak{p}}(\langle A \rangle_{C'_0}) \text{ is a subfield of } \mathfrak{f}.$$

Let $A_0 := \langle A \rangle_{C'_0}$; we notice that A_0 satisfies (P2) because of the assumption and (136).

Step 2. Suppose we have already defined A_i that satisfies Property (P2). At this step, we more or less get Property (P3) and give an indication on what C_i can be. Furthermore we introduce an auxiliary set A'_{i+1} and an auxiliary ring R'_{i+1} . This pair (A'_{i+1}, R'_{i+1}) will help us to enlarge $\pi_p(R_i)$ in a bounded number of steps.

Let R_i be the closure of the subring generated by A_i ; and let n_i be the largest positive integer in $[1, N]$ such that the first n_i grades $\text{gr}_{j,p}(R_i; \mathcal{O})$ of R_i have equal number of elements; that means Property (P1) holds. By Proposition 44 there is a positive integer $\overline{C}_i \ll_{d, \varepsilon_1} 1$ such that $\pi_{p^{n_i}}(\langle A_i \rangle_{\overline{C}_i}) \supseteq \pi_{p^{n_i}}(p^{\lceil n_i \varepsilon_2 \rceil} R_i)$. So for any positive integer C_i in $[\overline{C}_i, \Theta_{d, \varepsilon_1}(\overline{C}_i)]$ Property (P3) holds.

If $n_i \geq N\delta_0$, we let $C_i := \overline{C}_i$ and we are done. If not, we have to proceed and define A_{i+1} and C_i , and make sure that Property (P3) and Property (P4) hold. We also notice that

$$|\pi_p(R_0)| < |\pi_p(R_1)| < \cdots < |\pi_p(R_i)|$$

and $\pi_p(R_j)$'s are \mathfrak{f}_p -subspaces of $\pi_p(\mathcal{O})$. Hence

$$(137) \quad i \leq d.$$

Next we proceed as in Step 3 of proof of Proposition 44: let $m_i := \lceil n_i \varepsilon_2 \rceil$ and $s : \pi_{p^{n_i}}(p^{m_i} R_i) \rightarrow \langle A_i \rangle_{\overline{C}_i}$ be a section of $\pi_{p^{n_i}}$ which sends 0 to 0; that means for any $x \in \pi_{p^{n_i}}(p^{m_i} R_i)$ we have $\pi_{p^{n_i}}(s(x)) = x$ and $s(0) = 0$. Let

$$(138) \quad A'_{i+1} := s(p^{m_i})^{-1}(\langle A_i \rangle_{2\overline{C}_i} \cap p^{m_i} \mathcal{O}),$$

and R'_{i+1} be the closure of the ring generated by A'_{i+1} . Since $s(p^{m_i}) \in \langle A_i \rangle_{2\overline{C}_i}$, we deduce that $s(p^{m_i}) \langle A_i \rangle_{\overline{C}_i}$ is a subset of $\langle A_i \rangle_{2\overline{C}_i} \cap p^{m_i} \mathcal{O}$; and so

$$(139) \quad \langle A_i \rangle_{\overline{C}_i} \subseteq A'_{i+1};$$

and since $0, 1 \in A_i$, we deduce that $A_i \subseteq A'_{i+1}$ and $0, 1 \in A'_{i+1}$. Let n'_{i+1} be the largest positive integer such that the first n'_{i+1} -th grades of R'_{i+1} have equal sizes; that means

$$|\text{gr}_{0,p}(R'_{i+1}; \mathcal{O})| = \cdots = |\text{gr}_{n'_{i+1}-1,p}(R'_{i+1}; \mathcal{O})|.$$

Step 3. In this step, we show

$$(140) \quad \pi_{p^{n_i-m_i}}(A'_{i+1}) = \pi_{p^{n_i-m_i}}(R_i) = \pi_{p^{n_i-m_i}}(R'_{i+1});$$

and then we deduce that

$$(141) \quad n_i - m_i \leq n'_{i+1} \leq n_i.$$

To show (140), first we start with an argument similar to proof of Corollary 17, and prove that

$$(142) \quad \pi_{p^{n_i}}(R_i \cap p^{m_i} \mathcal{O}) = \pi_{p^{n_i}}(s(p^{m_i}) R_i).$$

Let $l : \pi_{p^{n_i-m_i}}(R_i) \rightarrow \pi_{p^{n_i}}(R_i \cap p^{m_i} \mathcal{O})$, $l(\pi_{p^{n_i-m_i}}(x)) := \pi_{p^{n_i}}(s(p^{m_i})x)$. One can see that l is a well-defined injection. On the other hand,

$$|\pi_{p^{n_i-m_i}}(R_i)| = \prod_{j=0}^{n_i-m_i-1} |\text{gr}_{j,p}(R_i; \mathcal{O})| = \prod_{j=m_i}^{n_i-1} |\text{gr}_{j,p}(R_i; \mathcal{O})| = |\pi_{p^{n_i}}(R_i \cap p^{m_i} \mathcal{O})|;$$

and so l is a bijection, which implies (142). By (142), we deduce $(R_i \cap p^{m_i} \mathcal{O}) + p^{n_i} \mathcal{O} = s(p^{m_i})R_i + p^{n_i} \mathcal{O}$; and so

$$s(p^{m_i})^{-1}(R_i \cap p^{m_i} \mathcal{O}) + p^{n_i - m_i} \mathcal{O} = R_i + p^{n_i - m_i} \mathcal{O},$$

which implies

$$(143) \quad \pi_{p^{n_i - m_i}}(A'_{i+1}) \subseteq \pi_{p^{n_i - m_i}}(s(p^{m_i})^{-1}(R_i \cap p^{m_i} \mathcal{O})) = \pi_{p^{n_i - m_i}}(R_i).$$

Since $\pi_{p^{n_i - m_i}}(R_i)$ is a ring, by (139) and (143) we deduce that

$$(144) \quad \pi_{p^{n_i - m_i}}(R_i) = \pi_{p^{n_i - m_i}}(R'_{i+1}).$$

Because of the existence of the section s , we have

$$(145) \quad s(p^{m_i})A'_{i+1} + p^{n_i} \mathcal{O} \supseteq p^{m_i} R_i + p^{n_i} \mathcal{O} = s(p^{m_i})R_i + p^{n_i} \mathcal{O},$$

which implies $\pi_{p^{n_i - m_i}}(A'_{i+1}) \supseteq \pi_{p^{n_i - m_i}}(R_i)$. And so by (143) and (144) we get (140). By (140) and Property (P1) for R_i , we have that $n'_{i+1} \geq n_i - m_i$. By (139), we have that $R_i \subseteq R'_{i+1}$; and so together with $n_i < N$ we deduce

$$(146) \quad |\mathrm{gr}_{n_i, p}(R'_{i+1}; \mathcal{O})| \geq |\mathrm{gr}_{n_i, p}(R_i; \mathcal{O})| > |\mathrm{gr}_{0, p}(R_i; \mathcal{O})| = |\mathrm{gr}_{0, p}(R'_{i+1}; \mathcal{O})|,$$

where the last equality is deduced from (144). By (146), we get that $n'_{i+1} \leq n_i$; and (141) follows.

Step 4. In this step, we define A_{i+1} , C_i , and α_i .

We start with a similar argument as in Step 3 and prove that

$$(147) \quad \pi_{p^{n'_{i+1}}}(R'_{i+1} \cap p^{\lfloor n'_{i+1}/2 \rfloor} \mathcal{O}) = \pi_{p^{n'_{i+1}}}(s(p^{m_i})^{-1} s(p^{\lfloor n'_{i+1}/2 \rfloor + m_i}) R'_{i+1}).$$

Notice that $m_i \leq \lfloor n'_{i+1}/2 \rfloor + m_i \leq n_i/2 + m_i < n_i$; and so $v_p(s(p^{m_i})^{-1} s(p^{\lfloor n'_{i+1}/2 \rfloor + m_i})) = \lfloor n'_{i+1}/2 \rfloor e$. Let

$$l' : \pi_{p^{\lfloor n'_{i+1}/2 \rfloor}}(R'_{i+1}) \rightarrow \pi_{p^{n'_{i+1}}}(R'_{i+1} \cap p^{\lfloor n'_{i+1}/2 \rfloor} \mathcal{O}), l'(\pi_{p^{\lfloor n'_{i+1}/2 \rfloor}}(x)) := \pi_{p^{n'_{i+1}}}(s(p^{m_i})^{-1} s(p^{\lfloor n'_{i+1}/2 \rfloor + m_i})x).$$

One can see that l' is a well-defined injective map. We also have

$$|\pi_{p^{\lfloor n'_{i+1}/2 \rfloor}}(R'_{i+1})| = \prod_{j=0}^{\lfloor n'_{i+1}/2 \rfloor} |\mathrm{gr}_{j, p}(R'_{i+1}; \mathcal{O})| = \prod_{j=\lfloor n'_{i+1}/2 \rfloor}^{n'_{i+1}} |\mathrm{gr}_{j, p}(R'_{i+1}; \mathcal{O})| = \pi_{p^{n'_{i+1}}}(R'_{i+1} \cap p^{\lfloor n'_{i+1}/2 \rfloor} \mathcal{O}).$$

Hence l' is a bijection, which implies (147). By (138), we have $s(p^{m_i})^{-1} s(p^{\lfloor n'_{i+1}/2 \rfloor + m_i}) \in A'_{i+1}$; therefore by $\lfloor n'_{i+1}/2 \rfloor \leq n_i - m_i$, $\pi_{p^{n_i - m_i}}(A'_{i+1}) = \pi_{p^{n_i - m_i}}(R'_{i+1})$ (see (140)), and (147), we get that

$$(148) \quad \pi_{p^{n'_{i+1}}}(R'_{i+1}) = \pi_{p^{n'_{i+1}}}(\langle A'_{i+1} \rangle_2).$$

Next we show that $\pi_{p^{n'_{i+1}+1}}(R'_{i+1}) = \pi_{p^{n'_{i+1}+1}}(\langle A'_{i+1} \rangle_{\mathcal{O}_{[K, \mathbb{Q}_p]}(1)})$. Let $c_0 := 2$ and V_0 be the zero \mathfrak{f}_p -vector space; we will recursively define an increasing sequence $\{c_j\}_j$ of positive integers and \mathfrak{f}_p -vector spaces V_j such that

- (*) either $\pi_{p^{n'_{i+1}+1}}(\langle A'_{i+1} \rangle_{c_j})$ is a ring or there is a subspace V_{j+1} of $\mathrm{gr}_{n'_{i+1}, p}(R'_{i+1}; \mathcal{O})$ which is a subset of $\pi_{p^{n'_{i+1}+1}}(\langle A'_{i+1} \rangle_{c_{j+1}} \cap p^{n'_{i+1}} \mathcal{O})$ and $\dim_{\mathfrak{f}_p} V_{j+1} = j + 1$.

Suppose $\pi_{p^{n'_{i+1}+1}}(\langle A'_{i+1} \rangle_{c_j})$ is not a ring; then by (148) there is $x_j \in R'_{i+1} \cap p^{n'_{i+1}} \mathcal{O} \cap \langle A'_{i+1} \rangle_{3c_j}$ such that $\pi_{p^{n'_{i+1}+1}}(x_j) \notin \pi_{p^{n'_{i+1}+1}}(\langle A'_{i+1} \rangle_{c_j})$. Hence by the graded structure of $\mathrm{gr}_p(R'_{i+1}; \mathcal{O})$ and (148), we have that $\mathfrak{f}_p x_j + V_j \subseteq \pi_{p^{n'_{i+1}+1}}(\langle A'_{i+1} \rangle_{1+4c_j})$. Hence $V_{j+1} := \mathfrak{f}_p x_j + V_j$ and $c_{j+1} := 1 + 4c_j$ satisfy (*).

Since $\mathrm{gr}_{n'_{i+1}, p}(R'_{i+1}; \mathcal{O})$ is an \mathfrak{f}_p -vector space of dimension at most $\dim_{\mathfrak{f}_p}(\mathcal{O}/p\mathcal{O}) = [K : \mathbb{Q}_p]$, we get that $\pi_{p^{n'_{i+1}+1}}(R'_{i+1}) = \pi_{p^{n'_{i+1}+1}}(\langle A'_{i+1} \rangle_{C'_i})$ where $C'_i \ll_d 1$. Let $s' : \pi_{p^{n'_{i+1}+1}}(R'_{i+1}) \rightarrow \langle A'_{i+1} \rangle_{C'_i}$ be a section of

$\pi_p^{n'_{i+1}+1}$; and

$$(149) \quad A''_{i+1} := s'(p^{n'_{i+1}})^{-1} \left(\langle A'_{i+1} \rangle_{2C'_i} \cap p^{n'_{i+1}} \mathcal{O} \right).$$

Hence

$$(150) \quad |\pi_p(A''_{i+1})| = |\text{gr}_{n'_{i+1}, p}(R'_{i+1}; \mathcal{O})| > |\text{gr}_{0, p}(R'_{i+1}; \mathcal{O})| = |\pi_p(A_i)|.$$

By (150), as in Step 1, we have that $\pi_p(\langle A''_{i+1} \rangle_{C_0})$ is a subfield of \mathfrak{f} . Let $A_{i+1} := \langle A''_{i+1} \rangle_{C_0}$. By (138)

$$(151) \quad s(p^{m_i})^{2C_0 C'_i} s'(p^{n'_{i+1}})^{C_0} A_{i+1} \subseteq \langle A_i \rangle_{4C_0 C'_i \overline{C}_i},$$

and $0, 1 \in A_{i+1}$. By (149) and $s'(p^{n'_{i+1}}) \in \langle A'_{i+1} \rangle_{C'_i}$, we get that

$$(152) \quad \langle A'_{i+1} \rangle_{C'_i} \subseteq A''_{i+1}.$$

By (139) and (152), we get that

$$(153) \quad A_i \subseteq A_{i+1}.$$

Therefore by (150), (151), and (153), we get that A_{i+1} , $C_i := 4C_0 C'_i \overline{C}_i$, and $\alpha_i := s(p^{m_i})^{2C_0 C'_i} s'(p^{n'_{i+1}})^{C_0}$ satisfy Property (P4) (Connection between sets). Since A_i satisfies Property (P2), by (153) we get that A_{i+1} also satisfies Property (P2) (Set conditions).

Step 5. Finishing the proof.

By (137), the above process stops in $i_0 \leq d+1$ steps, and we get $n_{i_0} \geq N\delta_0$. By Property (P4), inductively we get that for any integer j in $[0, i_0]$

$$\gamma_j A_j \subseteq \langle A \rangle_{\prod_{i=0}^{j-1} C_i},$$

for some $\gamma_j \in \mathcal{O}$ such that $v_p(\gamma_j) \ll_{d, \varepsilon_1} N\delta_0$. Hence

$$(154) \quad \gamma_{i_0} A_{i_0} \subseteq \langle A \rangle_{\prod_{i=0}^{i_0-1} C_i}.$$

By Property (P3) we have

$$\pi_p^{n_{i_0}}(\langle A_{i_0} \rangle_{C_{i_0}}) \supseteq \pi_p^{n_{i_0}}(p^{\lceil n_{i_0} \varepsilon_2 \rceil} R_{i_0});$$

and so by (154)

$$(155) \quad \pi_p^{n_{i_0} e + C_{i_0} v_p(\gamma_{i_0})} (\gamma_{i_0}^{C_{i_0}} p^{\lceil n_{i_0} \varepsilon_2 \rceil} R_{i_0}) \subseteq \pi_p^{n_{i_0} e + C_{i_0} v_p(\gamma_{i_0})} (\langle \gamma_{i_0} A_{i_0} \rangle_{C_{i_0}}) \subseteq \pi_p^{n_{i_0} e + C_{i_0} v_p(\gamma_{i_0})} (\langle A \rangle_{\prod_{i=0}^{i_0} C_i}).$$

Let $n := n_{i_0} e + C_{i_0} v_p(\gamma_{i_0})$, $m := \lceil n_{i_0} \varepsilon_2 \rceil e + C_{i_0} v_p(\gamma_{i_0})$, $a := p^{\lceil n_{i_0} \varepsilon_2 \rceil} \gamma_{i_0}^{C_{i_0}}$, and $C := \prod_{i=0}^{i_0} C_i$. Then

$$(156) \quad n \ll_{d, \varepsilon_1, \delta_0} N \ll_{d, \varepsilon_1, \delta_0} n - m, \quad (\text{Exponent conditions})$$

$$(157) \quad \pi_p^n(R_{i_0} a) \subseteq \pi_p^n(\langle A \rangle_C), \quad v_p(a) = m, \quad C \ll_{d, \varepsilon_1} 1, \quad (\text{Bounded generation})$$

$$(158) \quad |\text{gr}_{0, p}(R_{i_0}; \mathcal{O})| = \cdots = |\text{gr}_{(n-m)/e, p}(R_{i_0}; \mathcal{O})| \geq |\pi_p(R)|. \quad (\text{Grades equality})$$

Since $N \gg_{d, \varepsilon_1, \delta_0} 1$, by (158) (Grades equality) and Step 1 of proof of Proposition 44, there is a subfield K_0 of K with ring of integers \mathcal{O}_0 such that

$$\pi_p^{n-m-4e}(R_{i_0}) = \pi_p^{n-m-4e}(\mathcal{O}_0);$$

and so

$$(159) \quad \pi_p^{n-4e}(\mathcal{O}_0 a) = \pi_p^{n-4e}(R_{i_0} a) \subseteq \pi_p^{n-4e}(\langle A \rangle_C),$$

and $|\pi_p(\mathcal{O}_0)| \geq |\pi_p(R)| \geq |\mathfrak{f}|^{e\varepsilon_1}$. And the claim follows by (156), (157), (158), and (159). \square

Proof of Theorem 1. Step 1. (Small residue field) In [BG09, Appendix], Bourgain has essentially proved Theorem 1 for a given fixed prime p , with two short comings: (1) in [BG09, Appendix], it is assumed that p is a fixed prime that is *large* compared to d ; (2) In [BG09, Proposition 3.3], Bourgain only claims that a thick \mathbb{Z}_p -segment at certain scale can be generated in bounded number of steps.

Going through the argument in [BG09, Appendix], one can see that the largeness of p compared to d is used to ensure that K is not a widely ramified extension of \mathbb{Q}_p . In turn, this is used in [BG09, Section A.6] to describe the structure of certain subrings of \mathcal{O} . In fact [BG09, Section A.6] is the only place, where the largeness of p is used. So Proposition 19 and Step 1 of proof of Proposition 44 remove this obstruction.

Going through the proof of [BG09, Proposition 3.3] in [BG09, Section A.5], one can observe that one gets the stronger version as it is presented in Lemma 45.

Step 2. (Large residue field) As in [BG09] (see also [SG17, Section 2.3]), there is $A' \subseteq A$ such that $\pi_{\mathfrak{p}^N}(A')$ is an (m_0, \dots, m_{N-1}) -regular set, and

$$|\pi_{\mathfrak{p}^N}(A')| \geq \frac{|A|}{(2 \log |f|)^N} \geq |f|^{N\varepsilon/2},$$

for $|f| \gg_\varepsilon 1$. Let

$$\bar{n} := \max\{k \mid |\pi_{\mathfrak{p}^k}(A')| = m_0 \cdots m_{k-1} < |f|^{k\varepsilon/4}\}.$$

So for any $\bar{n} + 1 \leq l \leq N$, we have

$$(160) \quad \prod_{i=\bar{n}}^{l-1} m_i \geq \left(\prod_{i=0}^{l-1} m_i \right) \left(\prod_{i=0}^{\bar{n}-1} m_i \right)^{-1} \geq |f|^{l\varepsilon/4} |f|^{-\bar{n}\varepsilon/4} = |f|^{(l-\bar{n})\varepsilon/4}.$$

We also have

$$(161) \quad |f|^{N-\bar{n}} \geq \prod_{i=\bar{n}}^{N-1} m_i \geq \left(\prod_{i=0}^{N-1} m_i \right) \left(\prod_{i=0}^{\bar{n}-1} m_i \right)^{-1} \geq |f|^{N\varepsilon/2} |f|^{-\bar{n}\varepsilon/4}.$$

Therefore $M := N - \bar{n} \geq N\varepsilon/2 - \bar{n}\varepsilon/4 = N\varepsilon/2 - (N - M)\varepsilon/4 \geq N\varepsilon/4$. So there is a subset $B \subseteq \mathcal{O}$ such that

- (a) $\mathfrak{p}^{N'} B \subseteq A - A$ for $N' \leq N(1 - \varepsilon/4)$ (here N' is just \bar{n}),
- (b) $\pi_{\mathfrak{p}^M}(B)$ is a regular set for $M \geq N\varepsilon/4$,
- (c) $|\pi_{\mathfrak{p}^i}(B)| \geq |f|^{i\varepsilon/4}$ for any $1 \leq i \leq M$.

In particular, there is $\lambda \in (B - B) \cap \mathcal{O}^\times$. So replacing A with $\lambda^{-1}A$ and B with $\lambda^{-1}(B - B)$, we can and will assume that B contains 0 and 1. Notice that proving the claim of Theorem 1 for a unit multiple $\lambda^{-1}A$ of A implies the claim for A .

Hence by Lemma 45 there are positive integers M_1 and M_2 , $a \in \mathcal{O}$, and a subfield K_0 of K with ring of integers \mathcal{O}_0 such that

$$(162) \quad \pi_{\mathfrak{p}^{M_2}}(\mathcal{O}_0 a) \subseteq \pi_{\mathfrak{p}^{M_2}}(\langle B \rangle_C),$$

$$(163) \quad M_2 \ll_{d,\varepsilon} M \ll_{d,\varepsilon} M_2 - M_1, \quad v_{\mathfrak{p}}(a) = M_1,$$

$$(164) \quad |\pi_{\mathfrak{p}}(\mathcal{O}_0)| \geq |\pi_{\mathfrak{p}}(\mathcal{O})|^{\varepsilon/4}.$$

And so by (162) we get

$$\pi_{\mathfrak{p}^{M_2+CN'}}(\mathcal{O}_0(\mathfrak{p}^{N'} a)) \subseteq \pi_{\mathfrak{p}^{M_2+CN'}}(\langle \mathfrak{p}^{N'} B \rangle_C) \subseteq \pi_{\mathfrak{p}^{M_2+CN'}}(\langle A \rangle_C).$$

Let $N_1 := M_1 + CN'$ and $N_2 := M_2 + CN'$. Hence by (163), we get that N_1 and N_2 satisfy (3) for suitably chosen δ and enlarging C if necessary. And the claim follows. \square

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