

Expansion in perfect groups

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Abstract

Let Γ be a subgroup of $\mathrm{GL}_d(\mathbb{Z}[1/q_0])$ generated by a finite symmetric set S . For an integer q , denote by π_q the projection map $\mathbb{Z}[1/q_0] \rightarrow \mathbb{Z}[1/q_0]/q\mathbb{Z}[1/q_0]$. We prove that the Cayley graphs of $\pi_q(\Gamma)$ with respect to the generating sets $\pi_q(S)$ form a family of expanders when q ranges over square-free integers with large prime divisors if and only if the connected component of the Zariski-closure of Γ is perfect, i.e. it has no nontrivial Abelian quotients.

1 Introduction

Let \mathcal{G} be a graph, and for a set of vertices $X \subset V(\mathcal{G})$, denote by ∂X the set of edges that connect a vertex in X to one in $V(\mathcal{G}) \setminus X$. Define

$$c(\mathcal{G}) = \min_{X \subset V(\mathcal{G}), |X| \leq |V(\mathcal{G})|/2} \frac{|\partial X|}{|X|},$$

where $|X|$ denotes the cardinality of the set X . A family of graphs is called a family of expanders, if $c(\mathcal{G})$ is bounded away from zero for graphs \mathcal{G} that belong to the family. Expanders have a wide range of applications in computer science (see e.g. Hoory, Linial and Wigderson [37] for a survey on expanders) and recently they found remarkable applications in pure mathematics as well.

Let S be a symmetric (i.e. closed for taking inverses) subset of $\mathrm{GL}_d(\mathbb{Q})$ and let Γ be the group generated by S . For any positive integer q , let $\pi_q : \mathbb{Z} \rightarrow \mathbb{Z}/q\mathbb{Z}$ be the residue map. If the prime factors of q are large, π_q induces a homomorphism from Γ to $\mathrm{GL}_d(\mathbb{Z}/q\mathbb{Z})$. We denote this and all the similar maps by π_q also. In this article, we give a necessary and sufficient condition under which the family of Cayley graphs $\mathcal{G}(\pi_q(\Gamma), \pi_q(S))$ form expanders as q runs through square-free integers with large prime factors. Let us recall that if $S \subset G$ is a symmetric set of generators, then the Cayley graph $\mathcal{G}(G, S)$ of G with respect to the generating set S is defined to be the graph whose vertex set is G , and two vertices $x, y \in G$ are connected exactly if $y \in Sx$.

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Theorem 1. *Let $\Gamma \subseteq \mathrm{GL}_d(\mathbb{Z}[1/q_0])$ be the group generated by a symmetric set S . Then $\mathcal{G}(\pi_q(\Gamma), \pi_q(S))$ form a family of expanders when q ranges over square-free integers coprime to q_0 if and only if the connected component of the Zariski-closure of Γ is perfect.*

1.1 Motivation and related results

A result of the type of Theorem 1 was proved by Bourgain, Gamburd and Sarnak [11]. They proved that such Cayley graphs form expanders if $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is Zariski-dense. Their main motivation was to formulate and prove an affine sieve theorem. Moreover, they proved the affine sieve theorem for groups more general than SL_2 provided a conjecture of Lubotzky holds (see [11, Conjecture 1.5] and [47]) which is a special case of our Theorem 1. Following the ideas in [11] and using Theorem 1, in a forthcoming paper, Salehi Golsefidy and Sarnak [54] get a similar affine sieve theorem whenever the character group of the connected component of the Zariski-closure of Γ is trivial.

Following [11], now there is a rich literature of sieving applications of expander graphs not limited to number theory. We refer to the recent surveys of Kowalski [42] and Lubotzky [48] for more details on these developments. Besides sieving, results similar to the above theorem are useful in studying covers of hyperbolic 3-manifolds, see the paper of Long, Lubotzky and Reid [46] following the work of Lackenby [43].

The question about the expanding properties of mod q quotients was first studied only for “thick” groups, namely lattices in semisimple Lie groups. The first results used the representation theory of the underlying Lie group; property (T) in Kazhdan [38] and Margulis [49] and automorphic forms in e.g. [56], [18], [40] and [20]. Later Sarnak and Xue [55] developed a more elementary method. Kelmer and Silberman [39] combined this method with recent advances on automorphic forms to obtain a very general result on arbitrary arithmetic lattices. The advantage of these results over our method is that they give explicit and very good bounds.

Lubotzky was the first person who asked this question for a “thin” group in his famous 1-2-3 conjecture (see [47]).¹ Shalom [57], [58] obtained the first results which establish the expander property for quotients of certain non-lattices. A few years later Gamburd [27] showed that quotients of $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ are expanders if the Hausdorff dimension of the limit set of Γ is larger than $5/6$. The first paper which achieved a result which depend merely on the Zariski-closure of Γ was obtained by Bourgain and Gamburd [7]. Their assumptions were that the Zariski-closure of Γ is SL_2 , and the modulus q is prime. In the past four years, several articles appeared which extended that result, see [7]–[11], and [61]. However, there are still interesting questions to explore. For instance,

Question 2. Does the family of Cayley graphs $\mathcal{G}(\pi_q(\Gamma), \pi_q(S))$ form expanders as q runs through *any* positive integer with large prime factors if the connected

¹He asked this question for $S = \left\{ \begin{bmatrix} 1 & \pm 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pm 3 & 1 \end{bmatrix} \right\}$.

component of the Zariski-closure of Γ is perfect?

Bourgain and the second author [13] give an affirmative answer to this question when the Zariski-closure of Γ is $\mathbb{S}\mathbb{L}_d$.

Question 3. If $\Gamma \subseteq \mathrm{GL}_d(\mathbb{F}_p(t))$ is generated by a symmetric set S , then what is the necessary and sufficient condition such that $\mathcal{G}(\pi_{\mathfrak{p}}(\Gamma), \pi_{\mathfrak{p}}(S))$ form expanders as \mathfrak{p} runs through square-free polynomials with large degree prime factors?

Moreover, one might hope that the answer is positive even to the following very general question that was communicated to us by Alex Lubotzky:

Question 4 (Lubotzky). Let $\Gamma \subseteq \mathrm{GL}_n(A)$ is a finitely generated subgroup, where A is an integral domain which is generated by the traces of the elements of Γ . Is it true that if the Zariski-closure of Γ is semisimple, then the Cayley graphs $\mathcal{G}(\pi_{\mathfrak{a}}(\Gamma), \pi_{\mathfrak{a}}(S))$ form a family of expanders as \mathfrak{a} ranges through finite index ideals of A ?

We also mention that there are studies devoted to the problem of expansion with respect to random generators, see [16] and [17], and in the work of Breuillard and Gamburd [14] it is proved that except maybe for a set of primes p of zero density, $\mathrm{SL}_2(\mathbb{F}_p)$ are expanders with respect to any generators.

1.2 Groups defined over number fields

Let k be a number field and Γ be a finitely generated subgroup of $\mathrm{GL}_n(k)$. Let us also remark that Theorem 1 tells us under what condition the Cayley graphs of the square free quotients of Γ form expanders. To be precise, by the means of restriction of scalars, we can view Γ as a subgroup Γ' of $R_{k/\mathbb{Q}}(\mathrm{GL}_d)(\mathbb{Q}) \subseteq \mathrm{GL}_{rd}(\mathbb{Q})$, where $r = \dim_{\mathbb{Q}} k$. Now it is easy to see that $\mathcal{G}(\pi_{\mathfrak{q}}(\Gamma), \pi_{\mathfrak{q}}(S))$ form expanders as \mathfrak{q} runs through square free ideals of \mathcal{O}_k (the ring of integers in k) with large prime factors if and only if $\mathcal{G}(\pi_q(\Gamma'), \pi_q(S'))$ form expanders as q runs through square free integers with large prime factors. By Theorem 1, we know the necessary and sufficient condition under which the latter holds. In the following example, we present a finitely generated subgroup Γ of $\mathrm{GL}_d(\mathbb{Q}[i])$ whose Zariski-closure is Zariski connected and perfect but $\mathcal{G}(\pi_{\mathfrak{p}}(\Gamma), \pi_{\mathfrak{p}}(S))$ do not form expanders as \mathfrak{p} runs through prime ideals of $\mathbb{Z}[i]$. It shows that in general it is necessary to view Γ as a subgroup of $\mathrm{GL}_{rd}(\mathbb{Q})$ and then look at its Zariski-closure.

Example 5. Let h be a non-degenerate symplectic form on $V = \mathbb{Z}^m$. Let \mathbb{H} be the Heisenberg group associated with h . To be precise, $\mathbb{H}(\mathbb{Z})$ is the set $V \times \mathbb{Z}$ endowed with the group law

$$(v, t) \cdot (v', t') := (v + v', t + t' + h(v, v')).$$

From the definition it is clear that \mathbb{H} is a central extension of the group scheme \mathbb{V} associated with V . The action of the symplectic group $\mathrm{Sp}_{h,V}$ on \mathbb{V} can be

naturally extended to an action on \mathbb{H} by acting trivially on the center. Now let $\mathbb{G} = \mathrm{Sp}_{h,V} \times \mathbb{H}$ and

$$\Gamma := \{(\gamma, (v, t)) \in \mathbb{G}(\mathbb{Z}[i]) \mid \gamma \in \mathrm{Sp}_{h,V}(\mathbb{Z}), v \in V, t \in \mathbb{Z}[i]\}.$$

It is easy to see $\mathbb{G}_{\mathbb{Q}[i]} := \mathbb{G} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Q}[i])$ is a perfect Zariski-connected $\mathbb{Q}[i]$ -group and Γ is a finitely generated Zariski-dense subgroup of $\mathbb{G}_{\mathbb{Q}[i]}$. On the other hand, the Zariski-closure of Γ' in $R_{\mathbb{Q}[i]/\mathbb{Q}}(\mathbb{G})$ is isomorphic to $\mathbb{G} \times \mathbb{G}_a$, which is not perfect. Thus the Cayley graphs $\mathcal{G}(\pi_{\mathfrak{p}}(\Gamma), \pi_{\mathfrak{p}}(S))$ do not form expanders as \mathfrak{p} runs through primes in $\mathbb{Z}[i]$.

As we have seen in Example 5, in general, the connected component of the Zariski-closure \mathbb{G} of Γ in $\mathrm{GL}_d(k)$ might be perfect but the connected component of its Zariski-closure \mathbb{G}' in $R_{k/\mathbb{Q}}(\mathbb{G})(\mathbb{Q}) \subseteq \mathrm{GL}_{rd}(\mathbb{Q})$, where $r = \dim_{\mathbb{Q}} k$, might be not perfect. However it is easy to see that if \mathbb{G} is semisimple, so is \mathbb{G}' . Hence we get the following corollary:

Corollary 6. *Let $\Gamma \subseteq \mathrm{GL}_d(k)$ be the group generated by a symmetric set S , where k is a number field. If the Zariski-closure of Γ is semisimple, then $\mathcal{G}(\pi_{\mathfrak{q}}(\Gamma), \pi_{\mathfrak{q}}(S))$ form a family of expanders when \mathfrak{q} ranges over square-free ideals of the ring of integers \mathcal{O}_k in k with large prime factors.*

1.3 Outline of the argument

Similarly to most of the previous works done on this problem [7]–[11] and [61], first we prove *escape from proper subgroups* and then we show the occurrence of the *flattening phenomenon*.

Proposition 7 (Escape from subgroups). *Let \mathbb{G} be a Zariski-connected, perfect algebraic group defined over \mathbb{Q} . Let $S \subset \mathbb{G}(\mathbb{Q})$ be finite and symmetric. Assume that S generates a subgroup $\Gamma < \mathbb{G}(\mathbb{Q})$ which is Zariski dense in \mathbb{G} .*

Then there is a constant δ depending only on S , and there is a symmetric set $S' \subset \Gamma$ such that the following holds. For any square-free integer q which is relatively prime to the denominators of the entries of S , and for any proper subgroup $H < \pi_q(\Gamma)$ and for any even integer $l \geq \log q$, we have

$$\pi_q[\chi_{S'}^{(l)}](H) \ll [\pi_q(\Gamma) : H]^{-\delta}.$$

The notation used in this and the next proposition is explained in detail in Section 2. Now we only mention that $\chi_{S'}$ is the normalized counting (probability) measure on S' and $\chi_{S'}^{(l)}$ is the l -fold convolution of $\chi_{S'}$ with itself.

Proposition 8 (l^2 -flattening). *Let \mathbb{G} be a Zariski-connected, perfect algebraic group defined over \mathbb{Q} . Let $\Gamma < \mathbb{G}(\mathbb{Q})$ be a finitely generated Zariski-dense subgroup. Then for any $\varepsilon > 0$, there is some $\delta > 0$ depending only on ε and \mathbb{G} such that the following holds. Let q be a square-free integer which is relatively prime to the entries of the elements of Γ and let μ and ν be probability measures on $\pi_q(\Gamma)$ such that μ satisfies*

$$\|\mu\|_2 > |\pi_q(\Gamma)|^{-1/2+\varepsilon} \quad \text{and} \quad \mu(gH) < [\pi_q(\Gamma) : H]^{-\varepsilon}$$

for any $g \in \pi_q(\Gamma)$ and for any proper subgroup $H < \pi_q(\Gamma)$. Then

$$\|\mu * \nu\|_2 < \|\mu\|_2^{1/2+\delta} \|\nu\|_2^{1/2}.$$

We deduce Theorem 1 from the above propositions. The method to prove spectral gap using analogues of these propositions was discovered by Bourgain and Gamburd [7] building on ideas that go back to Sarnak and Xue [55]. Very briefly it goes as follows:

By Proposition 7, we can bound $\pi_q[\chi_{S'}^{(l)}](gH)$ which is the probability that the random walk after $l \approx \log q$ steps is in a coset of a proper subgroup H . In particular, taking $H = \{1\}$, we get $\|\chi_{S'}^{(l)}\|_2 \leq |\pi_q(\Gamma)|^\delta$. Now we can apply Proposition 8 and iterate it to get improved bounds. Finally the representation theory of $\mathbb{G}(\mathbb{Z}/q\mathbb{Z})$ gives a lower bound for the multiplicity of the eigenvalues of the adjacency matrix of the Cayley graph $\mathcal{G}(\pi_q(\Gamma), \pi_q(S))$. Then we can use a trace formula to deduce an upper bound for the eigenvalues.

The papers [7]–[11], [61] and the current work are all based on the above strategy. The difference between the proofs is in the way the analogues of these two propositions are proved.

We divided the proof of Proposition 7 into two parts. First, mainly using Nori’s result, we lift up the problem to Γ , and show that “small” lifts of a certain large subgroup of H is inside a proper algebraic subgroup \mathbb{H} . (The idea of using Nori’s theorem in this context is not new, it goes back to the paper of Bourgain and Gamburd [9].) Then we give a geometric description for being in a proper algebraic subgroup in the spirit of Chevalley’s theorem. To this end, we construct finitely many irreducible representations ρ_i of the semisimple quotient of \mathbb{G} . Then for any i we also give an algebraic family $\{\phi_{i,v}\}_v$ of affine transformation lifts of ρ_i to \mathbb{G} . And we show that a proper algebraic subgroup \mathbb{H} either fixes a line via ρ_i or fixes a point via $\phi_{i,v}$ for some i and v . In the second step, using some ideas of Tits, we construct certain “ping-pong players”, and show that, in the process of the random walk with respect to this set of generators, the probability of fixing either a particular line or a point in these finitely many algebraic families of affine representations is exponentially small.

In order to prove Proposition 8, first we prove a triple product theorem similar to Helfgott’s result [35], [36]. I.e. we show that if $A \subset \mathbb{G}(\mathbb{Z}/q\mathbb{Z})$ is suitably distributed among proper subgroup cosets (to be made precise, see Proposition 26) then $|A.A.A| \geq |A|^{1+\delta}$. Then the proposition can be deduced from the Balog-Szemerédi-Gowers Theorem just as in [7].

We comment on the new ideas of the current work compared to the previous results, especially to [61], where Theorem 1 was proved for $\mathbb{G} = R_{k/\mathbb{Q}}(\mathrm{SL}_d)$. We also indicate which of these ideas are relevant also when the Zariski closure \mathbb{G} of Γ is semisimple, since this case is of special interest.

Compared to [61], the “ping-pong argument” used in the current work is more flexible. In [61], the argument applies only for representations that are both proximal and irreducible. Whereas in the current paper we give a more self-contained argument that needs only irreducibility. This is significant because even in the semisimple case, it could be difficult to construct suitable

representations with both properties. Moreover, we do not rely on the result of Goldsheid and Margulis on proximality of Zariski dense subgroups. This allows us to work both in the Archimedean and the non-Archimedean setting which is needed to prove Theorem 1 for S -integers. (The theorem of Goldsheid and Margulis does not hold over p -adic fields.)

When \mathbb{G} is not semi-simple further new ideas are needed. The unipotent radical is in the kernel of any irreducible representation. In order to detect proper subgroups which surject onto the semisimple factor of \mathbb{G} , we introduce algebraic families of affine representations.

We also need to use more complicated constructions when we use Nori's result to lift subgroups of finite groups to algebraic subgroups. On the other hand we eliminate the use of the quantitative Nullstellensatz which was a tool in [61].

It was proved in [61] (see Proposition B in Section 4) that if the triple product theorem holds for a family of simple groups (which also satisfy some additional, more technical properties), then the triple product theorem is also true for their direct product. The proof of this result in [61] is closely related to the proof of the square-free sum-product theorem proved in [11]. The triple product theorem for finite simple groups of Lie type of bounded rank was achieved in a recent breakthrough of Breuillard, Green, Tao [15] and of Pyber, Szabó [52] independently. (See Theorem C in Section 4.) These results are used as black boxes in our paper. When \mathbb{G} is semisimple, then Proposition 8 almost immediately follows from these results. The new contribution of the current work in the proof of Proposition 8 is when \mathbb{G} is not semisimple. To this end, we have to deal with certain semidirect products and one can see some similarities with the work of Alon, Lubotzky and Wigderson [3].

We note that the all the constants appearing in the paper are effective. However, an explicit computation would be tedious especially since some of our references are non-explicit, too. In particular the paper of Nori [51] uses non-effective techniques, but it can be made effective using some results of computational algebraic geometry. We discuss this briefly in the Appendix.

All of our arguments are constructive, and the constants could be computed in a straightforward way, except for some of the proofs in Section 3.2. At those places, we prove the existence of certain objects by nonconstructive means. However, these objects can be found by an algorithm simply by checking countable many possibilities. The existence of the object implies that the algorithm terminates in finite time.

For example Proposition 21 claims the existence of a finite subset of Γ and certain subsets of vector spaces with certain properties. It is easy to see that we can choose those sets to be bounded by rational hyperplanes, so the data whose existence is claimed in the proposition can be found within a countable set. Since it is a finite computation to check the required properties, one can always find a suitable subset of Γ and the accompanied data by finite computation.

The other place is the proof of Proposition 20, where we show that the intersection of a collection of sets parametrized by an integer k is empty for some k . Although the proof does not give a clue how large k needs to be, but

we can always compute it by computing the intersection of the sets for every k until it becomes empty.

The organization of the paper is as follows. In Section 2 we introduce some notation. Section 3 is devoted to the proof of Proposition 7. In Section 4 we prove Proposition 8. In Section 5, we finish the proof of Theorem 1. Finally in the appendix the effectiveness of Nori's results [51] is showed.

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2 Notations

We introduce some notation that will be used throughout the paper. We use Vinogradov's notation $x \ll y$ as a shorthand for $|x| < Cy$ with some constant C . Let G be a group. The unit element of any multiplicatively written group is denoted by 1. For given subsets A and B , we denote their product-set by

$$A.B = \{gh \mid g \in A, h \in B\},$$

while the k -fold iterated product-set of A is denoted by $\prod_k A$. We write \tilde{A} for the set of inverses of all elements of A . We say that A is symmetric if $A = \tilde{A}$. The number of elements of a set A is denoted by $|A|$. The index of a subgroup H of G is denoted by $[G : H]$ and we write $H_1 \lesssim_L H_2$ if $[H_1 : H_1 \cap H_2] \leq L$ for some subgroups $H_1, H_2 < G$. We denote the center of G by $Z(G)$. If ρ is a representation of G , then we denote the underlying vector space by W_ρ and we denote by $(W_\rho)^G$ the set of points fixed by all elements of G . Occasionally (especially when a ring structure is present) we write groups additively, then we write

$$A + B = \{g + h \mid g \in A, h \in B\}$$

for the sum-set of A and B , $\sum_k A$ for the k -fold iterated sum-set of A and 0 for the unit element.

If μ and ν are complex valued functions on G , we define their convolution by

$$(\mu * \nu)(g) = \sum_{h \in G} \mu(gh^{-1})\nu(h),$$

and we define $\tilde{\mu}$ by the formula

$$\tilde{\mu}(g) = \mu(g^{-1}).$$

We write $\mu^{(k)}$ for the k -fold convolution of μ with itself. As measures and functions are essentially the same on discrete sets, we use these notions interchangeably, we will also use the notation

$$\mu(A) = \sum_{g \in A} \mu(g).$$

A probability measure is a nonnegative measure with total mass 1. Finally, the normalized counting measure on a finite set A is the probability measure

$$\chi_A(B) = \frac{|A \cap B|}{|A|}.$$

3 Escape from subgroups

In this section we prove Proposition 7. Some ideas are taken from [61] but there are substantial new difficulties especially due to the lack of proximality of the adjoint representation and because we also consider groups that are not semisimple.

We begin this section by recalling some results from the literature on the subgroup structure of $\pi_q(\Gamma)$. Then in Section 3.1, in order to solve the problem of escaping from proper subgroups of $\pi_q(\Gamma)$, we lift it up to a problem on escaping from certain proper subgroups of Γ . For that purpose, we consider “small” lifts of elements of H in Γ ; namely, for a square-free integer q and a subgroup $H < G_q$, we write

$$\mathcal{L}_\delta(H) := \{h \in \Gamma \mid \pi_q(h) \in H, \|h\|_{\mathcal{S}} < [G_q : H]^\delta\},$$

where $\|h\|_{\mathcal{S}} = \max_{p \in \mathcal{S} \cup \{\infty\}} \|h\|_p$ and $\|h\|_p$ is the operator norm on \mathbb{Q}_p^d . In the next section, we give a geometric description of the set $\mathcal{L}_\delta(H)$ in terms of its action in an irreducible representation. Then in Section 3.2, we present an argument to show that only a small fraction of the elements of Γ satisfy this geometric property. Finally, we combine these two results to get Proposition 7.

Let $\mathbb{G} \subseteq \mathbb{G}\mathbb{L}_d$ be a Zariski-connected \mathbb{Q} -group. Then it is well-known that its unipotent radical \mathbb{U} is also defined over \mathbb{Q} (e.g. see [59, Proposition 14.4.5]) and it has a Levi subgroup defined over \mathbb{Q} , i.e. a reductive subgroup \mathbb{L} defined over \mathbb{Q} such that \mathbb{G} is \mathbb{Q} -isomorphic to $\mathbb{L} \times \mathbb{U}$. If \mathbb{G} is a perfect group, i.e. $\mathbb{G} = [\mathbb{G}, \mathbb{G}]$, then clearly \mathbb{L} is a semisimple group. We say that \mathbb{G} is simply-connected if \mathbb{L} is simply-connected. If \mathbb{L} is a simply-connected \mathbb{Q} -group, then we can write \mathbb{L} as product of absolutely almost simple groups. The absolute Galois group permutes these factors. So there are number fields κ_i and absolutely almost simple κ_i -groups \mathbb{L}_i such that

$$\mathbb{L} \simeq \prod_{i=1}^m R_{\kappa_i/\mathbb{Q}}(\mathbb{L}_i)$$

as \mathbb{Q} -groups. By a result of Bruhat-Tits [60, Section 3.9], for large enough p , $\mathbb{L}(\mathbb{Z}_p)$ is a hyper-special parahoric subgroup and so $\mathbb{L}(\mathbb{F}_p)$ is a product of

quasi-simple groups. We also have that $\mathbb{U}(\mathbb{F}_p)$ is a finite p -group, and, for large enough p , $\mathbb{G}(\mathbb{F}_p) \simeq \mathbb{L}(\mathbb{F}_p) \ltimes \mathbb{U}(\mathbb{F}_p)$ and is a perfect group. Again as part of Bruhat-Tits theory, we know that $\mathbb{L}(\mathbb{F}_p)$ is generated by order p elements. (To be precise, we know that, for large enough p , the special fiber of \mathbb{L} over \mathbb{F}_p is a connected, simply connected, semisimple \mathbb{F}_p -group.) Thus, for large enough p , $\mathbb{G}(\mathbb{F}_p) = \mathbb{G}(\mathbb{F}_p)^+$, where G^+ is the subgroup generated by p -elements, for any subgroup G of $\mathrm{GL}_d(\mathbb{F}_p)$. As part of Nori's Strong Approximation [51, Theorem 5.4], we have that,

Theorem A. *Let $\mathbb{G} \subseteq \mathrm{GL}_d$ be a Zariski-connected, perfect, simply-connected \mathbb{Q} -group. Let Γ be a finitely generated Zariski-dense subgroup of $\mathbb{G}(\mathbb{Q})$; then there is a finite set \mathcal{S} of primes such that,*

1. *The closure of Γ in $\prod_{p \in \mathcal{P} \setminus \mathcal{S}} \mathbb{G}(\mathbb{Z}_p)$ is an open subgroup.*
2. *There is a constant p_0 depending on Γ such that for any square free integer q with prime factors larger than p_0 , $\pi_q(\Gamma) = \mathbb{G}(\mathbb{Z}/q\mathbb{Z})$.*
3. *There is a constant p_0 depending on \mathbb{G} and its embedding into GL_d , such that for any square free integer q with prime factors larger than p_0 ,*

$$\prod_{p|q} \pi_p : \mathbb{G}(\mathbb{Z}/q\mathbb{Z}) \xrightarrow{\sim} \prod_{p|q} \mathbb{G}(\mathbb{Z}/p\mathbb{Z}).$$

Moreover, for $p > p_0$, $\mathbb{G}(\mathbb{F}_p) = \mathbb{L}(\mathbb{F}_p) \ltimes \mathbb{U}(\mathbb{F}_p)$, $\mathbb{L}(\mathbb{F}_p)$ is a product of quasi-simple finite groups whose number of factors is bounded in terms of $\dim \mathbb{L}$.

If Γ is a finitely generated subgroup of $\mathbb{G}(\mathbb{Q})$, $\iota : \tilde{\mathbb{G}} \rightarrow \mathbb{G}$ is a \mathbb{Q} -isogeny, and $\tilde{\mathbb{G}} = \tilde{\mathbb{L}} \ltimes \mathbb{U}$ is simply connected, then it is easy to see that $\Gamma \cap \iota(\tilde{\mathbb{G}})$ is a finite index subgroup of Γ . Furthermore, for large enough p , $\iota(\tilde{\mathbb{G}}(\mathbb{Z}_p)) \subseteq \mathbb{G}(\mathbb{Z}_p)$ and the pre-image of the first congruence subgroup of $\mathbb{G}(\mathbb{Z}_p)$ is the first congruence subgroup of $\tilde{\mathbb{G}}(\mathbb{Z}_p)$. In particular, there is a square free number q_0 such that, for any q , $\pi_q(\Gamma) \simeq \pi_{(q, q_0)}(\Gamma) \times \prod_{p|q/(q, q_0)} \pi_p(\Gamma)$; moreover $\pi_p(\Gamma) = \iota(\tilde{\mathbb{L}}(\mathbb{F}_p)) \ltimes \mathbb{U}(\mathbb{F}_p)$ and $\iota(\tilde{\mathbb{L}}(\mathbb{F}_p))$ is a product of quasi-simple finite groups if p is large enough. Let us also add that $\mathbb{U}/[\mathbb{U}, \mathbb{U}]$ is a commutative unipotent \mathbb{Q} -group, and so it is a \mathbb{Q} -vector group, i.e. it is \mathbb{Q} -isomorphic to \mathbb{G}_a^M for some M (the logarithm and exponential maps give \mathbb{Q} -isomorphisms between a commutative unipotent \mathbb{Q} -group and its Lie algebra). Thus, for large enough p , $(\mathbb{U}/[\mathbb{U}, \mathbb{U}])(\mathbb{F}_p)$ is an \mathbb{F}_p -vector space. As $[\mathbb{U}, \mathbb{U}]$ is an \mathbb{F}_p -split unipotent algebraic group, $(\mathbb{U}/[\mathbb{U}, \mathbb{U}])(\mathbb{F}_p) = \mathbb{U}(\mathbb{F}_p)/[\mathbb{U}, \mathbb{U}](\mathbb{F}_p)$. The next lemma, shows that, for large enough p , we have $[\mathbb{U}, \mathbb{U}](\mathbb{F}_p) = [\mathbb{U}(\mathbb{F}_p), \mathbb{U}(\mathbb{F}_p)]$, and so overall we have that, for large enough p , $\mathbb{U}(\mathbb{F}_p)/[\mathbb{U}(\mathbb{F}_p), \mathbb{U}(\mathbb{F}_p)]$ is an \mathbb{F}_p -vector space. Let $\gamma_k(\mathbb{U})$ be the k -th lower central series, i.e. $\gamma_1(\mathbb{U}) = \mathbb{U}$ and $\gamma_{i+1}(\mathbb{U}) = [\mathbb{U}, \gamma_i(\mathbb{U})]$. It is well-known that, if \mathbb{U} is defined over \mathbb{Q} , then all of the lower central series are also defined over \mathbb{Q} .

Lemma 9. *Let \mathbb{U} be a unipotent \mathbb{Q} -algebraic group. Then, for any k and large enough p , $\gamma_k(\mathbb{U})(\mathbb{F}_p) = \gamma_k(\mathbb{U}(\mathbb{F}_p))$.*

Proof. As \mathbb{U} has a \mathbb{Q} -structure, there is a lattice Γ_U in $\mathbb{U}(\mathbb{R})$. In particular, it is a finitely generated, Zariski-dense subgroup of $\mathbb{U}(\mathbb{Q})$. Thus $\gamma_k(\Gamma_U)$ is Zariski-dense in $\gamma_k(\mathbb{U})$, for any k . By Nori's result, $\gamma_k(\Gamma_U)$ modulo p is the full group $\gamma_k(\mathbb{U})(\mathbb{F}_p)$, for large enough p , which finishes the proof. \square

For the rest of this section, \mathcal{S} is a finite set of primes such that $\Gamma \subseteq \mathrm{GL}_d(\mathbb{Z}_{\mathcal{S}})$, q will be a square-free integer, and we assume that it has no prime divisor less than a constant which depends on Γ . We write $G_q = \pi_q(\Gamma)$. For future reference we record the properties of G_q that we deduced above: For any square-free integer q with sufficiently large prime divisors, and for any sufficiently large prime p , we have

1. $G_q = \prod_{p|q} G_p$,
2. $G_p = \mathbb{G}(\mathbb{F}_p)^+ = L_p \times U_p$, where
 - (a) L_p is a product of quasi-simple finite groups of Lie type over finite fields which are extensions of \mathbb{F}_p ; moreover the number of the quasi-simple factors has an upper-bound independent of p ,
 - (b) U_p is a k -step nilpotent p -group, where k is independent of p , and $U_p/[U_p, U_p]$ is isomorphic to an \mathbb{F}_p -vector space; moreover $\log_p |U_p|$ is bounded independently of p .

Let us also recall from Nori's paper [51, Theorem B and C] that for large enough p , any subgroup H of $\mathrm{GL}_d(\mathbb{F}_p)$ satisfies the following properties:

3. There is a Zariski-connected algebraic subgroup \mathbb{H} of $\mathbb{G}\mathrm{L}_d$ defined over \mathbb{F}_p such that $\mathbb{H}(\mathbb{F}_p)^+ = H^+$.
4. There is a commutative subgroup F of H such that $H^+ \cdot F$ is a normal subgroup of H and $[H : H^+ \cdot F] < C$, where C just depends on d the size of the matrices.
5. There is a correspondence between p -elements of H and nilpotent elements of $\mathfrak{h}(\mathbb{F}_p)$, where \mathfrak{h} is the Lie algebra of \mathbb{H} ; moreover $\mathfrak{h}(\mathbb{F}_p)$ is generated by its nilpotent elements.

3.1 Description of subgroups

In this section we describe in geometric terms the set $\mathcal{L}_\delta(H)$ defined above. In fact what we show is that there is a subgroup H^\sharp of small index, such that $\mathcal{L}_\delta(H^\sharp)$ is contained in a certain proper algebraic subgroup of \mathbb{G} . It is well-known by Chevalley's theorem, that then there is a representation of \mathbb{G} in which $\mathcal{L}_\delta(H^\sharp)$ fixes a line that is not fixed by the whole group. For technical reasons we need that the representations come from a fixed finite family; therefore we construct them explicitly. In addition, the methods of the next section would require that the representations are irreducible, which is not possible to fulfill since \mathbb{G} is not necessarily semisimple. For this reason, besides the irreducible representations

we also consider homomorphisms into the group of affine transformations. Unfortunately, a finite family of such homomorphism is not rich enough to capture all possible subgroups. We need to consider uncountable families, where the linear part of the action is the same and the translation part can be parametrized by elements of an affine space. The precise formulation is contained in the next proposition that we will use later as a black box in the paper.

Proposition 10. *Let \mathbb{G} be a Zariski-connected perfect \mathbb{Q} -group, and let Γ be a Zariski-dense, finitely generated subgroup of $\mathbb{G}(\mathbb{Q})$. Then there are non-trivial irreducible representations ρ_i for $1 \leq i \leq m$ of \mathbb{G} and morphisms $\varphi_i : \mathbb{G} \times \mathbb{V}_i \rightarrow \text{Aff}(\mathbb{W}_{\rho_i})$ with the following properties:*

1. *For any i , \mathbb{V}_i is a (possibly 0-dimensional) affine space. \mathbb{V}_i , ρ_i and φ_i are defined over a local field \mathcal{K}_i ; and $\rho_i(\Gamma)$ is an unbounded subset of $\text{GL}(W_{\rho_i})$, where $W_{\rho_i} = \mathbb{W}_{\rho_i}(\mathcal{K}_i)$.*
2. *For any i and $0 \neq v \in \mathbb{V}_i(\mathcal{K}_i)$, $\varphi_{i,v} = \varphi_i(\cdot, v)$ is a group homomorphism to $\text{Aff}(W_{\rho_i})$ whose linear parts are ρ_i , and $\mathbb{G}(\mathcal{K}_i)$ does not fix any point of W_{ρ_i} via this action.*
3. *There are positive constants C and δ such that the following holds. Let $q = p_1 \cdots p_n$ be a square-free number, such that each p_i is a sufficiently large prime, and let H be a proper subgroup of $\pi_q(\Gamma)$. Then there is a subgroup H^\sharp of index at most C^n in H that satisfies one of the following two conditions:*
 - (a) *For some $1 \leq i \leq m$, there is $w \neq 0$ in W_{ρ_i} such that $\rho_i(h)([w]) = [w]$, for any $h \in \mathcal{L}_\delta(H^\sharp)$.*
 - (b) *For some $1 < i \leq m$, there is $v \in \mathbb{V}_i(\mathcal{K}_i)$ and $w \in W_{\rho_i}$ such that $\|v\| = 1$ and $\phi_{i,v}(h)(w) = w$, for any $h \in \mathcal{L}_\delta(H^\sharp)$.*

We will easily deduce Proposition 10 from the following more technical version.

Proposition 11. *Let \mathbb{G} and Γ be as in the setting of Proposition 10 and $\mathbb{G} = \mathbb{L} \rtimes \mathbb{U}$, where \mathbb{L} is a semisimple group and \mathbb{U} is a unipotent group. Then there are finitely many representations $\rho_1, \dots, \rho_{m'}, \psi_1, \dots, \psi_k$ of \mathbb{G} with the following properties:*

1. *For any i , $\mathbb{U} \subseteq \ker(\rho_i)$ and the restriction of ρ_i to \mathbb{L} is a non-trivial irreducible representation.*
2. *For any i , there is a sub-representation $\mathbb{W}_i^{(1)}$ of \mathbb{W}_{ψ_i} such that*
 - (a) *\mathbb{U} acts trivially on $\mathbb{W}_i^{(1)}$ and $\mathbb{W}_{\psi_i}/\mathbb{W}_i^{(1)}$.*
 - (b) *$\mathbb{W}_i^{(1)}$ is a non-trivial irreducible representation of \mathbb{L} that we denote by $\rho_{m'+i}$.*

(c) $\mathbb{W}_{\psi_i} = \mathbb{W}_i^{(1)} \oplus \mathbb{W}_i^{(2)}$, where $\mathbb{W}_i^{(2)} = \mathbb{W}_{\psi_i}^{\mathbb{L}}$ is the set of \mathbb{L} -invariant vectors.

3. For any i , there are local fields \mathcal{K}_i such that ρ_i is defined over \mathcal{K}_i and ψ_i are defined over $\mathcal{K}_{i+m'}$; moreover $\rho_i(\text{pr}(\Gamma))$, where pr is the projection to \mathbb{L} , is an unbounded subset of $\rho_i(\mathbb{L}(\mathcal{K}_i))$.

4. There are positive constants C and δ such that the following holds. Let $q = p_1 \cdots p_n$ be a square-free number, such that each p_i is a sufficiently large prime, and let H be a proper subgroup of $\pi_q(\Gamma)$. Then there is a subgroup H^\sharp of index at most C^n in H that satisfies one of the following two conditions:

(a) For some i , there is $w \neq 0$ in $W_{\rho_i} = \mathbb{W}_{\rho_i}(\mathcal{K}_i)$ such that $\rho_i(h)([w]) = [w]$, for any $h \in \mathcal{L}_\delta(H^\sharp)$.

(b) For some i , there is a vector $0 \neq w \in W_{\psi_i}$ such that $\psi_i(h)(w) = w$, for any $h \in \mathcal{L}_\delta(H^\sharp)$. Moreover there is no nonzero vector w' in W_{ψ_i} such that $\psi_i(\mathbb{G})(w') = w'$.

Proof of Proposition 10 assuming Proposition 11. For any $1 \leq i \leq m'$, we let ρ_i be the same as in Proposition 11. For these representations, we take $\mathbb{V}_i = 0$ and let $\varphi_i(g, 0) = \rho_i(g)$. For $1 \leq i \leq k$, we let $\rho_{m'+i}$ be the representation of \mathbb{L} on $\mathbb{W}_i^{(1)}$ and $\mathbb{V}_{m'+i} = \mathbb{W}_i^{(2)}$. For any $w_1 \in \mathbb{W}_i^{(1)}$, let

$$\varphi_{m'+i}(g, v)(w_1) := \psi_i(g)(w_1 + v) - v.$$

Notice that, since g acts trivially on $\mathbb{W}_{\psi_i}/\mathbb{W}_i^{(1)}$, $\varphi_{m'+i}(g, v) \in \text{Aff}(\mathbb{W}_i^{(1)})$, for any i .

With these choices, in order to complete the proof, it is enough to make the following observations. If for some $m' < i \leq m$ and $0 \neq v \in \mathbb{V}_i$, \mathbb{G} fixes a point $w_1 \in \mathbb{W}_{\rho_i}$, then by definition, $w_1 + v$ is fixed by \mathbb{G} . Therefore $w_1 + v = 0$, and so $w_1 = v = 0$, which is a contradiction. On the other hand, if $\psi_i(h)(w) = w$, where $w = w_1 + v$, $w_1 \in \mathbb{W}_i^{(1)}$ and $v \in \mathbb{W}_i^{(2)} = \mathbb{V}_i$, then $\varphi_{i,v}(h)(w_1) = w_1$. \square

We prove Proposition 11 in two steps. First we show that for an appropriate choice of δ and H^\sharp , the Zariski-closure of the group generated by $\mathcal{L}_\delta(H^\sharp)$ is a proper subgroup of \mathbb{G} . Then, for any proper closed subgroup of \mathbb{G} , we construct the desired representations. We start with some auxiliary lemmata describing the normal subgroups of G_p .

Lemma 12. *Let $L = \prod_{i=1}^m L^{(i)}$, where $L^{(i)}$ are quasi-simple groups. Then any normal subgroup H of L is of the form $\prod_{i \in I} L^{(i)} \times Z$, where I is a subset of $\{1, \dots, m\}$ and $Z \leq \prod_{i \notin I} Z(L^{(i)})$.*

Proof. For any i , either $\text{pr}_i(H)$, the projection onto $L^{(i)}$, is central or $\text{pr}_i(H) = L^{(i)}$. If (s_1, \dots, s_m) is in H , then, for any $g \in L^{(i)}$, $(1, \dots, 1, [g, s_i], 1, \dots, 1)$ is also in N . On the other hand, the group generated by $[g, s_i]$ is a normal

subgroup of $L^{(i)}$. If s_i is not central, then the above group cannot be central as $Z(L^{(i)}/Z(L^{(i)})) = \{1\}$, and so it is the full group $L^{(i)}$. The rest of the argument is straightforward. \square

Lemma 13. *Let L be a direct product of quasi-simple finite groups which acts on a finite nilpotent group U . Then any normal subgroup H of $L \times U$ is of the form $(H \cap L) \times (H \cap U)$ if the prime factors of $|U|$ are larger than an absolute constant depending only on the size of the center of L . Moreover $H \cap L$ acts trivially on $U/H \cap U$.*

Proof. Passing to $H/H \cap U \trianglelefteq L \times (U/H \cap U)$, we can and will assume that $H \cap U = \{1\}$. Thus projection to L induces an embedding and we get a map $\varphi : \text{pr}(H) \rightarrow U$, where $\text{pr} : L \times U \rightarrow L$ is the projection map, such that

$$H = \{(s, \varphi(s)) \mid s \in \text{pr}(H)\}.$$

One can easily check that φ is a 1-cocycle, i.e. $\varphi(s_1 s_2) = s_2^{-1} \varphi(s_1) s_2 \cdot \varphi(s_2)$. Furthermore, for any $u \in U$ and $s \in L$, we have

$$(1, u^{-1})(s, \varphi(s))(1, u) = (s, s^{-1} u^{-1} s \cdot \varphi(s) \cdot u) \in H;$$

thus $\varphi(s) = s^{-1} u^{-1} s \cdot \varphi(s) \cdot u$, for any $u \in U$ and $s \in L$. In particular, setting $s = s_2$ and $u = \varphi(s_1)^{-1}$, and then using the cocycle relation, we have

$$\varphi(s_2) \cdot \varphi(s_1) = s_2^{-1} \varphi(s_1) s_2 \cdot \varphi(s_2) = \varphi(s_1 s_2).$$

Since $\varphi(1) = 1$, by the above equation, we have that $\varphi(s^{-1}) = \varphi(s)^{-1}$. Therefore, by the above discussion, $\theta(s) = \varphi(s^{-1})$ defines a homomorphism from $\text{pr}(H)$ to U . On the other hand, $\text{pr}(H)$ is a normal subgroup of L , so by Lemma 12, if the prime factors of $|U|$ are larger than the size of the center of L , then θ is trivial, which finishes the proof of the first part. For the second part, it is enough to notice that $s u s^{-1} u^{-1}$ is in $H \cap U$, for any $s \in H \cap L$ and $u \in U$. \square

Corollary 14. *Let L be a product of quasi-simple finite groups, which acts on a finite nilpotent group U . Assume that $G = L \times U$ is a perfect group. If H is a normal subgroup of G and the projection of H onto L is surjective, then $H = G$.*

Proof. By Lemma 13, $H = L \times U'$, for a normal subgroup U' of U , and L acts trivially on U/U' . Thus $L \times U/U' \simeq L \times U/U'$ is not a perfect group unless $U = U'$. On the other hand, any quotient of G is perfect, which finishes the proof. \square

In the next step, for any proper subgroup H of G_q , we will find another subgroup containing H which is of product form and is of comparable size.

Lemma 15. *Let H be a proper subgroup of $G = \prod_{p \in \Sigma} G_p$, where*

0. Σ is a finite set of primes larger than 7,

1. $G_p = L_p \rtimes U_p$, where $L_p = \prod_i L_p^{(i)}$, $L_p^{(i)}$ are quasi-simple groups of Lie type over a finite field of characteristic p , and U_p is a p -group,
2. G_p is perfect,
3. $|G_p| \leq p^k$ for any $p \in \Sigma$, with a fixed k independent of p .

Then there is a positive number δ depending on k , such that

$$\prod_{p \in \Sigma} [G_p : \pi_p(H)] \geq [G : H]^\delta,$$

where π_p is the projection onto G_p .

Proof. We prove this by induction on $|G|$. Let $\Sigma' = \{p \in \Sigma \mid \pi_p(H) = G_p\}$. If Σ' is empty, then $[G_p : \pi_p(H)] \geq p$ for any p , as $\pi_p(H)$ is a proper subgroup of G_p and G_p is generated by p elements. Therefore

$$\prod_{p \in \Sigma} [G_p : \pi_p(H)] \geq \prod_{p \in \Sigma} p \geq \prod_{p \in \Sigma} |G_p|^{1/k} \geq [G : H]^{1/k}.$$

So we shall assume that Σ' is non-empty. For any p , $H_p := G_p \cap H$ is a normal subgroup of $\pi_p(H)$; in particular, H_p is a normal subgroup of G_p when $p \in \Sigma'$. Let

$$G'_p = \begin{cases} G_p & \text{if } p \notin \Sigma', \\ G_p/H_p & \text{if } p \in \Sigma', \end{cases} \quad \text{and}$$

$$H' = H / \prod_{p \in \Sigma'} H_p \subseteq \prod_{p \in \Sigma} G'_p =: G'.$$

If H_p is not trivial for some $p \in \Sigma'$, then $|G'| < |G|$. Moreover, by Lemma 13 and Corollary 14, we can apply the induction hypothesis, and we get that

$$[G : H]^\delta = [G' : H']^\delta \leq \prod_{p \in \Sigma} [G'_p : \pi_p(H')] = \prod_{p \in \Sigma} [G_p : \pi_p(H)],$$

and we are done. Thus, without loss of generality, we can and will assume that H_p is trivial for any $p \in \Sigma'$.

Let $p_0 \in \Sigma'$. Since H_{p_0} is trivial and $\pi_{p_0}(H) = G_{p_0}$, there is an epimorphism from $H' := \pi_{\Sigma \setminus \{p_0\}}(H)$ to G_{p_0} and H' is isomorphic to H . Let N be the kernel of this epimorphism. By the induction hypothesis, we have that

$$\prod_{p \in \Sigma \setminus \{p_0\}} [G_p : \pi_p(N)] \geq \left[\prod_{p \in \Sigma \setminus \{p_0\}} G_p : N \right]^\delta.$$

On the other hand, $|H| = |H'| = |G_{p_0}| \cdot |N|$; so

$$\prod_{p \in \Sigma \setminus \{p_0\}} [G_p : \pi_p(N)] \geq [G : H]^\delta. \quad (1)$$

By (1), it is clear that, if we have $\pi_p(N) = \pi_p(H)$ for all p , then we are done. Thus we can assume that $\pi_p(N) \neq \pi_p(H)$ for some p .

Since $H'/N \simeq G_{p_0}$, it is clear that, for any $p \neq p_0$, $\pi_p(H)/\pi_p(N)$ is a homomorphic image of G_{p_0} . Thus, by the Jordan-Hölder Theorem, either G_{p_0} and $\pi_p(H)$ have some composition factor in common or $\pi_p(H) = \pi_p(N)$. On the other hand, the composition factors of G_p are the cyclic group of order p and $L_p^{(i)}$. In particular, if p and p' are distinct primes and either p or p' is larger than 7, then G_p and $G_{p'}$ do not have any composition factor in common. So $\pi_p(H) = \pi_p(N)$ if $p \in \Sigma' \setminus \{p_0\}$. So, for any $p_0 \in \Sigma'$, there is $p \in \Sigma \setminus \Sigma'$ such that $|L_{p_0}^{(i)}|$ divides $|G_p|$, for some i . In particular, p_0 divides $|G_p|$. Thus we have that

$$\prod_{p \in \Sigma'} p \mid \prod_{p \in \Sigma \setminus \Sigma'} |G_p|.$$

Therefore $\prod_{p \in \Sigma'} p \leq \prod_{p \in \Sigma \setminus \Sigma'} p^k$. Now it is straightforward to show that

$$\prod_{p \in \Sigma} [G_p : \pi_p(H)] \geq [G : H]^{\frac{1}{k(k+1)}},$$

and we are done. \square

After these preparations we are ready to make the first step towards the proof of Proposition 11. Recall that we try to describe the set $\mathcal{L}_\delta(H)$ in a geometric way, which set is the set of "small lifts" of H to $\mathbb{G}(\mathbb{Z})$. The next Proposition shows that for every $H < G_q$, we can find a subgroup H^\sharp for which $\mathcal{L}_\delta(H^\sharp)$ is contained in a proper algebraic subgroup of \mathbb{G} .

Proposition 16. *There are positive numbers δ and C for which the following holds:*

Take any proper subgroup H of G_q , where q is a square free integer with large prime factors. Then one can find a proper algebraic subgroup $\mathbb{H} < \mathbb{G}$ defined over \mathbb{Q} , and a subgroup H^\sharp of H , such that

1. $\mathcal{L}_\delta(H^\sharp) \subset \mathbb{H}(\mathbb{Q})$,
2. $[H : H^\sharp] \leq C^n$,

where n is the number of prime factors of q .

Proof. We begin the proof by constructing H^\sharp . Let q'' be the product of prime factors of q such that $G_p \neq \pi_p(H)$. By Nori's result (see 4. on page 10), for large enough p , there is a commutative subgroup F_p of $\pi_p(H)$ such that $H_p^\sharp := \pi_p(H)^+ \cdot F_p$ is a normal subgroup of $\pi_p(H)$ and $[\pi_p(H) : H_p^\sharp] < C$, where C just depends on the size of the matrices. Let $H_{q''}^\sharp = \prod_{p|q''} H_p^\sharp$ and

$$H^\sharp = \{h \in H \mid \pi_{q''}(h) \in H_{q''}^\sharp\}.$$

It is clear that $[H : H^\sharp] \leq [\prod_{p|q''} \pi_p(H) : H_{q''}^\sharp] \leq C^n$, where n is the number of prime factors of q . We will see that without loss of generality we can replace

q by q'' and H by $H_{q''}^\sharp$. By Lemma 15 and the discussions in the beginning of Section 3, we have that

$$[G_q : H]^{\delta_1} \leq [G_{q''} : \prod_{p|q''} \pi_p(H)].$$

Hence

$$[G_q : H^\sharp]^{\delta_1} \leq [G_{q''} : H_{q''}^\sharp].$$

Therefore $\mathcal{L}_\delta(H^\sharp) \subseteq \mathcal{L}_{\delta/\delta_1}(H_{q''}^\sharp)$. So, without loss of generality, we assume that

1. $H = \prod_{p|q} H_p$, where H_p is a proper subgroup of G_p .
2. $H^\sharp = \prod_{p|q} H_p^\sharp$.

Consider an embedding $\mathbb{G} \subseteq \mathbb{GL}_d$ defined over \mathbb{Q} . Below we will show that for each prime $p|q$ there is a polynomial $P_p \in (\mathbb{Z}/p\mathbb{Z})[x_{1,1}, \dots, x_{d,d}]$ of degree at most 4 such that all elements $g \in H_p^\sharp$ satisfy this equation, i.e. $P_p(g) = 0$, but not all elements of G_p do. Now we show that this easily implies the first part of the proposition with some $\delta > 0$. To do this, we first show that there is a polynomial $P \in \mathbb{Q}[x_{1,1}, \dots, x_{d,d}]$ which vanishes on $\mathcal{L}_\delta(H^\sharp)$ but not on all of \mathbb{G} . Consider the usual degree 4 monomial map: $\Psi : \mathbb{GL}_d \rightarrow \mathbb{A}_{\binom{d^2+4}{4}}$. Denote by D the dimension of the linear subspace of $\mathbb{A}_{\binom{d^2+4}{4}}$ spanned by $\Psi(\mathbb{G})$. We need to show that $\Psi(\mathcal{L}_\delta(H^\sharp))$ spans a subspace of dimension lower than D . Suppose the contrary, and let $g_1, \dots, g_D \in \Psi(\mathcal{L}_\delta(H^\sharp))$ which are linearly independent. We can consider the g_i as column vectors, and form a $\binom{d^2+4}{4} \times D$ matrix. By independence this has a nonzero $D \times D$ subdeterminant, whose entries are all less than

$$\frac{1}{D!} q^{\frac{1}{D(|S|+1)}}$$

in the $\|\cdot\|_S$ -norm, if we choose δ sufficiently small. Recall that S is the set of primes which occur in the denominators of elements of Γ . Now, the value of this subdeterminant is a nonzero rational number less than $q^{1/(|S|+1)}$, whose denominator is less than $q^{|S|/(|S|+1)}$. This implies that the projection mod p of this determinant is still nonzero for some $p|q$, which contradicts the existence of P_p to be demonstrated below.

So far we showed that for sufficiently small δ , $\mathcal{L}_\delta(H^\sharp)$ is contained in a proper subvariety $\mathbb{X} \subseteq \mathbb{G}$. By [24, Proposition 3.2], there is an integer N such that if $A \subseteq \mathbb{G}(\mathbb{Q})$ is a finite symmetric set generating a Zariski dense subgroup of \mathbb{G} , then $\prod_N A \not\subseteq \mathbb{X}(\mathbb{Q})$. This implies that $\mathcal{L}_{\delta/N}(H^\sharp)$ is contained in a proper algebraic subgroup. We note that the proof of [24, Proposition 3.2] gives that N depends only on the dimension, the degree and the number of irreducible components of \mathbb{X} . The proof in [24, Proposition 3.2] is based on the idea, that by Zariski density, one can find an element $g \in A$ such that $\mathbb{X} \cap g\mathbb{X}$ is either of lower dimension or contains less components of maximal dimension than \mathbb{X} . N is the number of iterations we need to make to get a trivial intersection. It is

clear that we can keep track of the dimension, the number of components and the degree of the varieties that arise this way. Hence the procedure terminates in N steps controlled by those parameters only.

It remains to show our claim about the existence of the polynomials P_p . In what follows, let \mathfrak{g} be the Lie algebra of \mathbb{G} and \mathfrak{g}^* its dual.

We consider the adjoint representation of \mathbb{G} and its dual on these spaces, respectively. For large enough p , these actions reduce to the action of $\mathbb{G}(\mathbb{F}_p)$ on $\mathfrak{g}(\mathbb{F}_p)$ and $\mathfrak{g}^*(\mathbb{F}_p)$; moreover $\mathfrak{g}^*(\mathbb{F}_p) = \mathfrak{g}(\mathbb{F}_p)^*$. There is a natural non-degenerate bilinear form on $\mathfrak{g} \otimes \mathfrak{g}^*$, which is the linear extension of the following map

$$\langle v_1 \otimes f_1, v_2 \otimes f_2 \rangle := f_1(v_2)f_2(v_1).$$

(It is worth mentioning that this bilinear map is \mathbb{G} -invariant.) It is also well-known that $\mathfrak{g} \otimes \mathfrak{g}^*$ is isomorphic to $\text{End}(\mathfrak{g})$ as a \mathbb{G} -module, where \mathbb{G} acts on $\text{End}(\mathfrak{g})$ via the conjugation by the adjoint representation. We denote both of these representations by ρ .

Clearly, we can assume that any prime divisor $p|q$ is sufficiently large, hence $\langle \cdot, \cdot \rangle$ induces a non-degenerate bilinear form on $\mathfrak{g}(\mathbb{F}_p) \otimes \mathfrak{g}^*(\mathbb{F}_p)$.

For any x and y in $\mathfrak{g} \otimes \mathfrak{g}^*$, let $\eta_{x,y}$ be a polynomial in d^2 variables with coefficients in $\mathbb{Z}[1/q_0]$, which is defined as follows,

$$\eta_{x,y}(g) := \langle \rho(g)(x), y \rangle,$$

We show that for a prime divisor p of q , we can find some $g_0 \in S$ and x and y in $\mathfrak{g} \otimes \mathfrak{g}^*$ such that $\eta_{x,y}(g) = 0$ modulo p , for any $h \in H_p^\sharp$, and $\eta_{x,y}(g_0) = 1$. Since $P_p := \eta_{x,y}$ is of degree at most 4, this proves our claim, and hence the proposition.

Since $\langle \cdot, \cdot \rangle$ is a non-degenerate bilinear form on $\mathfrak{g}(\mathbb{F}_p) \otimes \mathfrak{g}(\mathbb{F}_p)^*$, it is enough to show that there is a proper subspace of $\mathfrak{g}(\mathbb{F}_p) \otimes \mathfrak{g}(\mathbb{F}_p)^*$ which is H_p^\sharp -invariant, but not $\mathbb{G}(\mathbb{F}_p)^+$ -invariant.

If H_p^+ is not a normal subgroup of $\mathbb{G}(\mathbb{F}_p)^+$, then clearly, by Nori's result (see 3. and 5. on page 10 in Section 3), $\mathfrak{h}(\mathbb{F}_p)$ is H_p^\sharp -invariant, but not $\mathbb{G}(\mathbb{F}_p)^+$ -invariant. So $\mathfrak{h}(\mathbb{F}_p) \otimes \mathfrak{g}^*(\mathbb{F}_p)$ has the desired property.

Now, let us assume that H_p^+ is a normal subgroup. Since it is a proper normal subgroup of $\mathbb{G}(\mathbb{F}_p)^+$, by Corollary 14, the projection $\text{pr}(H_p^+)$ of H_p^+ to $\mathbb{L}(\mathbb{F}_p)^+$ is a proper normal subgroup. On the other hand, we know that

$$\mathbb{L}(\mathbb{F}_p)^+ \simeq \prod_{i=1}^m \prod_{\mathfrak{p} \in \mathcal{P}(k_i), \mathfrak{p}|p} \mathbb{L}_i(\mathbb{F}_{\mathfrak{p}})^+,$$

and $\mathbb{L}_i(\mathbb{F}_{\mathfrak{p}})^+$ is quasi-simple, for any i and \mathfrak{p} . Hence, by Lemma 12, there is a non-empty subset I of possible indices (i, \mathfrak{p}) such that

$$\text{pr}(H_p^+) \subseteq \prod_I \mathbb{L}_i(\mathbb{F}_{\mathfrak{p}})^+ \times \prod_{I^c} Z(\mathbb{L}_i(\mathbb{F}_{\mathfrak{p}})^+),$$

where I^c is the complement of I in the set of all the possible indices. Thus we have that $\text{pr}(H_p^+) = \prod_I \mathbb{L}_i(\mathbb{F}_{\mathfrak{p}})^+$ as H_p^+ is generated by p -elements. We also

notice that $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u}$, where \mathfrak{l} is the Lie algebra of \mathbb{L} and \mathfrak{u} is the Lie algebra of \mathbb{U} . Moreover

$$\mathfrak{g}(\mathbb{F}_p) = \left[\bigoplus_{i=1}^m \bigoplus_{\mathfrak{p} \in \mathcal{P}(k_i, \mathfrak{p}|p)} \mathfrak{l}_i(\mathbb{F}_p) \right] \oplus \mathfrak{u}(\mathbb{F}_p),$$

where \mathfrak{l}_i is the Lie algebra of \mathbb{L}_i . For any possible indices (i, \mathfrak{p}) , let $F_{i, \mathfrak{p}}$ be the projection of F_p (defined at the beginning of this proof) to $\mathbb{L}_i(\mathbb{F}_p)^+$. We notice that $\mathfrak{l}_i(\mathbb{F}_p)$ is an irreducible $\mathbb{L}_i(\mathbb{F}_p)^+$ -module. Therefore, if $\mathfrak{l}_i(\mathbb{F}_p)$ is not an irreducible $F_{i, \mathfrak{p}}$ -module, for some (i, \mathfrak{p}) , then one can easily get a proper subspace of $\mathfrak{g}(\mathbb{F}_p)$ which is invariant under H_p^\sharp , but not under $\mathbb{G}(\mathbb{F}_p)^+$ and finish the argument as before. So, without loss of generality, we assume that $\mathfrak{l}_i(\mathbb{F}_p)$ is an irreducible $F_{i, \mathfrak{p}}$ -module. Since $F_{i, \mathfrak{p}}$ is a commutative group, the \mathbb{F}_p -span $E_{i, \mathfrak{p}}$ of its image in $\text{End}(\mathfrak{l}_i(\mathbb{F}_p))$ is a field extension of \mathbb{F}_p of degree $\dim_{\mathbb{F}_p} \mathfrak{l}_i(\mathbb{F}_p)$. (In particular, by the Double Centralizer Theorem, the centralizer of $E_{i, \mathfrak{p}}$ in $\text{End}(\mathfrak{l}_i(\mathbb{F}_p))$ is itself.) Now we consider the subspace W of $\text{End}(\mathfrak{g}(\mathbb{F}_p))$ consisting of elements x with the following properties,

1. $x(\mathfrak{u}(\mathbb{F}_p)) \subseteq \mathfrak{u}(\mathbb{F}_p)$
2. $x(\mathfrak{l}_i(\mathbb{F}_p)) \subseteq \mathfrak{l}_i(\mathbb{F}_p) \oplus \mathfrak{u}(\mathbb{F}_p)$ if $(i, \mathfrak{p}) \in I$,
3. $\exists y \in E_{i, \mathfrak{p}}, \forall l_i \in \mathfrak{l}_i(\mathbb{F}_p) : x(l_i) - y(l_i) \in \mathfrak{u}(\mathbb{F}_p)$ if $(i, \mathfrak{p}) \notin I$.

We claim that W is H_p^\sharp -invariant, but not $\mathbb{G}(\mathbb{F}_p)^+$ -invariant. Let $g \in \mathbb{G}(\mathbb{F}_p)^+$ and $x \in W$; then $\text{Ad}(g)^{-1}x\text{Ad}(g)$ clearly satisfies the first and second conditions. It is straightforward to check that $\text{Ad}(g)^{-1}x\text{Ad}(g)$ satisfies the third condition for all x if and only if

$$\text{Ad}(g_{i, \mathfrak{p}})^{-1}E_{i, \mathfrak{p}}\text{Ad}(g_{i, \mathfrak{p}}) = E_{i, \mathfrak{p}}, \quad (2)$$

for any $(i, \mathfrak{p}) \notin I$, where $g_{i, \mathfrak{p}}$ is the projection of g onto $\mathbb{L}_i(\mathbb{F}_p)^+$. So clearly W is H_p^\sharp -invariant. On the other hand, any element in

$$N(E_{i, \mathfrak{p}}) = \{x \in \text{GL}(\mathfrak{l}_i(\mathbb{F}_p)) \mid x^{-1}E_{i, \mathfrak{p}}x = E_{i, \mathfrak{p}}\}$$

induces a Galois element and $E_{i, \mathfrak{p}}$ is a maximal subfield of $\text{End}(\mathfrak{l}_i(\mathbb{F}_p))$. Therefore $[N(E_{i, \mathfrak{p}}) : E_{i, \mathfrak{p}}^*] \leq \dim_{\mathbb{F}_p} \mathfrak{l}_i(\mathbb{F}_p)$. Thus $\mathbb{G}(\mathbb{F}_p)^+$ cannot leave W invariant if p is large enough, as we wished. \square

The next proposition is the source of the desired representations claimed in Proposition 11.

Proposition 17. *Let $\mathbb{G} = \mathbb{L} \ltimes \mathbb{U}$ be a perfect group, where \mathbb{L} is a semisimple group and \mathbb{U} is a unipotent group. There are finitely many representations $\rho_1, \dots, \rho_m, \psi_1, \dots, \psi_k$ of \mathbb{G} with the following properties:*

1. For any i , $\mathbb{U} \subseteq \ker(\rho_i)$ and the restriction of ρ_i to \mathbb{L} is a non-trivial irreducible representation.

2. For any i , there is a sub-representation $\mathbb{W}_i^{(1)}$ of \mathbb{W}_{ψ_i} such that
- (a) \mathbb{U} acts trivially on $\mathbb{W}_i^{(1)}$ and $\mathbb{W}_{\psi_i}/\mathbb{W}_i^{(1)}$.
 - (b) $\mathbb{W}_i^{(1)}$ is a non-trivial irreducible representation of \mathbb{L} that we denote by $\rho_{m'+i}$.
 - (c) $\mathbb{W}_{\psi_i} = \mathbb{W}_i^{(1)} \oplus \mathbb{W}_i^{(2)}$, where $\mathbb{W}_i^{(2)} = \mathbb{W}_{\psi_i}^{\mathbb{L}}$ is the set of \mathbb{L} -invariant vectors.
3. For any proper subgroup \mathbb{H} of \mathbb{G} , one of the following holds:
- (a) For some i , there is $w \neq 0$ in \mathbb{W}_{ρ_i} such that $\rho_i(\mathbb{H})([w]) = [w]$.
 - (b) For some i , there is $0 \neq w \in \mathbb{W}_{\psi_i}$ such that $\psi_i(\mathbb{H})(w) = w$. Moreover there is no non-zero vector w' in \mathbb{W}_{ψ_i} such that $\psi_i(\mathbb{G})(w') = w'$.

Proof. We divide the argument into several cases. First we consider the case, where the projection of \mathbb{H} onto \mathbb{L} is not surjective. In the second case we will assume that the group generated by its projection to \mathbb{U} is a proper subgroup of \mathbb{U} , and the third case finishes the argument. In each step, we introduce only finitely many representations which satisfy the desired properties.

In the first case, without loss of generality, we can assume that $\mathbb{G} = \mathbb{L}$ is a semisimple group and \mathbb{H} is a proper subgroup. If \mathbb{H} is not a normal subgroup, then, in the representation $\wedge^{\dim \mathfrak{h}} \text{Ad}$, $[w] = \wedge^{\dim \mathfrak{h}} \mathfrak{h} \in W$ is \mathbb{H} -invariant, but it is not \mathbb{L} -invariant. Since \mathbb{L} is semisimple, we can take a decomposition of W into irreducible components. Since $[w]$ is not \mathbb{L} -invariant, its projection to one of the non-trivial irreducible components is not zero, and so this representation satisfies the condition 3(a).

Notice that this process introduced only finitely many representations which satisfy the properties of ρ_i .

If \mathbb{H} is a proper normal subgroup and $\mathbb{L} = \prod_{i=1}^{m_0} \mathbb{L}_i$, where \mathbb{L}_i is an absolutely almost simple group, then there is a proper subset I of indices such that

$$\mathbb{H} \subseteq \prod_{i \in I} \mathbb{L}_i \times Z\left(\prod_{i \notin I} \mathbb{L}_i\right).$$

Consider the action of \mathbb{L} on the Lie algebra \mathfrak{l}_i of \mathbb{L}_i via the adjoint representation of \mathbb{L}_i , for any $i \notin I$. Clearly any line in this representation is fixed by \mathbb{H} and not by \mathbb{G} , which finishes the proof of the first case.

We notice that if \mathbb{H} is a proper subgroup of \mathbb{G} , then $\mathbb{H}[\mathbb{U}, \mathbb{U}]$ is also a proper subgroup of \mathbb{G} . So, without loss of generality, we assume that \mathbb{U} is a vector group, i.e. isomorphic to $\mathbb{G}_a^{k_0}$, for some k_0 .

Now we assume that the projection of \mathbb{H} onto \mathbb{L} is surjective, but the group generated by its projection to \mathbb{U} is a proper subgroup. So, without loss of generality, we assume that $\mathbb{H} = \mathbb{L} \times \widetilde{\mathbb{U}'}$, where $\widetilde{\mathbb{U}'}$ is a proper subgroup of \mathbb{U} . We consider $\widetilde{\mathbb{W}} = \mathbb{U}$ as an \mathbb{L} -space (and $\widetilde{\mathbb{W}'} = \mathbb{U}'$ is a proper \mathbb{L} -subspace), and take its decomposition into homogeneous subspaces, i.e.

$$\widetilde{\mathbb{W}} = \widetilde{\mathbb{W}}_1 \oplus \widetilde{\mathbb{W}}_2 \oplus \cdots \oplus \widetilde{\mathbb{W}}_n,$$

where $\text{Hom}_{\mathbb{L}}(\widetilde{\mathbb{W}}_i, \widetilde{\mathbb{W}}_j) = 0$ if $i \neq j$, and $\widetilde{\mathbb{W}}_i \simeq \mathbb{W}_i^{m_i}$, where \mathbb{W}_i is an irreducible \mathbb{L} -space. Here $\text{Hom}_{\mathbb{L}}(\widetilde{\mathbb{W}}_i, \widetilde{\mathbb{W}}_j)$ denotes the space of \mathbb{L} -equivariant linear maps from $\widetilde{\mathbb{W}}_i$ to $\widetilde{\mathbb{W}}_j$. Since $\widetilde{\mathbb{W}}$ is a proper \mathbb{L} -subspace, its projection to at least one of the homogeneous spaces is proper. Therefore, without loss of generality, we assume that $\widetilde{\mathbb{W}} \simeq \mathbb{W}^m$, where \mathbb{W} is an irreducible \mathbb{L} -space. We call a map f from $\widetilde{\mathbb{W}}$ to \mathbb{W} an affine map if

$$f(t_1\tilde{w}_1 + t_2\tilde{w}_2) = t_1f(\tilde{w}_1) + t_2f(\tilde{w}_2),$$

for any \tilde{w}_1 and \tilde{w}_2 in $\widetilde{\mathbb{W}}$ and $t_1 + t_2 = 1$. Let $\text{Aff}(\widetilde{\mathbb{W}}, \mathbb{W})$ be the set of all affine maps. If f is an affine map, then there are $f_{\text{lin}} \in \text{Hom}(\widetilde{\mathbb{W}}, \mathbb{W})$ and $w \in \mathbb{W}$ such that

$$f(x) = f_{\text{lin}}(x) + w,$$

for any $x \in \widetilde{\mathbb{W}}$; f_{lin} is called the linear part of f . Let $\text{Aff}_{\mathbb{L}}(\widetilde{\mathbb{W}}, \mathbb{W})$ be the set of affine maps whose linear part is in $\text{Hom}_{\mathbb{L}}(\widetilde{\mathbb{W}}, \mathbb{W})$. Therefore

$$\text{Aff}_{\mathbb{L}}(\widetilde{\mathbb{W}}, \mathbb{W}) = \text{Con}(\widetilde{\mathbb{W}}, \mathbb{W}) \oplus \text{Hom}_{\mathbb{L}}(\widetilde{\mathbb{W}}, \mathbb{W}),$$

where $\text{Con}(\widetilde{\mathbb{W}}, \mathbb{W})$ is the space of the constant functions. We claim that the representation ψ of $\mathbb{G} = \mathbb{L} \ltimes \widetilde{\mathbb{W}}$ on $\text{Aff}_{\mathbb{L}}(\widetilde{\mathbb{W}}, \mathbb{W})$, defined by

$$\psi(l, \tilde{w})(f)(x) := l \cdot f(l^{-1} \cdot x - \tilde{w}),$$

satisfies our desired conditions. Alternatively, we can say that if $f(x) = f_{\text{lin}}(x) + w$, then

$$\psi(l, \tilde{w})(f)(x) = f_{\text{lin}}(x) + l \cdot w - l \cdot f_{\text{lin}}(\tilde{w}).$$

Both of the subspaces $\text{Con}(\widetilde{\mathbb{W}}, \mathbb{W})$ and $\text{Hom}_{\mathbb{L}}(\widetilde{\mathbb{W}}, \mathbb{W})$ are \mathbb{L} -invariant. Moreover, \mathbb{L} acts trivially on $\text{Hom}_{\mathbb{L}}(\widetilde{\mathbb{W}}, \mathbb{W})$ and $\text{Con}(\widetilde{\mathbb{W}}, \mathbb{W})$ is isomorphic to \mathbb{W} as an \mathbb{L} -space, and, in particular, it is irreducible. It is also clear that $\widetilde{\mathbb{W}}$ acts trivially on $\text{Con}(\widetilde{\mathbb{W}}, \mathbb{W})$ and $\text{Aff}_{\mathbb{L}}(\widetilde{\mathbb{W}}, \mathbb{W})/\text{Con}(\widetilde{\mathbb{W}}, \mathbb{W})$.

Now since $\widetilde{\mathbb{W}}$ is a proper \mathbb{L} -subspace of $\widetilde{\mathbb{W}}$, there is $0 \neq f \in \text{Hom}_{\mathbb{L}}(\widetilde{\mathbb{W}}, \mathbb{W})$ such that $\widetilde{\mathbb{W}} \subseteq \ker(f)$. Thus, by the definition of ψ , $\psi(l, \tilde{w}')(f) = f$, for any $(l, \tilde{w}') \in \mathbb{L} \ltimes \widetilde{\mathbb{W}}$. Now assume that, for some $w \in \mathbb{W}$ and $f_{\text{lin}} \in \text{Hom}_{\mathbb{L}}(\widetilde{\mathbb{W}}, \mathbb{W})$, $f_w(x) = f_{\text{lin}}(x) + w$ is \mathbb{G} -invariant. Then f_w is a constant function as it is invariant under any translation, i.e. $f_{\text{lin}} = 0$. Hence w is \mathbb{L} -invariant and so w is also 0. Therefore this representation satisfies all the desired properties.

Now, we assume that the projection of \mathbb{H} to \mathbb{L} is surjective and the group generated by the projection of \mathbb{H} to \mathbb{U} generates \mathbb{U} . By Corollary 14, \mathbb{H} is not a normal subgroup of \mathbb{G} . (In fact, Corollary 14 was stated for finite groups, however, the proof works verbatim for algebraic groups, as well.) Hence, again, in the representation $\tilde{\rho} = \wedge^{\dim \mathfrak{h}} \text{Ad}$, $[w_0] = \wedge^{\dim \mathfrak{h}} \mathfrak{h} \in \widetilde{\mathbb{W}}$ is \mathbb{H} -invariant, but it is not \mathbb{G} -invariant. We claim that \mathbb{H} does not have any character, and therefore w_0 is fixed by \mathbb{H} . Let χ be a character of \mathbb{H} ; then $\mathbb{H} \cap \mathbb{U} \subseteq \ker(\chi)$ as $\mathbb{U} \cap \mathbb{H}$ is a

unipotent group. So χ factors through a character of $\mathbb{H}/(\mathbb{H} \cap \mathbb{U}) \simeq \mathbb{L}$. Since \mathbb{L} is semisimple, χ is trivial.

Since \mathbb{U} is a unipotent and normal subgroup of \mathbb{G} ,

$$\widetilde{\mathbb{W}} \supseteq (\tilde{\rho}(\mathbb{U}) - 1)(\widetilde{\mathbb{W}}) \supseteq \cdots \supseteq (\tilde{\rho}(\mathbb{U}) - 1)^{n_0+1}(\widetilde{\mathbb{W}}) = 0,$$

and for any i , $(\tilde{\rho}(\mathbb{U}) - 1)^i(\widetilde{\mathbb{W}})$ is a \mathbb{G} -space. Let $k + 1$ be the smallest possible integer such that the projection of w_0 to $\widetilde{\mathbb{W}}/\langle(\tilde{\rho}(\mathbb{U}) - 1)^{k+1} \cdot \widetilde{\mathbb{W}}\rangle$ is not \mathbb{G} -invariant. Notice that k is definitely positive as $\mathbb{G} = \mathbb{H} \cdot \mathbb{U}$, w_0 is \mathbb{H} -invariant and \mathbb{U} acts trivially on $\widetilde{\mathbb{W}}/\langle(\tilde{\rho}(\mathbb{U}) - 1) \cdot \widetilde{\mathbb{W}}\rangle$. After going to the quotient space, we can and will assume that $\langle(\tilde{\rho}(\mathbb{U}) - 1)^{k+1} \cdot \widetilde{\mathbb{W}}\rangle = 0$, i.e. \mathbb{U} acts trivially on $\langle(\tilde{\rho}(\mathbb{U}) - 1)^k \cdot \widetilde{\mathbb{W}}\rangle$.

Let $\widehat{\mathbb{W}}$ be the subspace of $\widetilde{\mathbb{W}}$ such that

$$\langle\langle(\tilde{\rho}(\mathbb{U}) - 1)^k \cdot \widetilde{\mathbb{W}}\rangle\rangle^{\mathbb{G}} = \widehat{\mathbb{W}}/\langle(\tilde{\rho}(\mathbb{U}) - 1)^k \cdot \widetilde{\mathbb{W}}\rangle,$$

i.e. $\widehat{\mathbb{W}} = \{w \in \widetilde{\mathbb{W}} \mid \forall g \in \mathbb{G}, \tilde{\rho}(g)(w) - w \in \langle(\tilde{\rho}(\mathbb{U}) - 1)^k \cdot \widetilde{\mathbb{W}}\rangle\}$. By the above argument, $w_0 \in \widehat{\mathbb{W}}$.

Now take a decomposition $\mathbb{W}_1 \oplus \cdots \oplus \mathbb{W}_n$ of $\langle(\tilde{\rho}(\mathbb{U}) - 1)^k \cdot \widetilde{\mathbb{W}}\rangle$ into irreducible \mathbb{L} -spaces. Since \mathbb{L} is a semisimple group and it acts trivially on $\widehat{\mathbb{W}}/\langle(\tilde{\rho}(\mathbb{U}) - 1)^k \cdot \widetilde{\mathbb{W}}\rangle$, there is a subspace \mathbb{W}_0 such that $\widehat{\mathbb{W}} = \mathbb{W}_0 \oplus \mathbb{W}_1 \oplus \cdots \oplus \mathbb{W}_n$ and \mathbb{L} acts trivially on \mathbb{W}_0 . We claim that the projection of w_0 to one of the non-trivial irreducible components is non-zero. Otherwise, w_0 is \mathbb{L} -invariant and so it is also invariant by the image of the projection of \mathbb{H} onto \mathbb{U} . By our assumptions on \mathbb{H} , we conclude that w_0 is \mathbb{G} -invariant, which is a contradiction. So, for some i , we have that the projection of w_0 to

$$\mathbb{W} = \widehat{\mathbb{W}}/\oplus_{j \neq i} \mathbb{W}_j$$

is not \mathbb{G} -invariant. Now let $\mathbb{W}^{(1)} = \oplus_j \mathbb{W}_j/\oplus_{j \neq i} \mathbb{W}_j$ and $\mathbb{W}^{(2)} = \mathbb{W}^{\mathbb{L}}/\mathbb{W}^{\mathbb{G}}$. Clearly

$$\mathbb{W}/\mathbb{W}^{\mathbb{G}} = \mathbb{W}^{(1)} \oplus \mathbb{W}^{(2)},$$

\mathbb{U} acts trivially on $\mathbb{W}^{(1)}$ and $\mathbb{W}/\mathbb{W}^{(1)}$, and $\mathbb{W}^{(1)}$ is an irreducible \mathbb{L} -space; moreover \mathbb{H} fixes $\bar{w}_0 = w_1 + w_2$, where \bar{w}_0 is the image of w_0 in $\mathbb{W}/\mathbb{W}^{\mathbb{G}}$, $w_1 \in \mathbb{W}^{(1)}$ and $w_2 \in \mathbb{W}^{(2)}$. Moreover, if $\bar{w}' \in (\mathbb{W}/\mathbb{W}^{\mathbb{G}})^{\mathbb{G}}$, then $g \cdot w' - w' \in \mathbb{W}^{\mathbb{G}} \subseteq \mathbb{W}^{\mathbb{L}} = \mathbb{W}^{(2)}$. On the other hand, \mathbb{G} acts trivially on $\mathbb{W}/\mathbb{W}^{(1)}$, and so $g \cdot w' - w' \in \mathbb{W}^{(1)}$. Therefore overall, we have that $w' \in \mathbb{W}^{\mathbb{G}}$, i.e. $\bar{w}' = 0$ as we wished. \square

Remark 18. From the proof of Proposition 17, it is clear that, if \mathbb{G} is defined over \mathbb{Q} , then there is a number field κ such that all the desired representations ρ_i and ψ_i are defined over κ .

Lemma 19. *Let Γ be a Zariski-dense subgroup of $\mathbb{G} \subseteq \mathrm{GL}_d$, a Zariski-connected \mathbb{Q} -group, such that $\Gamma \subseteq \mathbb{G}(\mathbb{Z}_S)$. Let ρ be a non-trivial representation of \mathbb{G} which is defined over a number field κ . Then there are $p \in S \cup \{\infty\}$ and a place \mathfrak{p} of κ such that,*

1. \mathfrak{p} divides p , i.e. $\mathbb{Q}_{\mathfrak{p}}$ is a subfield of $\kappa_{\mathfrak{p}}$.

2. $\rho(\Gamma)$ is an unbounded subset of $\rho(\mathbb{G}(\kappa_{\mathfrak{p}}))$.

Proof. If this is not the case, then $\rho(\Gamma)$ is bounded in $\rho(\mathbb{G}(\kappa_{\mathfrak{p}}))$, for any $\mathfrak{p} \in \mathcal{P}(\kappa)$. Hence $\rho(\Gamma)$ is a finite group. On the other hand, $\rho(\Gamma)$ is Zariski-dense in $\rho(\mathbb{G})$. Moreover we know that $\rho(\mathbb{G})$ is Zariski-connected. Thus ρ is trivial which is a contradiction. \square

Proof of Proposition 11. This is a direct consequence of Proposition 16, Proposition 17, Remark 18 and Lemma 19. \square

3.2 A ping-pong argument

We recall the notation from Proposition 10. \mathbb{G} is a Zariski-connected perfect \mathbb{Q} -group, and Γ is a Zariski dense subgroup of $\mathbb{G}(\mathbb{Q})$. We are given finitely many irreducible representations ρ_i for $1 \leq i \leq m$ each defined over a local field \mathcal{K}_i , and $\rho_i(\Gamma)$ is unbounded. Furthermore, for each i we are given an affine space \mathbb{V}_i and a morphism $\varphi_i : \mathbb{G} \times \mathbb{V}_i \rightarrow \text{Aff}(W_{\rho_i})$. Often we think about φ_i as homomorphisms from \mathbb{G} to $\text{Aff}(W_{\rho_i})$ parametrized by the elements of \mathbb{V}_i . Then we also write $\varphi_i(\cdot, v) = \varphi_{i,v}(\cdot)$. We also recall that for any i and $0 \neq v \in \mathbb{V}_i(\mathcal{K}_i)$, $\varphi_{i,v} : \mathbb{G}(\mathcal{K}_i) \rightarrow \text{Aff}(W_{\rho_i})$ is a homomorphism whose linear part is ρ_i , and no point of W_{ρ_i} is fixed under the action of $\mathbb{G}(\mathcal{K}_i)$. Our aim in this section is to prove that if we modify our generating set in an appropriate way, then only a small fraction of our group satisfy a condition like 3.(a) or 3.(b) in Proposition 10.

Let A be a subset of a group that generates freely a subgroup. A reduced word over A is a product of the form $g_1 \cdots g_l$, where $g_i \in A$ or $g_i^{-1} \in A$ and $g_i g_{i+1} \neq 1$ for any i . We write $B_l(A)$ (or simply B_l) for the set of reduced words over A of length l .

Proposition 20. *Let notation be as above. Then there is a set $A \subset \Gamma$ generating freely a subgroup Γ' , which satisfies the following properties. Write $S' = A \cup A$. Then for any i and for any vector $w \in W_{\rho_i}$, we have*

$$|\{g \in B_l | \rho_i(g)[w] = [w]\}| < |B_l|^{1-c},$$

furthermore, for any i , $v \in \mathbb{V}_i(\mathcal{K}_i)$ with $\|v\| = 1$ and for $w \in W_{\rho_i}$ we have

$$|\{g \in B_l | \varphi_{i,v}(g)w = w\}| < |B_l|^{1-c},$$

where c is a constant depending on S and on the representations.

The rest of this section is devoted to the proof of this proposition and in Section 3.3 we combine it with Proposition 10 to get Proposition 7. First we construct a desirable set of generators.

Proposition 21. *Let notation be as above. Then there is a symmetric set $S' \subset \Gamma$, and a number $R > 0$ and for each $g \in S'$ and $1 \leq i \leq m$ there are two sets $K_g^{(i)} \subset U_g^{(i)} \subset W_{\rho_i}$ such that the following hold.*

1. For each i and g we have $\rho_i(g)(U_g^{(i)}) \subset K_g^{(i)}$.
2. For each i , any vector $w \neq 0 \in W_{\rho_i}$ is contained in $U_g^{(i)}$ for at least two elements $g \in S'$.
3. For each i and for any elements $g_1, g_2 \in S'$ we have $K_{g_1}^{(i)} \subset U_{g_2}^{(i)}$ unless $g_1 g_2 = 1$.
4. For each i and for any elements $g_1, g_2 \in S'$ we have $K_{g_1}^{(i)} \cap K_{g_2}^{(i)} = \emptyset$ unless $g_1 = g_2$.
5. For each i , $v \in \mathbb{V}_i(\mathcal{K}_i)$ with $\|v\| = 1$ and $g \in S'$ we have that if $w \in U_g^{(i)}$ and $\|w\| > R$, then $\varphi_{i,v}(g)w \in K_g^{(i)}$ and $\|\varphi_{i,v}(g)w\| > \|w\|$.
6. For each i , $v \in \mathbb{V}_i(\mathcal{K}_i)$ with $\|v\| = 1$ and $w \in W_{\rho_i}$, there are at least two elements $g_1, g_2 \in S'$ such that $\varphi_{i,v}(g_1)w \neq w$ and $\varphi_{i,v}(g_2)w \neq w$.

The construction of the set S' relies on the notion of quasi-projective transformation introduced by Furstenberg [26] and further studied by Goldsheid and Margulis [29] and Abels, Margulis, and Soifer [1]. We use a slightly different notion also used by Cano and Seade [19], which suits better our purposes. Let $b \in \text{Mat}_d(\mathcal{K})$ be a not necessarily invertible linear transformation, where \mathcal{K} is a local field. Write $V^+(b) = \text{Im}(b)$ and $V^-(b) = \text{Ker}(b)$. Denote by $\mathbb{P}(\mathcal{K}^d)$ the projective space, and in general we denote by $\mathbb{P}(\cdot)$ the projectivization of a concept. Then $\mathbb{P}(b) : \mathbb{P}(\mathcal{K}^d) \setminus \mathbb{P}(V^-(b)) \rightarrow \mathbb{P}(\mathcal{K}^d)$ is a partially defined map on the projective space, and we call it a quasi-projective transformation.

Consider a sequence $\{g_i\}_{i=1}^\infty \subset \text{GL}_d(\mathcal{K})$. It is easy to see (see e.g. [19, Proposition 2.1]) that it contains a subsequence, still denoted by $\{g_i\}_{i=1}^\infty$ such that

$$\lim_{i \rightarrow \infty} g_i / \|g_i\| = b \quad (3)$$

uniformly for some linear transformation $b : \mathcal{K}^d \rightarrow \mathcal{K}^d$. Here and everywhere below $\|\cdot\|$ denotes a fixed submultiplicative matrix-norm. Moreover, this implies that

$$\lim_{i \rightarrow \infty} \mathbb{P}(g_i) = \mathbb{P}(b)$$

uniformly on compact subsets of $\mathbb{P}(\mathcal{K}^d) \setminus \mathbb{P}(V^-(b))$. Let $\Gamma \leq \text{GL}_d(\mathcal{K})$ be a group and denote by $\bar{\Gamma}$ the set of maps b for which (3) holds for some sequence $\{g_i\}_{i=1}^\infty \subseteq \Gamma$. The following lemma, crucial for us, is statement b, in [1, Lemma 4.3]. For completeness we give the proof.

Lemma 22. *Let $\{g_i\}_{i=1}^\infty, \{h_i\}_{i=1}^\infty \subseteq \text{GL}_d(\mathcal{K})$ be two sequences such that*

$$\lim_{i \rightarrow \infty} g_i / \|g_i\| = b_1 \quad \text{and} \quad \lim_{i \rightarrow \infty} h_i / \|h_i\| = b_2 \quad (4)$$

for some linear transformations b_1, b_2 . If $b_1 b_2 \neq 0$, then

$$\lim_{i \rightarrow \infty} g_i h_i / \|g_i h_i\| = \lambda b_1 b_2$$

for some nonzero $\lambda \in \mathcal{K}$.

Proof. Since the convergences in (4) are uniform, we have

$$\lim_{i \rightarrow \infty} \frac{g_i}{\|g_i\|} \circ \frac{h_i}{\|h_i\|} = b_1 b_2.$$

Observe the following: If $\{\gamma_i\}_{i=1}^{\infty} \subseteq \mathrm{GL}_d(\mathcal{K})$ and γ_i/λ_i converge to a nonzero linear transformation for a sequence of scalars $\{\lambda_i\}_{i=1}^{\infty} \subset \mathcal{K}$, then $\gamma_i/\|\gamma_i\|$ is convergent, too. This proves the lemma. \square

This lemma implies that if $b_1, b_2 \in \overline{\Gamma}$, and $b_1 b_2 \neq 0$, then $\lambda b_1 b_2 \in \overline{\Gamma}$. This property is crucial for us. Denote by r the minimum of the ranks of the elements in $\overline{\Gamma}$. If $b_1 \in \overline{\Gamma}$ is of rank r and if $V^+(b_1) \cap V^-(b_2) \neq \{0\}$ for some $b_2 \in \overline{\Gamma}$, then $b_1 b_2 = 0$, whence $V^+(b_1) \subset V^-(b_2)$.

We will use the following lemma to construct the first element of S' . This lemma is a variant of [1, Lemma 5.15].

Lemma 23. *Let \mathbb{G} be a Zariski-connected algebraic group, and let ρ_1, \dots, ρ_m be irreducible representations defined over local fields \mathcal{K}_i . Let $\Gamma \leq \mathbb{G}(\mathbb{Q})$ be Zariski-dense. For each $1 \leq i \leq m$ denote by r_i the minimal rank of an element in $\overline{\rho_i(\Gamma)}$.*

Then for each $1 \leq i \leq m$ there is a $b_i \in \overline{\rho_i(\Gamma)}$ and there is a sequence of elements $\{h_j\}_{j=1}^{\infty} \subseteq \Gamma$ such that the following hold.

1. *For each $1 \leq i \leq m$, $V^+(b_i) \cap V^-(b_i) = \{0\}$ and $\dim V^+(b_i) = r_i$.*
2. *For each $1 \leq i \leq m$, we have*

$$\lim_{j \rightarrow \infty} \rho_i(h_j) / \|\rho_i(h_j)\| = b_i.$$

Proof. Let $1 \leq k \leq m$ and assume that $\{h_j\}_{j=1}^{\infty} \subseteq \Gamma$ is a sequence such that for each i we have $\rho_i(h_j) / \|\rho_i(h_j)\| \rightarrow b_i$ for some linear transformation b_i , and 1. holds for $i < k$. We show below that we can replace $\{h_j\}_{j=1}^{\infty}$ with another sequence such that 1. holds for $i = k$ as well. Then the Lemma follows by induction.

Let $\{h'_j\}_{j=1}^{\infty} \subseteq \Gamma$ be a sequence such that $\rho_k(h'_j) / \|\rho_k(h'_j)\| \rightarrow b'_k$, where b'_k is a linear transformation of rank r_k . By taking a subsequence we can assume that $\rho_i(h'_j) / \|\rho_i(h'_j)\| \rightarrow b'_i$ for some linear transformation b'_i for all $1 \leq i \leq m$. Take two elements $g_1, g_2 \in \Gamma$. We consider the sequence

$$\{\tilde{h}_j = g_1 h'_j g_2 h_j\}_{j=1}^{\infty}$$

By Lemma 22 we get that for all $1 \leq i \leq m$ we have

$$\rho_i(\tilde{h}_j) / \|\rho_i(\tilde{h}_j)\| \rightarrow \lambda_i \rho_i(g_1) b'_i \rho_i(g_2) b_i$$

provided $\rho_i(g_1) b'_i \rho_i(g_2) b_i \neq 0$. Then for each i , there is a nonempty Zariski-open subset X_i of $\mathbb{G}(\mathcal{K}_i)$ such that for $g_2 \in X_i$ we have

$$\rho_i(g_2)(V^+(b_i)) \not\subseteq V^-(b'_i).$$

The Zariski-openness is clear and non-emptiness follows from the irreducibility of ρ_i . (A more detailed argument for a similar statement will be given in the proof of Lemma 24). Now take $g_2 \in \Gamma \cap \bigcap X_i$. Then $\rho_i(g_1)b'_i\rho_i(g_2)b_i \neq 0$ no matter how we choose g_1 . Similarly, there is a nonempty Zariski-open subset X'_i of $\mathbb{G}(\mathcal{K}_i)$ such that for $g_1 \in X'_i$, we have

$$\rho_i(g_1)(V^+(b'_i\rho_i(g_2)b_i)) \not\subseteq V^-(b'_i\rho_i(g_2)b_i). \quad (5)$$

Take $g_1 \in \Gamma \cap \bigcap X'_i$. For $i \leq k$, the rank of $\rho_i(g_1)b'_i\rho_i(g_2)b_i$ is r_i , and then (5) implies that

$$V^+(\rho_i(g_1)b'_i\rho_i(g_2)b_i) \cap V^-(\rho_i(g_1)b'_i\rho_i(g_2)b_i) = \{0\}$$

by the remarks after Lemma 22, which we wanted to show. \square

Let $\{g_i\}_{i=1}^n \subseteq \mathrm{GL}_d(\mathcal{K})$ be a sequence such that

$$\lim_{i \rightarrow \infty} g_i / \|g_i\| = b \quad \text{and} \quad \lim_{i \rightarrow \infty} g_i^{-1} / \|g_i^{-1}\| = \tilde{b}$$

for some non-invertible $b, \tilde{b} \in \mathrm{Mat}_d(\mathcal{K})$. Let $w \in V^+(b)$ and assume to the contrary that $w \notin V^-(\tilde{b})$. Then there is some vector $u_1 \in \mathcal{K}^d$ such that

$$\lim_{i \rightarrow \infty} g_i(u_1) / \|g_i\| = w.$$

By uniform convergence, we then have

$$\lim_{i \rightarrow \infty} \frac{g_i^{-1}(g_i(u_1) / \|g_i\|)}{\|g_i^{-1}\|} = u_2.$$

for some nonzero $u_2 \in \mathcal{K}^d$. This implies that $\|g_i\| \cdot \|g_i^{-1}\|$ is bounded which contradicts to the non-invertibility of b . Therefore we can conclude that $V^+(b) \subset V^-(\tilde{b})$ and $V^+(\tilde{b}) \subset V^-(b)$.

In the proof of Proposition 21 we will use Lemma 23 to produce an element $g_0 \in \Gamma$ with certain nice properties, and then we will define A to be a set of appropriate conjugates of it, whom we will find using the following two lemmata.

Lemma 24. *Let \mathbb{G} be a Zariski-connected algebraic group defined over a local field \mathcal{K} , and let ρ be an irreducible representation of it. Let $V_1^+, V_1^-, V_2^+, V_2^- \subseteq W_\rho$ be subspaces such that $V_1^+ \cap V_1^- = V_2^+ \cap V_2^- = \{0\}$ and $V_1^+ \subseteq V_2^-$ and $V_2^+ \subseteq V_1^-$. Let M be an integer and denote by $X \subseteq \mathbb{G}(\mathcal{K})^M$ the set of M -tuples (g_1, \dots, g_M) such that the following hold. If we have $\rho(g_\alpha)(V_i^+) \subseteq \rho(g_\beta)(V_j^-)$ for some $1 \leq i, j \leq 2$ and $1 \leq \alpha, \beta \leq M$, then $\alpha = \beta$ and $i + j = 3$. Then X is a nonempty Zariski-open set.*

Proof. Let v_1, \dots, v_r be a basis for V_1^+ and let ψ_1, \dots, ψ_r be a basis for the space of functionals vanishing on V_2^- . Then the condition $\rho(g_1)(V_1^+) \subseteq \rho(g_2)(V_2^-)$ is equivalent to the equations $\langle \rho(g_1)v_j, \rho(g_2)^*\psi_i \rangle = 0$ for $1 \leq i, j \leq r$. The other

conditions can be described in terms of algebraic equations similarly, whence the Zariski-openness follows.

It is clear that there is an M -tuple (g_1, \dots, g_M) for which the single condition $\rho(g_1)(V_1^+) \not\subseteq \rho(g_2)(V_2^-)$ is satisfied. For example we can take $g_2 = 1$ pick a vector $w_1 \in V_1^+$ and choose g_1 in such a way that $\rho(g_1)w_1 \notin V_2^-$, the existence of g_1 follows from irreducibility. It is a similar argument to show that the other constraints can be satisfied, so X , being the intersection of finitely many nonempty Zariski-open sets, is nonempty. \square

Lemma 25. *Let \mathbb{G} be a Zariski-connected algebraic group, and let ρ be an irreducible representation of it. Let \mathbb{V} be an affine space and let $\varphi : \mathbb{G} \times \mathbb{V} \rightarrow \text{Aff}(\mathbb{W}_\rho)$ be a morphism such that the linear part of φ_v is ρ . Assume that for some $0 \neq v \in \mathbb{V}$ and $w \in W_\rho$ there is an element $g \in \mathbb{G}(\mathcal{K})$ such that $\varphi_v(g)w \neq w$.*

Then for $M \geq 2 \dim(W_\rho) + \dim(\mathbb{V}) + 1$, there is a nonempty Zariski-open set $X \subset \mathbb{G}(\mathcal{K})^M$ such that if $(g_1, \dots, g_M) \in X$ then the following hold.

1. *Let $w \in W_\rho$ and $W \subsetneq W_\rho$ be a proper linear subspace. Then for any set of indices $I \subset \{1, \dots, M\}$ with $|I| = 2 \dim(W_\rho) - 1$, there is some $i \in I$ such that $\rho(g_i)w \notin W$.*
2. *Let $v \in \mathbb{V}(\mathcal{K})$, $w \in W_\rho$ and $W \subset W_\rho$ be an affine subspace, then for any set of indices $I \subset \{1, \dots, M\}$ with $|I| = 2 \dim(W_\rho) + \dim(\mathbb{V}) + 1$, there is some $i \in I$ such that $\varphi_v(g_i)w \notin W$.*

Proof. We only show that property 1. can be satisfied, 2. is similar, and then we can take the intersection of the two sets. Moreover, it is enough to show that 1. can be satisfied for the index set $I = \{1, \dots, 2 \dim(W_\rho) - 1\}$. Consider the algebraic variety

$$\mathbb{P}(W_\rho) \times \mathbb{P}(W_\rho^*) \times \mathbb{G}(\mathcal{K})^M.$$

Consider also the subvariety

$$Y = \{([w], [\psi], g_1, \dots, g_M) : \langle \rho(g_i)(w), \psi \rangle = 0 \text{ for } 1 \leq i \leq 2 \dim(W_\rho) - 1\}.$$

By the irreducibility of ρ it follows that for $([w], [\psi]) \in \mathbb{P}(W_\rho) \times \mathbb{P}(W_\rho^*)$ fixed, the variety

$$Y_{w, \psi} = \{g \in \mathbb{G}(\mathcal{K}) : \langle \rho(g)(w), \psi \rangle = 0\}$$

is a proper subvariety of $\mathbb{G}(\mathcal{K})$, hence $\dim(Y_{w, \psi}) \leq \dim(\mathbb{G}) - 1$. This implies that the fiber of Y over $([w], [\psi])$ is of codimension at least $2 \dim(W_\rho) - 1$ in $\mathbb{G}(\mathcal{K})^M$. Now let Z be the Zariski-closure of the image of Y under the projection map

$$\mathbb{P}(W_\rho) \times \mathbb{P}(W_\rho^*) \times \mathbb{G}(\mathcal{K})^M \rightarrow \mathbb{G}(\mathcal{K})^M.$$

Then $\dim(Z) \leq \dim(Y)$, hence Z is a proper subvariety, and by construction its complement satisfies 1. for $I = \{1, \dots, 2 \dim(W_\rho) - 1\}$. \square

Proof of Proposition 21. If ρ is a representation of \mathbb{G} , then write $\tilde{\rho}$ for the representation that associates the transpose inverse of $\rho(g)$ for every $g \in \mathbb{G}$. Apply Lemma 23 to the representations $\rho_1, \dots, \rho_m, \tilde{\rho}_1, \dots, \tilde{\rho}_m$. We get a sequence $\{h_i\}_{i=1}^\infty \subset \Gamma$ and linear transformations $b_1^{(1)}, \dots, b_m^{(1)}, b_1^{(2)}, \dots, b_m^{(2)}$ with the following properties. For each $1 \leq i \leq m$, we have

$$\lim_{j \rightarrow \infty} \rho_i(h_j) / \|\rho_i(h_j)\| = b_i^{(1)} \quad \text{and} \quad \lim_{j \rightarrow \infty} \rho_i(h_j^{-1}) / \|\rho_i(h_j^{-1})\| = b_i^{(2)}.$$

Furthermore we have that $\dim(V^+(b_i^{(1)})) = \dim(V^+(b_i^{(2)})) = r_i$ and $V^+(b_i^{(j)}) \cap V^-(b_i^{(j)}) = \{0\}$. Here and everywhere below r_i denotes the minimal rank of the elements of $\overline{\rho_i(\Gamma)}$. By the remarks preceding Lemma 24 we see that $V^+(b_i^{(j)}) \subset V^-(b_i^{(3-j)})$ for $j = 1, 2$. Let d be the maximum of the dimensions of the representation spaces W_{ρ_i} and parameter spaces \mathbb{V}_i . Apply Lemma 24 with $M = 3d + 2$ for each ρ_i and for the subspaces $V_j^+ = V^+(b_i^{(j)})$ and $V_j^- = V^-(b_i^{(j)})$, $j = 1, 2$. This way we get Zariski-open subsets $X_i \subset \mathbb{G}(\mathcal{K}_i)^M$. Also apply Lemma 25 for the representations ρ_i and for the morphisms φ_i , this gives Zariski-open subsets $X'_i \subset \mathbb{G}(\mathcal{K}_i)^M$. Since Γ is Zariski dense, we get elements $g_1, \dots, g_M \in \Gamma$ such that $(g_1, \dots, g_M) \in X_i \cap X'_i$ for all i , hence they have the following properties. Recall that if $c_1, c_2 \in \rho_i(\Gamma)$, and $\dim(V^+(c_1)) = r_i$, then either $V^+(c_1) \cap V^-(c_2) = \{0\}$ or $V^+(c_1) \subset V^-(c_2)$. For each i we have

$$\rho_i(g_\alpha)(V^+(b_i^{(j)})) \cap \rho_i(g_\beta)(V^-(b_i^{(k)})) = \{0\}$$

for every $1 \leq j, k \leq 2$ and $1 \leq \alpha, \beta \leq M$, except for $\alpha = \beta$ and $i + j = 3$. Using 1. in Lemma 25 with $W = V^-(b_i^{(1)})$, we also have that

$$\rho_i(g_{\alpha_1})(V^-(b_i^{(1)})) \cap \dots \cap \rho_i(g_{\alpha_{2d-1}})(V^-(b_i^{(1)})) = \{0\}$$

for any $1 \leq \alpha_1 < \dots < \alpha_{2d-1} \leq M$.

We show that if we set $A = \{g_1 h_j g_1^{-1}, \dots, g_M h_j g_M^{-1}\}$ and j is large enough then we can choose the sets $K_g^{(i)}$ and $U_g^{(i)}$ in such a way that the proposition holds. At this point we fix i , and omit the corresponding indices everywhere.

For a set $X \subset \mathbb{P}(\mathcal{K}^d)$ denote by $B_\varepsilon(X)$ the set of points which are of distance at most ε from X with respect to any fixed metric which induces the standard topology on $\mathbb{P}(\mathcal{K}^d)$. Let $\varepsilon > 0$ be sufficiently small, so that when V_1, \dots, V_l are among the subspaces $\rho(g_k)(V^\pm(b^{(j)}))$, then

$$B_\varepsilon(\mathbb{P}(V_1)) \cap \dots \cap B_\varepsilon(\mathbb{P}(V_l)) \neq \emptyset$$

only if $\mathbb{P}(V_1) \cap \dots \cap \mathbb{P}(V_l) \neq \emptyset$. For $g = g_k h_j g_k^{-1} \in A$ define

$$\begin{aligned} K_g &= \{w \in \mathcal{K}^d \setminus \{0\} \mid [w] \in B_\varepsilon(\mathbb{P}(\rho(g_k)(V^+(b^{(1)}))))\} \quad \text{and} \\ U_g &= \{w \in \mathcal{K}^d \setminus \{0\} \mid [w] \notin B_\varepsilon(\mathbb{P}(\rho(g_k)(V^-(b^{(1)}))))\}. \end{aligned}$$

We define K_g and U_g in a similar manner for $g \in \tilde{A}$, but we use $b^{(2)}$ instead of $b^{(1)}$. Properties 2., 3. and 4. can be deduced immediately from the properties

of the spaces $\rho(g_k)(V^+(b^{(1)}))$ and $\rho(g_k)(V^-(b^{(1)}))$ provided ε is small enough. We remark that property 4. follows from property 3., since by construction $K_g \cap U_{g^{-1}} = \emptyset$.

Property 1. holds if j is large enough, since $\mathbb{P}(\rho(g_k h_j g_k^{-1}))$ converges to $\mathbb{P}(\rho(g_k) b^{(1)} \rho(g_k^{-1}))$ uniformly on compact subsets of $\mathbb{P}(W_\rho) \setminus \mathbb{P}(V^-(b^{(1)}))$. For property 5, we note that there is a constant $c > 0$ depending on ε such that

$$\|\rho(g_k h_j g_k^{-1})(w)\| > c \|\rho(g_k h_j g_k^{-1})\| \cdot \|w\|$$

for $w \in U_g$. Since $\lim_{j \rightarrow \infty} \|\rho(g_k h_j g_k^{-1})\| = \infty$, we have $c \|\rho(g_k h_j g_k^{-1})\| > 1$ for large j . Then for $\|w\| > R$ large and $v \in \mathbb{V}(\mathcal{K})$, $\|v\| = 1$, the translation component of $\varphi_v(g_k h_j g_k^{-1})$ is negligible compared to the linear part, and property 5 follows. Here we used that the unit ball in $\mathbb{V}(\mathcal{K})$ is compact, hence for j fixed, the translation part of $\varphi_v(g_k h_j g_k^{-1})$ is bounded in v . Finally, denote by $W \subset W_\rho$ the possibly empty affine subspace that consist of the fixed points of $\varphi_v(h_j)$. Then the set of fixed points of $\varphi_v(g_k h_j g_k^{-1})$ is $g_k(W)$. From this we see that property 6. follows from part 2. of Lemma 25. \square

Proof of Proposition 20. Let $A \subset \Gamma$ be the set that we constructed in Proposition 21, and let $K_g^{(i)}$ and $U_g^{(i)}$ be the corresponding sets. In what follows we fix i and omit the corresponding indices.

We deal with the two parts separately, first we estimate the size of the set $\{g \in B_l | \rho(g)[w] = [w]\}$. For $0 \leq k < l$ denote by X_k the set of those reduced words $g_l \cdots g_1 \in B_l$ for which $\rho(g_k \cdots g_1)w \in U_{g_{k+1}}$, and k is the smallest index with this property. We write X_l for those words which are not contained in any of the X_k with $k < l$. Let $g_l \cdots g_1 \in X_k$. We remark that by the properties of the sets U_g and K_g , we have

$$\rho(g_{k+1} \cdots g_1)w \in K_{g_{k+1}} \subset U_{g_{k+2}}.$$

In fact, by induction we can conclude that $\rho(g_j \cdots g_1)w \in K_{g_j}$ for $j > k$. Assume further that $\rho(g_l \cdots g_1)[w] = [w]$. Then we also have

$$\rho(g_{j+1}^{-1} \cdots g_l^{-1})[w] = \rho(g_j \cdots g_1)[w] \in \mathbb{P}(K_{g_j})$$

for $j > k$. Since the sets K_g are disjoint, we see that $[w]$ determines g_j uniquely for $j > k$. Indeed, once g_l, \dots, g_{j+1} are known, they determine which of the sets $\mathbb{P}(K_{g_j})$ does $\rho(g_{j+1}^{-1} \cdots g_l^{-1})[w]$ belong to. On the other hand we know that for $j \leq k$, we have $\rho(g_{j-1} \cdots g_1)w \notin U_{g_j}$. Since $\rho(g_{j-1} \cdots g_1)w$ is covered by at least two of the sets U_g , we have at most $|S'| - 2$ possibilities for g_j . Therefore we have

$$|\{g \in B_l | \rho(g)[w] = [w]\} \cap X_k| \leq (|S'| - 2)^k,$$

from where the first part of the proposition follows easily.

Now we give an estimate for $|\{g \in B_l | \varphi_v(g)w = w\}|$. We show that there is an integer k such that for any $v \in \mathbb{V}(\mathcal{K})$ with $\|v\| = 1$, $w \in W_\rho$ and $g \in S'$ there is a reduced word $\omega \in B_k$ of length k with the following properties. The first letter of ω is not g , and we have $\|\varphi_v(\omega)w\| > R$, $\|\varphi_v(\omega)w\| > \|w\|$

and $\varphi_v(\omega)w \in K_{g'}$, where g' is the last letter of ω . If $|w| > R$, this is easy, since there are at least two letters $g' \in S'$ such that $w \in U_{g'}$. We can also make an existing word longer, since we can preserve the required properties no matter how we continue it as long as it stays reduced. Consider the case when $\|w\| \leq R$. Denote by $D \subset W_\rho$ the solid ball of radius R . To each reduced word ω we associate a set $E_\omega \subset \{v \in \mathbb{V}(\mathcal{K}) : \|v\| = 1\} \times W_\rho$ defined by

$$E_\omega = \{(v, x) : x \in \varphi_v(\omega^{-1})(D)\}.$$

We need to show that there is a number k such that

$$\bigcap_{\omega \in B_k : \omega \text{ does not contain } g} E_\omega = \emptyset.$$

Since the E_ω are compact, it is enough to show that

$$\bigcap_{\omega : \omega \text{ does not contain } g} E_\omega = \emptyset.$$

We show that for each $v_0 \in \mathbb{V}(\mathcal{K})$ with $\|v_0\| = 1$,

$$(\{v_0\} \times W_\rho) \cap \bigcap_{\omega : \omega \text{ does not contain } g} E_\omega = \emptyset$$

Assume to the contrary that there are at least two points

$$(v_0, w_1), (v_0, w_2) \in \{v_0\} \times W_\rho \cap \bigcap_{\omega : \omega \text{ does not contain } g} E_\omega.$$

Then $w_1 - w_2 \in U_h$ for some $g \neq h \in S'$. Property 5 in Proposition 21 implies that there is some $c > 1$ such that $\|\rho(h)w\| > c\|w\|$ for all $w \in U_h$. Then $2R > \|\varphi_{v_0}(h^l)w_1 - \varphi_{v_0}(h^l)w_2\| > c^l\|w\|$, a contradiction. Assume to the contrary that

$$\{v_0\} \times W_\rho \cap \bigcap_{\omega : \omega \text{ does not contain } g} E_\omega = \{(v_0, w)\}$$

for a point $w \in W_\rho$. Then w is fixed by all elements of S' except maybe for g , which contradicts to property 6 in Proposition 21. So far we showed all required properties of ω except that $\varphi_v(\omega)w$ belong to the right set $K_{g'}$. However, by properties 2 and 5 in Proposition 21 there are at least two letters that we can append to ω to fulfill these last requirement as well. For at least one of these two, the word stays reduced.

Now consider a reduced word $g_l \cdots g_1$ for which $\varphi_v(g_l \cdots g_1)w = w$. Then we also have for all $1 \leq j < l/k$

$$\varphi_v(g_{jk} \cdots g_1 g_l \cdots g_{j(k+1)}) (\varphi_v(g_{jk} \cdots g_1)w) = \varphi_v(g_{jk} \cdots g_1)w. \quad (6)$$

The above argument shows that out of the $(|S'| - 1)^k$ possibilities for $g_{(j+1)k} \cdots g_{jk+1}$, there is at least one for which (6) does not hold since the vector on the left

hand side is longer than the one on the right. Although $g_{jk} \cdots g_1 g_l \cdots g_{jk+1}$ may not be reduced, if $j < l/k - 1$, we still get a reduced word ending with $g_{(j+1)k} \cdots g_{jk+1}$ after all possible reductions. This shows that

$$|\{g \in B_l | \varphi_v(g)w = w\}| < \left(\frac{(|S'| - 1)^k - 1}{(|S'| - 1)^k} \right)^{l/k-2} |B_l|$$

giving the second half of the proposition. \square

3.3 Proof of Proposition 7

We show that the proposition holds if S' is the set of generators constructed in Proposition 20. Let q be a square-free integer and $H < \pi_q(\Gamma)$ a subgroup. Denote by q_1 the product of those prime factors of q which are large enough so that Propositions 10 and 20 hold. Let $H_1 = \pi_{q_1}(H) < \pi_{q_1}(\Gamma)$. Then clearly

$$[\pi_{q_1}(\Gamma) : H_1] \geq q_1/q [\pi_q(\Gamma) : H] \gg [\pi_q(\Gamma) : H],$$

hence if we show the claim for q_1 and H_1 , it will follow for q and H as well with a worse implied constant. From now on we assume that $q = q_1$.

Let H^\sharp be the subgroup corresponding to H in Proposition 10. Let $l \leq c_1 \log[\pi_q(\Gamma) : H]$ be an integer, where c_1 is a sufficiently small constant. Let $h \in \Gamma$ be such that $\pi_q(h) \in H^\sharp$ and $h \in B_l$, where B_l is the set of reduced words of length l over the alphabet S' . If c_1 is small enough, then $\|h\| < [\pi_q(\Gamma) : H^\sharp]^\delta$ with the same δ for which Proposition 10 holds. Then by definition, $h \in \mathcal{L}_\delta(H^\sharp)$. If we combine Propositions 10 and 20, then we get

$$|B_l \cap \mathcal{L}_\delta(H^\sharp)| < |B_l|^{1-c_2}$$

for some $c_2 > 0$.

Write $|S'| = 2M$ Set $P_k(l) = \chi_{S'}^{(2k)}(\omega)$, where $\omega \in B_l$. Since $|B_l| = 2M(2M - 1)^{l-1}$ for $l \geq 1$,

$$1 = P_k(0) + \sum_{l \geq 1} |B_l| P_k(l). \quad (7)$$

By a result of Kesten [41, Theorem 3.], we have

$$\limsup_{k \rightarrow \infty} (P_k(0))^{1/k} = (2M - 1)/M^2.$$

From general properties of Markov chains (see [62, Lemma 1.9]) it follows that

$$P_k(0) \leq \left(\frac{2M - 1}{M^2} \right)^k.$$

Since $\chi_{S'}^{(2k)}$ is symmetric, we have $P_k(0) = \sum_g [\chi_{S'}^{(k)}(g)]^2$, hence $P_k(l) \leq P_k(0)$

for all l by the Cauchy-Schwartz inequality. Now we can write for $k \leq c_1 \log q/2$:

$$\begin{aligned}
\chi_{S'}^{(2k)}(\mathcal{L}_\delta(H^\sharp)) &= \sum_l |B_l \cap \mathcal{L}_\delta(H^\sharp)| P_k(l) \\
&\leq \sum_l |B_l|^{1-c_2} P_k(l) \\
&\leq \sum_{l \leq k/10} (2M)^l \left(\frac{2M-1}{M^2} \right)^k \\
&\quad + (2M-1)^{-c_2 k/10} \sum_{l \geq k/10} |B_l| P_k(l) \\
&< \frac{(2M)^{11k/10+1}}{M^{2k}} + (2M-1)^{-c_2 k/10}.
\end{aligned}$$

The inequality between the third and fourth lines follows from (7).

Note that $\pi_q[\chi_{S'}^{(2k)}](H^\sharp)$ is non-increasing with k . Let $g \in \pi_q(H)$, since S' is symmetric, we have

$$\pi_q[\chi_{S'}^{(2k)}](gH^\sharp)^2 \leq \pi_q[\chi_{S'}^{(4k)}](H^\sharp).$$

Thus

$$\pi_q[\chi_{S'}^{(2k)}](H) \leq C^n \left(\pi_q[\chi_{S'}^{(4k)}](H^\sharp) \right)^{1/2},$$

and this finishes the proof, since any positive power of q dominates C^n if q is large enough.

4 Growth of product-sets

In this section we aim to prove Proposition 8, but first we recall some results we will need later on. We fix a square-free integer $q = p_1 \cdots p_n$, and assume that each prime divisor p_i of q is bigger than some large but fixed constant. Then as we saw at the beginning of Section 3, we have

$$G := \pi_q(\Gamma) = G_{p_1} \times \cdots \times G_{p_n},$$

where $G_{p_i} = \pi_{p_i}(\Gamma) = L_{p_i} \times U_{p_i}$ and L_{p_i} is a product of quasi-simple groups generated by their elements of order p_i and U_{p_i} is a p_i -group. Furthermore, we have that $U_{p_i}/[U_{p_i} : U_{p_i}]$ is isomorphic to $(\mathbb{F}_{p_i}^{d_i}, +)$ for some integers d_i which are bounded independently of q . We write $L = L_{p_1} \times \cdots \times L_{p_n}$ and $U = U_{p_1} \times \cdots \times U_{p_n}$. The following result is a statement about abstract groups satisfying certain assumptions that we will recall later in Section 4.2. We will also show there that the group L satisfy these assumptions with parameters that are independent of q .

Proposition B ([61, Proposition 14]). *Let G be a group satisfying (A0)–(A5). For any $\varepsilon > 0$ there is a $\delta > 0$ depending only on ε and the constants in (A0)–(A5) such that the following holds. If $A \subseteq G$ is symmetric such that*

$$|A| < |G|^{1-\varepsilon} \quad \text{and} \quad \chi_A(gH) < [G : H]^{-\varepsilon} |G|^\delta$$

for any $g \in G$ and any proper $H < G$, then $|\prod_3 A| \gg |A|^{1+\delta}$.

Assumptions (A0)–(A5) are more or less straightforward to check except for (A4) which basically boils down to showing Proposition B for quasi-simple groups that are the direct factors of L_{p_i} . This statement was first proved by Helfgott for $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ [35] and for $\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z})$ [36], and later it was extended by Dinai [22] for $\mathrm{SL}_2(\mathbb{F}_q)$ for an arbitrary finite field \mathbb{F}_q . Now the statement is known for all finite simple groups of Lie type due to a recent breakthrough by Breuillard, Green, Tao [15], and Pyber, Szabó [52]. We can either use [52, Theorem 4] or [15, Corollary 2.4]. The statement in the formulation of [52, Theorem 4] is the following.

Theorem C. *Let L be a simple group of Lie type of rank r and A a generating set of L . Then either $\prod_3 A = L$ or $|\prod_3 A| \gg |A|^{1+\varepsilon_0}$, where ε_0 and the implied constant depend only on r .*

The following useful Lemma is based on the Balog-Szemerédi-Gowers Theorem and it is implicitly contained in [7].

Lemma D ([61, Lemma 15]). *Let μ and ν be two probability measures on an arbitrary group G and let $K > 2$ be a number. If*

$$\|\mu * \nu\|_2 > \frac{\|\mu\|_2^{1/2} \|\nu\|_2^{1/2}}{K}$$

then there is a symmetric set $A \subset G$ with

$$\frac{1}{K^R \|\mu\|_2^2} \ll |A| \ll \frac{K^R}{\|\mu\|_2^2},$$

$$|\prod_3 A| \ll K^R |A| \quad \text{and}$$

$$\min_{g \in A} (\tilde{\mu} * \mu)(g) \gg \frac{1}{K^R |A|},$$

where R and the implied constants are absolute.

Using the Lemma it is very easy to deduce Proposition 8 from the following

Proposition 26. *Let \mathbb{G} be a Zariski-connected perfect algebraic group defined over \mathbb{Q} . Let $\Gamma < \mathbb{G}(\mathbb{Q})$ be a finitely generated Zariski-dense subgroup. Then for any $\varepsilon > 0$, there is some $\delta > 0$ depending only on ε and \mathbb{G} such that the following holds. Let q be a square-free integer without small prime factors.*

If $A \subseteq G_q$ is symmetric such that

$$|A| < |G_q|^{1-\varepsilon} \quad \text{and} \quad \chi_A(gH) < [G_q : H]^{-\varepsilon} |G_q|^\delta$$

for any $g \in G_q$ and any proper $H < G_q$, then $|\prod_3 A| \gg |A|^{1+\delta}$.

We defer the proof to the following sections, and now we show how it implies the

Proof of Proposition 8. Assume that the conclusion of the proposition fails, i.e. that there is an ε such that for any δ , there is a q and there are probability measures μ and ν with

$$\|\mu\|_2 > |G_q|^{-1/2+\varepsilon} \quad \text{and} \quad \mu(gH) < [G_q : H]^{-\varepsilon}$$

for any $g \in G_q$ and for any proper $H < G_q$, and yet

$$\|\mu * \nu\|_2 \geq \|\mu\|_2^{1/2+\delta} \|\nu\|_2^{1/2}.$$

Take $K = \|\mu\|_2^{-\delta}$ in Lemma D. Note that by the third property of the set A , we have

$$\chi_A(gH) \ll K^R \tilde{\mu} * \mu(gH) \leq K^R \max_{h \in G_q} \mu(hH) \ll |G_q|^{R\delta} [G_q : H]^{-\varepsilon}.$$

Now $|\prod_3 A| \ll K^R |A|$ contradicts Proposition 26, if δ is small enough. In fact, when q contains small prime factors, Proposition 26 does not apply, but we still get a contradiction for $\pi_{q'}(A)$ and the group $G_{q'}$, where q' is the product of the prime factors of q which are not too small for Proposition 26. Also note that when q is smaller than a fixed constant we can get the contradiction by the trivial inequality $|\prod_3 A| \geq |A| + 1$. \square

4.1 Growth in the unipotent radical

As mentioned before, Proposition B and Theorem C together imply Proposition 26, when \mathbb{G} is semisimple. When the unipotent radical \mathbb{U} is nontrivial, we need to do some work which is carried out in this section. Recall the definition of G and L from the beginning of Section 4. Denote by pr the projection homomorphism $G \rightarrow L$.

The purpose of this section is to prove the following

Proposition 27. *For every $\varepsilon > 0$ there is an integer C such that the following holds. Let $A \subseteq G$ be a symmetric set such that $\text{pr}(A) = L$, and*

$$\chi_A(gH) < [G : H]^{-\varepsilon} |G|^{1/C}$$

for any $g \in G$ and any proper $H < G$. Then $\pi_{q_1}[\prod_C A] = G_{q_1}$ for some $q_1 > q^{1-\varepsilon}$.

We need a couple of Lemmata. Let $\widehat{G} = \widehat{G}_1 \times \dots \times \widehat{G}_n$ be a direct product of groups. For each i , let $\beta_i : \widehat{G}_i \rightarrow \widehat{L}$ be a given homomorphism into a group \widehat{L} . Denote by $\beta : \widehat{G} \rightarrow \widehat{L}$ the homomorphism induced by the β_i in the obvious way. For each i write $\text{pr}_i : \widehat{G} \rightarrow \widehat{G}_i$ for the projection homomorphisms. We introduce the following distance for two elements $g_1, g_2 \in \widehat{G}$:

$$d(g_1, g_2) = \sum_{i: \text{pr}_i(g_1) \neq \text{pr}_i(g_2)} \log |\text{Ker}(\beta_i)|. \quad (8)$$

Lemma 28. *Let $A \subseteq \widehat{G}$ be a symmetric set with $\beta(A) = \widehat{L}$ and $1 \in A$. Assume that for every $g \in A$, $\beta(g) = 1$ we have $d(1, g) \leq \varepsilon \log |\text{Ker}(\beta)|$ for some $\varepsilon > 0$. Then A can be covered with at most $2^n |\text{Ker}(\beta)|^{25\varepsilon}$ cosets of a subgroup of \widehat{G} of order at most $|\widehat{L}|$.*

We will apply this Lemma in the following setting: We will have normal subgroups $N_{p_i} \trianglelefteq U_{p_i}$ which are normal in G_{p_i} as well, and we will set $\widehat{G}_i = G_{p_i}/N_{p_i}$ and $\widehat{L} = L$. The homomorphism β_i will be the projection $L_{p_i} \times (U_{p_i}/N_{p_i}) \rightarrow L_{p_i}$. The purpose of the lemma is to find an element g in a product-set of A which is in the kernel of pr , but have a large conjugacy class. In a subsequent lemma we will recover the normal subgroup generated by g in the product-set of a bounded number of copies of A . This will allow us to increase N_{p_i} , and proceed to the next step of the iteration.

Proof of Lemma 28. Let $\psi : \widehat{L} \rightarrow \widehat{G}$ be a map such that $\beta \circ \psi = \text{Id}$, and $\psi(\widehat{L}) \subseteq A$, this is possible due to the assumption $\beta(A) = \widehat{L}$. By assumption, we have $d(g^{-1}\psi(\beta(g)), 1) < \varepsilon \log |\text{Ker}(\beta)|$ for any $g \in A$, hence

$$d(\psi(\beta(g)), g) < \varepsilon \log |\text{Ker}(\beta)|. \quad (9)$$

Moreover, for any $g, h \in \widehat{L}$, we have

$$\begin{aligned} d(\psi(g)\psi(h), \psi(gh)) &< \varepsilon \log |\text{Ker}(\beta)| \quad \text{and} \\ d(\psi(g)^{-1}, \psi(g^{-1})) &< \varepsilon \log |\text{Ker}(\beta)|. \end{aligned}$$

These two inequalities mean that ψ is an $\varepsilon |\text{Ker}(\beta)|$ -homomorphism of type II with respect to d in the sense of Farah, see [25, Section 1]. Then by [25, Theorem 2.1], there is a homomorphism $\varphi : \widehat{L} \rightarrow \widehat{G}$ such that

$$d(\psi(g), \varphi(g)) < 24\varepsilon \log |\text{Ker}(\beta)|$$

for every $g \in \widehat{L}$. Combining this with (9), we get that for every element $g \in A$, there is $h \in \varphi(\widehat{L})$ such that $d(g, h) < 25\varepsilon \log |\text{Ker}(\beta)|$. By definition, this means that there is an index set $I \subset \{1, \dots, n\}$ such that $\text{pr}_i(g) = \text{pr}_i(h)$ if $i \notin I$ and $\prod_{i \in I} |\text{Ker}(\beta_i)| < |\text{Ker}(\beta)|^{25\varepsilon}$. For a fixed I , the elements g which satisfy this condition can be covered by at most $|\text{Ker}(\beta)|^{25\varepsilon}$ cosets of the group $\varphi(\widehat{L})$. This proves the claim, because we have 2^n possibilities for I . \square

As already promised, we show that we can recover the normal subgroup generated by the element constructed in the previous lemma. We need to introduce more notation. Let p_1, \dots, p_n be primes and with the notation as above, assume that \widehat{G}_i is generated by its p_i -elements. Assume that $\widehat{G}_i = \widehat{L}_i \rtimes \widehat{U}_i$ is a semidirect product and $\widehat{L} = \widehat{L}_1 \times \dots \times \widehat{L}_n$ and that the kernel of β_i is \widehat{U}_i . We assume further that \widehat{U}_i is isomorphic to $(\mathbb{F}_{p_i}^{d_i}, +)$. Then writing \widehat{U}_i additively, we can associate to it an \mathbb{F}_{p_i} -vector space M_i which is an \widehat{L}_i -module such that the action $v \mapsto g \cdot v$ of $g \in \widehat{L}_i$ on M_i descends from the conjugation action $u \mapsto gug^{-1}$ of $g \in \widehat{G}_i$ on \widehat{U}_i . We note that since \widehat{U}_i is commutative, its action by conjugation is trivial on itself, so the above is well-defined. Write $\widehat{U} = \widehat{U}_1 \times \dots \times \widehat{U}_n$.

Lemma 29. *Let $p_1, \dots, p_n, \widehat{G}_i, \widehat{L}_i, \widehat{U}_i$ and M_i satisfy the above assumptions. Furthermore, assume that no M_i contain a one dimensional composition factor. Let $A \subseteq \widehat{G}$ be a symmetric set with $1 \in A$ and $\beta(A) = \widehat{L}$ and let $g \in A$ be any element with $\beta(g) = 1$. Denote by N the smallest normal subgroup of \widehat{G} that contains g .*

Then there is a constant c depending only on $\max d_i$ such that

$$\prod_c A \supseteq N.$$

The proof of Lemma 29 requires

Lemma 30. *Let p be a prime and let $H \subseteq \text{GL}(M)$ be a group generated by its p -elements, where M is a vector space of dimension d over \mathbb{F}_p and $p \gg d$. Assume that no non-zero vector is fixed by H , i.e. the trivial representation is not a subrepresentation of M . Then there is a constant $c = c(d)$, only depending on d , such that $\sum_c H \cdot v$ contains a non-zero H -subspace, for any $0 \neq v \in M$.*

Here and everywhere below $H \cdot v$ denotes the orbit of v under the action of H on M .

Proof. Let $g \in H$ be an element of order p . Then $x = g - 1 \in \text{End}(M)$ is a nilpotent element. Let k be the largest integer such that x^k is not zero. It is clear that k is at most d . So

$$\begin{aligned} \sum_{2^d} H \ni (g^j - 1)^k &= ((1+x)^j - 1)^k && \forall 0 \leq j \leq p-1 \\ &= \left(x \sum_{i=0}^{j-1} \binom{j}{i+1} x^i \right)^k && \text{since } x^{k+1} = 0. \end{aligned}$$

Since any element of \mathbb{F}_p can be written as the sum of at most k k -th powers, we have that $\sum_{2^d} H \supseteq \mathbb{F}_p x^k$. Thus $\sum_{d^2} H \supseteq \mathbb{F}_p H x^k H$, and therefore

$$\sum_{d^3} H \supseteq \langle x^k \rangle,$$

where $\langle x^k \rangle$ is the ideal generated by x^k in $\mathcal{A} = \mathbb{F}_p[H]$, the \mathbb{F}_p -span of H in $\text{End}(M)$.

Now, we prove the lemma by induction on d . If $\mathcal{A} \cdot v$ is a proper H -subspace, we get the claim by the induction hypothesis. If not, then $\langle x^k \rangle \cdot v$ is a non-zero H -subspace of $\sum_{d^3} H \cdot v$, as we wished. Note that, if $\langle x^k \rangle \cdot v = 0$ and $\mathcal{A} \cdot v = M$, then $\langle x^k \rangle \cdot M = 0$, which is a contradiction as M is a faithful \mathcal{A} -module. \square

Corollary 31. *Let p be a prime and let $H \subseteq \text{GL}(M)$ be a group generated by its p -elements, where M is a vector space of dimension d over \mathbb{F}_p and $p \gg d$. Assume that none of the composition factors of M is one-dimensional. Then there is a constant $c = c(d)$, depending only on d , such that $\sum_c H \cdot v$ is equal to the H -subspace generated by v .*

Proof. Using Lemma 30, one can easily prove this, by induction on d . \square

Proof of Lemma 29. Since $\widehat{U} = \widehat{U}_1 \times \cdots \times \widehat{U}_n$ is a normal subgroup of \widehat{G} , we get a homomorphism θ from \widehat{G} to $\text{Aut}(\widehat{U})$, \widehat{G} acting on \widehat{U} by conjugation. As \widehat{U} is commutative, θ factors through \widehat{L} and we get back the action of \widehat{L}_i on \widehat{U}_i , for any i . Moreover, θ commutes with the projection homomorphisms pr_i .

Let $g = (u_1, \dots, u_n)$, where $u_i \in \widehat{U}_i$, for any i . Denote by N_i the normal subgroup generated by u_i in \widehat{G}_i . Translating Corollary 31 to the language of multiplicative groups, we get a constant c , depending only on $\max d_i$, such that

$$\prod_c \{h(u_1, \dots, u_n)h^{-1} : h \in A\} \supseteq N_1 \times \cdots \times N_n.$$

It is clear that $N_1 \times \cdots \times N_n$ is a normal subgroup of \widehat{G} which contains g . Thus

$$\prod_{3c} A = \prod_c A.A.\widetilde{A} \supseteq N,$$

as we wished. \square

The above lemmata allows us to deal with the case when U is commutative. In the general case we will work with $U/[U, U]$, and recover a subset of U which projects onto $U/[U, U]$. Then we will use the following lemma to recover U . This lemma is very similar to the main idea behind the papers [28] and [21].

Lemma 32. *Let \widehat{U} be a finite k -step nilpotent group generated by m elements, and let $A \subseteq \widehat{U}$ be a subset such that $A.[\widehat{U}, \widehat{U}] = \widehat{U}$. Then $\prod_{C(k,m)} A = \widehat{U}$.*

Proof. Consider the lower central series $\widehat{U} = \Gamma_1 \triangleright \Gamma_2 \triangleright \cdots \triangleright \Gamma_{k+1} = \{1\}$ defined by $\Gamma_{i+1} = [\widehat{U}, \Gamma_i]$. Then for $1 \leq i \leq k$, $K_i = \Gamma_i/\Gamma_{i+1}$ is a commutative group. It is well known (see [63, Corollary 1.12]) that for any i, j we have $[\Gamma_i, \Gamma_j] \subseteq \Gamma_{i+j}$ and for any $x, y, z \in U$ we have the identities (see [63, equations 1.4 and 1.5]):

$$\begin{aligned} [x, yz] &= [x, z][x, y]^z \\ [xy, z] &= [x, z]^y[y, z], \end{aligned}$$

where $x^y = y^{-1}xy$. Therefore the maps

$$\varphi_i : K_1 \times K_i \rightarrow K_{i+1}$$

defined by

$$\varphi_i(g\Gamma_2, h\Gamma_{i+1}) = [g, h]\Gamma_{i+2}$$

are well-defined, and they are homomorphisms in both variables.

We show that for any i

$$\prod_m \varphi_i(K_1, K_i) = K_{i+1}. \quad (10)$$

Let x_1, \dots, x_m be generators for K_1 . Then any element of K_{i+1} is of the form

$$\varphi_i(x_1^{a_{1,1}} \cdots x_m^{a_{1,m}}, y_1) \cdots \varphi_i(x_1^{a_{i,1}} \cdots x_m^{a_{i,m}}, y_i)$$

for some $a_{\cdot, \cdot} \in \mathbb{Z}$ and $y_j \in K_i$. Using that φ_i is a homomorphism in the first variable, we can expand this, then we can collect the factors containing x_k using

the commutativity of K_{i+1} and finally we use that φ_i is a homomorphism in the second variable and get that the above is equal to

$$\varphi_i(x_1, y_1^{a_{1,1}} \cdots y_l^{a_{l,1}}) \cdots \varphi_i(x_m, y_1^{a_{1,m}} \cdots y_l^{a_{l,m}}).$$

This proves the claim.

To prove the lemma, we note that the above claim implies that if $(\prod_C A)\Gamma_{i+1} \supseteq \Gamma_i$, then

$$(\prod_{m(2C+2)} A)\Gamma_{i+2} \supseteq \Gamma_{i+1}.$$

This proves the lemma by induction, and approximating an element successively in \widehat{U}/Γ_i for larger and larger values of i . \square

Proof of Proposition 27. Let A, G, L, U and pr be the same as in the proposition. First we prove the proposition in the case, when U is commutative. Then each U_{p_i} is isomorphic to $(\mathbb{F}_{p_i}^{d_i}, +)$ for some integers d_i . Denote $d = \max d_i$. For each p_i , we give a sequence of normal subgroups

$$\{1\} = N_{p_i}^{(0)} \trianglelefteq N_{p_i}^{(1)} \trianglelefteq \cdots \trianglelefteq N_{p_i}^{(l)} \trianglelefteq U_{p_i}$$

such that each of them is a normal subgroup in G_{p_i} as well. Write $N^{(m)} = N_{p_1}^{(m)} \times \cdots \times N_{p_n}^{(m)}$. They will satisfy the following properties:

$$(\prod_{C_1} A)N^{(k-1)} \supseteq N^{(k)} \tag{11}$$

$$[N^{(k)} : N^{(k-1)}] \geq [U : N^{(k-1)}]^{\varepsilon/100d} \tag{12}$$

$$[U : N^{(l)}] < q^\varepsilon, \tag{13}$$

where C_1 is a constant depending only on d .

Assume that $m > 0$, and $N_{p_i}^{(k)}$ is defined for $k < m$ and they satisfy (11) and (12). If $[U : N^{(m-1)}] < q^\varepsilon$, then we can set $l = m - 1$, and we are done. Assume the contrary. To apply Lemma 28 we take $\widehat{G}_i = G_{p_i}/N_{p_i}^{(m-1)}$, $L = \widehat{L}$ and we let $\beta_i : \widehat{G}_i \rightarrow \widehat{L}$ be the homomorphism induced by pr . Consider the group $\widehat{G} = G/N^{(m-1)}$ and the set $\overline{A} = AN^{(m-1)} \subset \widehat{G}$. By assumption, $A \subset G$ cannot be covered with less than $|\text{Ker}(\beta)|^\varepsilon |G|^{-1/C}$ cosets of a subgroup of G of index $|\text{Ker}(\beta)|$. We can assume that C is so large that $|G|^{1/C} < |\text{Ker}(\beta)|^{\varepsilon/2}$ and q is so large that $2^n < |\text{Ker}(\beta)|^{\varepsilon/4}$. Then \overline{A} cannot be covered by less than $2^n |\text{Ker}(\beta)|^{\varepsilon/4}$ cosets of a subgroup of \widehat{G} of order $|\widehat{L}|$. Using Lemma 28 we find an element $g \in \overline{A} \cdot \overline{A} \cdot \overline{A}$ such that $\beta(g) = 1$ and $d(g, 1) > \varepsilon \log |\text{Ker}(\beta)|/100$, where $d(\cdot, \cdot)$ is defined by (8). Let $N_{p_i}^{(m)}$ be the smallest normal subgroup of G_{p_i} that contains $N_{p_i}^{(m-1)}$ and $\pi_{p_i}(g)$. Then (11) follows from Lemma 29 applied for $\widehat{G} = G/N^{(m-1)}$ and $A = \overline{A} \cdot \overline{A} \cdot \overline{A}$. However, we need to check the condition that the L_{p_i} -modules M_i defined in the paragraph preceding Lemma 29 do not contain one dimensional composition factors. Suppose the contrary. We can assume that one of the M_i contains the trivial representation as a subrepresentation, actually for this purpose we might need to enlarge $N^{(m-1)}$. By [34, Theorem A], it follows that M_i is completely reducible, hence we can write

$M_i = M'_i \oplus M''_i$ as the sum of L_{p_i} -modules such that the action on M'_i is trivial. Then there is a proper normal subgroup $N \triangleleft U_{p_i}$ of G_{p_i} corresponding to M''_i such that G_{p_i} acts trivially on U_{p_i}/N , hence G_{p_i}/N is isomorphic to the direct product $L_{p_i} \times (U_{p_i}/N)$. This contradicts to the assumption that \mathbb{G} and hence G_p is perfect.

Let q' be the product of primes p_i for which $N_{p_i}^{(m-1)} \neq N_{p_i}^{(m)}$. Then

$$q' \geq e^{d(g,1)/d} > |\text{Ker}(\beta)|^{\varepsilon/100d},$$

since $|\text{Ker}(\beta_i)| \leq p_i^d$. The groups U_{p_i} are p_i -groups, hence $[N^{(m)} : N^{(m-1)}] \geq q'$. This implies (12), since $[U : N^{(k-1)}] = |\text{Ker}(\beta)|$. Therefore we proved equations (11)–(13).

Equations (11) and (13) together imply that there is an integer $q_1 > q^{1-\varepsilon}$ such that

$$\pi_{q_1}(\prod_{lC_1} A) \supseteq U_{q_1}.$$

Since $\text{pr}(A) = L$ and $G_{q_1} = L_{q_1}U_{q_1}$, this proves the proposition when U is commutative.

In the general case, running the above argument for the group $G/[U, U]$, we get

$$\pi_{q_1}(\prod_{lC_1} A) \cdot [U_{q_1}, U_{q_1}] \supseteq U_{q_1}.$$

Then Lemma 32 applied for the group $\widehat{U} = U_{q_1}$ and for the set $\pi_{q_1}(\prod_{lC_1} A) \cap U_{q_1}$ finishes the proof. \square

It is worth mentioning that the result from [34] depends on the classification of the finite simple groups. But we do not really need this result as the involved representations are coming from a representation over \mathbb{Q} and therefore, for large enough p , the picture modulo p is similar to the picture over \mathbb{Q} .

4.2 Assumptions (A0)–(A5) for $\mathbb{L}(\mathbb{Z}/q\mathbb{Z})$

We list the assumptions mentioned in Proposition B. When we say that something depends on the constants appearing in the assumptions (A1)–(A5) we mean C and the function $\delta(\varepsilon)$ for which (A4) holds.

- (A0) $L = L_1 \times \cdots \times L_n$ is a direct product, and the collection of the factors satisfy (A1)–(A5) for some sufficiently large constant C .
- (A1) There are at most C isomorphic copies of the same group in the collection.
- (A2) Each L_i is quasi-simple and we have $|Z(L_i)| < C$.
- (A3) Any non-trivial representation of L_i is of dimension at least $|L_i|^{1/C}$.
- (A4) For any $\varepsilon > 0$, there is a $\delta > 0$ such that the following holds. If μ and ν are probability measures on L_i satisfying

$$\|\mu\|_2 > |L_i|^{-1/2+\varepsilon} \quad \text{and} \quad \mu(gH) < |L_i|^{-\varepsilon}$$

for any $g \in L_i$ and for any proper $H < L_i$, then

$$\|\mu * \nu\|_2 \ll \|\mu\|_2^{1/2+\delta} \|\nu\|_2^{1/2}. \quad (14)$$

(A5) For some $m < C$, there are classes $\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_m$ of subgroups of L_i having the following properties.

- (i) $\mathcal{H}_0 = \{Z(L_i)\}$.
- (ii) Each \mathcal{H}_j is closed under conjugation by elements of L_i .
- (iii) For each proper $H < L_i$, there is an $H^\sharp \in \mathcal{H}_j$, for some j , with $H \lesssim_C H^\sharp$.
- (iv) For every pair of subgroups $H_1, H_2 \in \mathcal{H}_j$, $H_1 \neq H_2$, there is some $j' < j$ and $H^\sharp \in \mathcal{H}_{j'}$ for which $H_1 \cap H_2 \lesssim_C H^\sharp$.

One may think about (A5) that there is a notion for dimension of the subgroups of L_i .

In this section, we will check these assumptions. In the beginning of Section 3, we have already checked (A1) and (A2). By a result of V. Landazuri and G. Seitz [44], we also know that (A3) holds.

Assume that (A4) does not hold for L_i ; then there is an ε such that for any δ one can find probability measures on L_i with the following properties:

$$\|\mu\|_2 > |L_i|^{-1/2+\varepsilon}, \quad (15)$$

$$\mu(gH) < |L_i|^{-\varepsilon}, \quad \forall g \in G, \forall H \lesssim G, \quad (16)$$

$$\|\mu * \nu\|_2 \geq \|\mu\|_2^{1/2+\delta} \|\nu\|_2^{1/2}. \quad (17)$$

One can easily see that ε is less than $1/2$. Choose δ such that $R\delta$ is less than $\varepsilon/(1/2 - \varepsilon)$ and $2\varepsilon_0/(1 + \varepsilon_0)$, where ε_0 is the constant from Theorem C and R is the constant from Lemma D.

By Lemma D and (17), there is a symmetric subset A of L_i such that,

$$\|\mu\|_2^{-2+\delta'} \ll |A| \ll \|\mu\|_2^{-2-\delta'}, \quad (18)$$

$$|\Pi_3 A| \ll \|\mu\|_2^{-\delta'} |A|, \quad (19)$$

$$\min_{s \in A} (\tilde{\mu} * \mu)(s) \gg \frac{1}{\|\mu\|_2^{-\delta'} |A|}, \quad (20)$$

where $\delta' = R\delta$ and the implied constants are absolute.

First we claim that A is a generating set of L_i . If not, it generates a proper subgroup H . Hence, on one hand, we have that

$$\begin{aligned} (\tilde{\mu} * \mu)(A) &\leq (\tilde{\mu} * \mu)(H) = \sum_{h \in H} (\tilde{\mu} * \mu)(h) \\ &= \sum_{h \in H} \sum_{g \in L_i} \mu(g) \mu(gh) \\ &= \sum_{g \in L_i} \mu(g) \mu(gH) < |L_i|^{-\varepsilon} \quad \text{by (16)}. \end{aligned} \quad (21)$$

On the there hand

$$\begin{aligned} (\tilde{\mu} * \mu)(A) &\gg \|\mu\|_2^{\delta'} && \text{by (20),} \\ &> |L_i|^{\delta'(-1/2+\varepsilon)} && \text{by (15).} \end{aligned} \quad (22)$$

We get a contradiction, by (21), (22) and $\delta' < \varepsilon/(1/2 - \varepsilon)$.

Assume $\Pi_3 A \neq L_i$; then since A is a generating set of L_i , by Theorem C, $|\Pi_3 A| \gg |A|^{1+\varepsilon_0}$. Hence, by (19) and (18), we have that

$$\|\mu\|_2^{(-2+\delta')\varepsilon_0} \ll |A|^{\varepsilon_0} \ll \|\mu\|_2^{-\delta'}. \quad (23)$$

We get a contradiction by (23) and $\delta' < 2\varepsilon_0/(1 + \varepsilon_0)$.

So we have $\Pi_3 A = L_i$. By (19), (18) and (15), we have

$$|L_i| = |\Pi_3 A| \ll \|\mu\|_2^{-\delta'} |A| \ll \|\mu\|_2^{-2-2\delta'} < |L_i|^{(1/2-\varepsilon)(2+2\delta')},$$

which is a contradiction by $\delta' < \varepsilon/(1/2 - \varepsilon)$. Overall we showed that (A4) holds for L_i .

As L_i is a quasi-simple finite group over a finite field which is of a bounded degree extension of its prime field, property (A5) is a direct consequence of [61, Proposition 24].

4.3 Proof of Proposition 26

Let N be a constant such that $|G_p| < p^N$ for all p . We consider two cases. The first case is when $|\text{pr}(A)| < q^{-\varepsilon/10NC}|L|$, where C is a constant such that any nontrivial representation of L_p is of dimension at least $p^{1/C}$ (cf. assumption (A3) in section 4.2). As we have seen in section 4.2, the group L satisfies the assumptions (A0)–(A5), hence Proposition B is applicable. Then we get $|\prod_3 \text{pr}(A)| \gg |\text{pr}(A)|^{1+\delta}$. By the pigeonhole principle A contains at least $|A|/|\text{pr}(A)|$ elements of a coset of $\text{Ker}(\text{pr})$. Then it follows that $|\prod_4 A| \gg |\text{pr}(A)|^\delta |A|$. Note that $|\text{pr}(A)| > |L|^\varepsilon |G_q|^{-\delta}$ by the assumption we made in the proposition on the set A . This proves the proposition in the first case (see (24) below).

Now we consider the second case, i.e. when $|\text{pr}(A)| \geq q^{-\varepsilon/10NC}|L|$. Then by [11, Lemma 5.2], there is a set $A' \subset \text{pr}(A)$ and integers K_j such that for every $g \in A'$ we have

$$|\{x \in L_{p_j} : \exists h \in A' \text{ s.t. } \pi_{p_1 \dots p_{j-1}}(h) = \pi_{p_1 \dots p_{j-1}}(g) \text{ and } \pi_{p_j}(h) = x\}| = K_j$$

and the integers K_j satisfy

$$|A'| = \prod K_j \geq [\prod (2 \log p_j)^{-1}] |A|.$$

Denote by q_2 the product of primes p_j for which $K_j \geq p_j^{-1/3C} |L_j|$. Then $q/q_2 < q^{\varepsilon/3N}$, if all the primes p_j are sufficiently large. By a theorem of Gowers

[30] (see also [50, Corollary 1]) it follows that if $B_1, B_2, B_3 \subseteq L_{p_j}$ are sets with $|B_i| \geq p_j^{-1/3C} |L_j|$, $i = 1, 2, 3$, then $B_1.B_2.B_3 = L_{p_j}$. This implies that

$$\text{pr}(\pi_{q_2}(A.A.A)) = L_{q_2}.$$

For more details see the argument on [61, pp. 26]. Now using Proposition 27 for the set $\pi_{q_2}(A.A.A)$ we get an integer $q_1|q_2$ with $q_1 > q^{1-\varepsilon/2N}$ such that $\pi_{q_1}[\prod_C A] = G_{q_1}$ for some constant C independent of q . Thus $|\prod_C A| > q^{-\varepsilon/2}|G_q|$. It is a general fact (see the proof of [35, Lemma 2.2]) that

$$|\prod_C A| < \left(\frac{|A.A.A|}{|A|} \right)^{C-2} |A| \quad (24)$$

whenever A is a symmetric set in a group. This finishes the proof.

5 Proof of Theorem 1

5.1 Necessity

Let us first show the necessary part. Let \mathbb{G} be the Zariski-closure of Γ . Denote by \mathbb{G}° , the connected component of \mathbb{G} , and let $\Gamma^\circ = \mathbb{G}^\circ \cap \Gamma$. It is clear that Γ° is a normal finite-index subgroup of Γ , and so Γ° is also generated by a finite set S° . We start by showing that $\mathcal{G}(\pi_q(\Gamma^\circ), \pi_q(S^\circ))$ form expanders as q runs through square free integers with large prime factors assuming $\mathcal{G}(\pi_q(\Gamma), \pi_q(S))$ form expanders. To this end, first we show that Γ° is a ‘‘congruence’’ subgroup, i.e. it contains a congruence kernel $\Gamma(q) = \ker(\Gamma \xrightarrow{\pi_q} G_q)$ if the prime factors of q are large enough. To prove this claim, we notice that \mathbb{G}° and the quotient map $\iota : \mathbb{G} \rightarrow \mathbb{G}/\mathbb{G}^\circ$ are defined over \mathbb{Q} . Hence $\iota(\Gamma(q)) = (\iota(\Gamma))(q)$ for any q with large prime factors. On the other hand, since $\mathbb{G}/\mathbb{G}^\circ$ is a finite \mathbb{Q} -group, $(\iota(\Gamma))(q) = 1$ for any q with large prime factors, which completes the argument of the our claim. Now it is pretty easy to show that $\mathcal{G}(\pi_q(\Gamma^\circ), \pi_q(S^\circ))$ form expanders as q runs through square free integers with large prime factors. For the sake of completeness we present one argument: it is well-known that our desired condition holds if and only if the Haar measure is the only finitely additive Γ° -invariant measure on $\widehat{\Gamma}^\circ$, where $\widehat{\Gamma}^\circ$ is the profinite completion of Γ° with respect to $\{\Gamma^\circ \cap \Gamma(q)\}$. By the above discussion $\widehat{\Gamma}^\circ$ is a finite-index open subgroup of $\widehat{\Gamma}$ the profinite closure of Γ with respect to $\{\Gamma(q)\}$; thus one can easily deduce our claim. As a consequence we get a uniform upper bound on $|\pi_q(\Gamma^\circ)/[\pi_q(\Gamma^\circ), \pi_q(\Gamma^\circ)]|$. On the other hand, $[\mathbb{G}^\circ, \mathbb{G}^\circ]$ and the quotient map $\iota' : \mathbb{G}^\circ \rightarrow \mathbb{G}^\circ/[\mathbb{G}^\circ, \mathbb{G}^\circ]$ are defined over \mathbb{Q} . Hence again we have that ι' and π_q commute with each other for any q with large prime factors. Thus one can complete the proof of the necessary part using the facts that Γ° is Zariski-dense in \mathbb{G}° and \mathbb{G}° does not have any proper open subgroup.

5.2 Sufficiency

Next we show that the condition that the connected component of the Zariski closure of Γ is perfect is sufficient for the Cayley graphs to form a family of

expanders. The argument which shows this using Propositions 7 and 8 is based on the ideas of Sarnak and Xue [55] and Bourgain and Gamburd [7] and it is common to all of the papers [7]–[11] and [61]. In the previous section we have already remarked that $\Gamma^\circ = \Gamma \cap \mathbb{G}^\circ$ is finitely generated. Using Proposition 7 for Γ° , we get a symmetric set $S' \subset \Gamma^\circ$ such that if q is square-free and coprime to the denominators of the entries in the elements of S , $H \leq \pi_q(\Gamma)$ and l is an integer with $l > \log q$, then

$$\pi_q[\chi_{S'}^{(l)}](H) \ll [\pi_q(\Gamma^\circ) : H]^{-\delta}.$$

We show that the Cayley graphs $\mathcal{G}(\pi_q(\Gamma^\circ), \pi_q(S'))$ are expanders and later we will see that this implies the statement of the theorem. Denote by $T = T_q$ the convolution operator by $\chi_{\pi_q(S')}$ in the regular representation of $\pi_q(\Gamma^\circ)$. I.e. we write $T(\mu) = \chi_{\pi_q(S')} * \mu$ for $\mu \in l^2(\pi_q(\Gamma^\circ))$. We will show that there is a constant $c < 1$ independent of q such that the second largest eigenvalue of T is less than c . By a result of Dodziuk [23]; Alon [2]; and Alon and Milman [4] (see also [37, Theorem 2.4]) this then implies that $\mathcal{G}(\pi_q(\Gamma^\circ), \pi_q(S'))$ is a family of expanders.

Consider an eigenvalue λ of T , and let μ be a corresponding eigenfunction. Consider the irreducible representations of $\pi_q(\Gamma^\circ)$; these are subspaces of $l^2(\pi_q(\Gamma^\circ))$ invariant under T . Denote by ρ the irreducible representation that contains μ . Recall from Section 3 that

$$\pi_q(\Gamma^\circ) = \prod_{p|q \text{ prime}} \pi_p(\Gamma^\circ).$$

We only consider the case when the kernel of ρ does not contain $\pi_p(\Gamma^\circ)$ for any $p|q$, otherwise we can consider the quotient of $\pi_q(\Gamma^\circ)$ by $\pi_p(\Gamma^\circ)$, and we can replace q by a smaller integer. Then ρ is the tensor-product of nontrivial representations of the groups $\pi_p(\Gamma^\circ)$, hence the dimension of ρ is at least $|\pi_q(\Gamma^\circ)|^\varepsilon$ for some $\varepsilon > 0$ (cf. assumption (A3) in Section 4.2 and note that by Corollary 14 the semisimple part can not be contained in $\text{Ker}(\rho)$). This in turn implies that the multiplicity of λ in T is at least $|\pi_q(\Gamma^\circ)|^\varepsilon$, since the regular representation $l^2(\Gamma/\Gamma_q)$ contains $\dim(\rho)$ irreducible components isomorphic to ρ .

Using this bound for the multiplicity, we can bound λ^{2l} by computing the trace of T^{2l} in the standard basis:

$$\lambda^{2l} \leq |\pi_q(\Gamma^\circ)|^{-\varepsilon} \text{Tr}(T^{2l}) = |\pi_q(\Gamma^\circ)|^{-\varepsilon} |\pi_q(\Gamma^\circ)| \|\pi_q[\chi_{S'}^{(l)}]\|_2^2,$$

where $\|\cdot\|_2$ denotes the l^2 norm over the finite set $\pi_q(\Gamma^\circ)$. This proves the theorem, if we can show that

$$\|\pi_q[\chi_{S'}^{(l)}]\|_2 \ll |\pi_q(\Gamma^\circ)|^{-1/2+\varepsilon/4} \quad (25)$$

for some $l \ll \log q$.

First apply Proposition 7 with $H = \{1\}$. It gives $\pi_q[\chi_{S'}^{(l)}](1) \ll |\pi_q(\Gamma^\circ)|^{-\varepsilon}$ for $l > \log q$ and for some $\varepsilon > 0$. If l is even then $\pi_q[\chi_{S'}^{(l)}](1) > \pi_q[\chi_{S'}^{(l)}](g)$ for any

$g \in \pi_q(\Gamma)$ by the Cauchy-Schwartz inequality and the definition of convolution (recall that S is symmetric). Then we get the estimate

$$\|\pi_q[\chi_{S'}^{(l)}]\|_2 \ll |\pi_q(\Gamma^\circ)|^{-\varepsilon/2}.$$

Observe that if we repeatedly apply Proposition 8 for the measures $\mu = \nu = \pi_q[\chi_{S'}^{(2^k l)}]$, then we get (25) in finitely many steps. To justify the use of Proposition 8, we remark that since S' is symmetric, we have

$$\left(\pi_q[\chi_{S'}^{(2^k l)}](gH)\right)^2 \leq \pi_q[\chi_{S'}^{(2^{k+1} l)}](H)$$

that can be bounded using Proposition 7. This shows that $\mathcal{G}(\pi_q(\Gamma^\circ), \pi_q(S'))$ are expanders indeed.

To finish the proof we show the same for the family $\mathcal{G}(\pi_q(\Gamma), \pi_q(S))$. Write $c = c(\mathcal{G}(\pi_q(\Gamma^\circ), \pi_q(S')))$, recall the definition from the introduction. Assume that the elements of S' are the product of at most m elements of S . Consider a set $A = V(\mathcal{G}) = \pi_q(\Gamma)$ of vertices with $|A| \leq |V(\mathcal{G})|/2$, and denote by $N_k(A)$ the set of vertices that can be joined to an element of A by a path of length at most k in $\mathcal{G}(\pi_q(\Gamma), \pi_q(S))$. I.e. by definition

$$N_k(A) = \left(\prod_k S\right).A.$$

Also, it is easy to see that $|N_k(A)| \leq |S|^{k-1}|\partial A| + |A|$, so it is enough to give a lower bound on $|N_k(A)|$. We clearly have

$$|N_{|\Gamma/\Gamma^\circ|}(A)| \geq |\Gamma/\Gamma^\circ| \max_{g \in \pi_q(\Gamma)} |A \cap g\pi_q[\Gamma^\circ]|.$$

This finishes the proof if say $|A \cap \pi_q(\Gamma^\circ)| < |A|/(2|\Gamma/\Gamma^\circ|)$ or if $|A \cap \pi_q(\Gamma^\circ)| > 3|\pi_q(\Gamma^\circ)|/4$. If both of these inequalities fail, then by the expander property of $\mathcal{G}(\pi_q(\Gamma^\circ), \pi_q(S'))$ already proved, we conclude that

$$N_m(A) > |A| + c/|S'| \min\{|A|/(2|\Gamma/\Gamma^\circ|), |\pi_q(\Gamma^\circ)|/4\}$$

which proves the theorem.

Remark 33. The above proof implies a variant of Proposition 8 that is useful in some applications. Compare the statement below with [12, Lemma 2 in Section 7]. Let q be a square-free integer and \mathbb{G} be a Zariski-connected, perfect algebraic group defined over \mathbb{Q} , and write $G = \mathbb{G}(\mathbb{Z}/q\mathbb{Z})$. For every $\varepsilon > 0$, there is a $\delta > 0$ such that the following hold: Let μ be a probability measure which satisfies the following version of the assumptions in Proposition 8 for some $\varepsilon > 0$. I.e.

$$\|\mu\|_2 > |G|^{-1/2+\varepsilon} \quad \text{and} \quad \mu(gH) < [G : H]^{-\varepsilon}|G|^\delta$$

for any $g \in G$ and for any proper subgroup $H < G$. Let $f \in l^2(G)$ be a complex valued function on the group G such that

$$\sum_{g \in a\mathbb{G}(\mathbb{Z}/q'\mathbb{Z})} f(g) = 0$$

for all $a \in G$ and $q'|q$ with $q' \neq 1$. This condition is equivalent to saying that f is orthogonal to those irreducible subrepresentations in the regular representation of G that factor through $G/\mathbb{G}(\mathbb{Z}/q'\mathbb{Z})$ for some $q' \neq 1$. Then using the argument in the proof above, we can write

$$\|\mu * f\|_2 < q^{-\delta} \|f\|_2 \quad (26)$$

for some $\delta > 0$ depending on ε and \mathbb{G} . Indeed, repeated application of Proposition 8 shows the analogue of (25) for $\mu^{(L)}$ in place of $\pi_q[\chi_{S'}^{(l)}]$ for some integer L which depend on ε and \mathbb{G} . Combining this with the lower bounds for multiplicities of the eigenvalues we get the inequality (26). We also note that the statement in this remark also holds if we consider a group G which satisfies the assumptions (A0)–(A5) listed in Section 4.2 instead of taking $G = \mathbb{G}(\mathbb{Z}/q\mathbb{Z})$.

A Appendix: Effectivization of Nori's paper

In this section, we address the non-effective parts of Nori's argument in [51] and present alternative effective arguments. Most of the arguments in the mentioned article are effective. We only need to present effective proofs of [51, Proposition 2.7, Lemma 2.8 and Theorem 5.1]. It should be said that in this article by effective we mean that there is an algorithm to find the implied constants. Alternatively one can say the mentioned functions are recursively defined. We should say that these results are far from the best possible. In fact using the classification of finite simple groups, Guralnick [34, Theorem D] showed that if $p > \max\{n + 2, 11\}$, then Nori's statement hold for any subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ which is generated by its p -elements and *has no normal p -subgroup*. Unfortunately this last condition does not allow us to apply this sharp result.

Before stating the main results of this section, we recall very few terms from [51] and refer the reader to the mentioned article for the undefined terms.

Here R always denotes a finitely presented integral domain unless otherwise mentioned.

Definition 34 (Definition 2.2 in [51]). An R -submodule L of $M_n(R)$ is called a k -strict Lie subalgebra of $M_n(R)$ if

1. L is a Lie ring.
2. There is a submodule L' of $M_n(R)$ such that

$$M_n(R) = L \oplus L',$$

and L' is locally free of rank $n^2 - k$.

Definition 35 (Definition 2.5 in [51]). Let \mathbb{U}_n , \mathbb{N}_n and $\mathbb{Y}_{n,k}$ be the schemes which represent the following functors from $\mathbb{Z}[\frac{1}{(2n-1)!}]$ -algebras A to sets:

1. $\mathbb{N}_n(A) := \{x \in M_n(A) \mid x^n = 0\}$,

2. $\mathbb{U}_n(A) := \{x \in M_n(A) \mid (x - 1)^n = 0\}$,
3. $\mathbb{Y}_{n,k}(A) := \{\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{N}_n(A) \mid L_{\mathbf{x}} \text{ is a } k\text{-strict Lie subalgebra}\}$,
where $L_{\mathbf{x}} = Ax_1 + \dots + Ax_k$.

Definition 36. Let L be a k -strict Lie subalgebra of $M_n(A)$ and \mathbb{H} be a closed subgroup-scheme of $(\mathbb{GL}_n)_A := \mathbb{GL}_n \times \text{Spec}A$. Let \mathbb{L} be the A -scheme which represents the functor $S \mapsto S \otimes A$ defined for all commutative A -algebras. Let us define two closed subschemes of $(\mathbb{GL}_n)_A$:

$$e(\mathbb{L}^{(n)}) := \exp(\mathbb{L} \cap (\mathbb{N}_n)_A), \quad \mathbb{H}^{(u)} := \mathbb{H} \cap (\mathbb{U}_n)_A,$$

for any $\mathbb{Z}[1/(2n-1)!]$ -algebra A .

Definition 37 (Definition 2.3 and Remark 2.18 in [51]). Let L and \mathbb{H} be as in Definition 36. Then (L, \mathbb{H}) is called an acceptable pair if the following hold:

1. The projection $\mathbb{H} \rightarrow \text{Spec}(A)$ is a smooth morphism with all the fibers connected.
2. $\text{Lie}(\mathbb{H}/A) = L$.
3. $(e(\mathbb{L}^{(n)}))_{\text{red}} = (\mathbb{H}^{(u)})_{\text{red}}$.

In this case, L or \mathbb{H} are called acceptable.

In this section, let $\underline{X} = \{X_1, \dots, X_m\}$ and $R[\underline{X}]$ be the ring of polynomials in the variables X_1, \dots, X_m with coefficients in the ring R . If F is a subset of a ring R , then $\langle F \rangle$ denotes the ideal generated by F in R .

Here are the main results of this section:

Lemma 38 (Effective version of Lemma 2.8 in [51]). *Let R be a computable noetherian integral domain with quotient field K and $\mathbf{z} \in \mathbb{Y}_{n,k}(R)$. If $L_{\mathbf{z}} \otimes_R K$ is acceptable, then we can algorithmically find a non-zero element $g \in R$ such that $L_{\mathbf{z}} \otimes_R R_g \subseteq M_n(R_g)$ is also acceptable.*

(For the definition of a computable ring, see [5, Chapter 4.6].)

Lemma 39 (Effective version of Proposition 2.7 in [51]). *For a given k, n , we can give presentations of finitely many integral domains R_i and algorithmically find elements $\mathbf{z}_i \in \mathbb{Y}_{n,k}(R_i)$ such that*

1. $L_{\mathbf{z}_i}$ is acceptable if $\text{char}(R_i) = 0$.
2. $\mathbf{z}_i : \text{Spec}(R_i) \rightarrow \mathbb{Y}_{n,k}$ is a locally closed immersion and

$$\mathbb{Y}_{n,k} = \bigsqcup_i \mathbf{z}_i(\text{Spec}(R_i)).$$

In order to prove Lemma 38, we need to show the following effective version of certain results from EGA [31].

Theorem 40 (Effective version of Theorem 9.7.7 (i) and Theorem 12.2.4 (iii) in [31]). *Let R be a computable integral domain. Let $F = \{f_1, \dots, f_l\} \subseteq R[\underline{X}]$ and $A = R[\underline{X}]/\langle F \rangle$. If the generic fiber of the projection map $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is smooth and geometrically irreducible, then one can compute a non-zero element $g \in R$ such that the projection map $\text{Spec}(A_g) \rightarrow \text{Spec}(R_g)$ is smooth and all of its fibers are geometrically irreducible.*

Finally we shall use Lemma 38 and Lemma 39 to get the following:

Theorem 41 (Effective version of a special case of Theorem 5.1 in [51]). *Let \mathbb{G} be a perfect, Zariski-connected, simply-connected \mathbb{Q} -group. Let $\Gamma \subseteq \mathbb{G}(\mathbb{Q})$ be a Zariski-dense subgroup generated by a finite symmetric set S . Then one can effectively find $p_0 = p_0(S)$ such that for any $p > p_0$ one has $\pi_p(\Gamma) = \mathbb{G}(\mathbb{F}_p)$.*

A.1 Proof of Theorem 40.

In this section, first we show the “generic flatness” in Lemma 42 and the “generic smoothness” in Lemma 45. Then we reduce the general case of Theorem 40 to the hyperplane case and finish it as in [31].

Lemma 42. *Let R be a computable noetherian integral domain. Let K be the quotient field of R . Let $F = \{f_1, \dots, f_l\} \subseteq R[\underline{X}]$ and $A = R[\underline{X}]/\langle F \rangle$. Then we can algorithmically find a non-zero element $g \in R$ such that*

1. A_g is a free R_g -module.
2. $\langle F \rangle_g = \langle F \rangle_K \cap R_g[\underline{X}]$, where $\langle F \rangle_g$ (resp. $\langle F \rangle_K$) is the ideal generated by F in $R_g[\underline{X}]$ (resp. $K[\underline{X}]$).
3. All the fibers of the projection map $\text{Spec}(A_g) \rightarrow \text{Spec}(R_g)$ are equidimensional.

Proof. For any ordering of X_i , we compute the Gröbner basis of $\langle F \rangle_K$ and multiply all the head coefficients which appear in the process. Let $g \in R$ be the product of these head coefficients. Now the basic information on the Gröbner basis gives us the first and the second parts.

Now without loss of generality we can and will assume that $\{X_1, \dots, X_d\}$ is a maximal independent set modulo $\langle F \rangle_K$. Since the head coefficients of the Gröbner basis are units in R_g , for any $\mathfrak{p} \in \text{Spec}(R_g)$, $\{X_1, \dots, X_d\}$ is a maximal independent set modulo $\langle F \rangle_{k(\mathfrak{p})} \subseteq k(\mathfrak{p})[\underline{X}]$, where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Hence, by [5, Theorem 9.27], we have that the Gelfand-Kirillov dimension of $A \otimes k(\mathfrak{p})$ is d , which proves the third part. \square

Definition 43. For $F = \{f_1, \dots, f_l\} \subseteq R[\underline{X}]$, Let $J_e(F)$ denote the ideal generated by the $e \times e$ minors of $[\partial f_i / \partial X_j]$ in $R[\underline{X}]/\langle F \rangle$.

Lemma 44. *Let \bar{k} be an algebraically closed field and $F = \{f_1, \dots, f_l\} \subseteq \bar{k}[\underline{X}]$. Let $A = \bar{k}[\underline{X}]/\langle F \rangle$. Then the projection map $\text{Spec}(A) \rightarrow \text{Spec}(\bar{k})$ is smooth if and only if $J_{m-d}(F) = A$, where $m = \#\underline{X}$ and $d = \dim A$.*

Proof. This is essentially Jacobi's criteria. \square

Lemma 45. *Let R be a computable noetherian integral domain, $F = \{f_1, \dots, f_l\}$ be a subset of $R[\underline{X}]$, and $A = R[\underline{X}]/\langle F \rangle$. If the generic fiber of the projection map $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is smooth, then one can compute a non-zero element $g \in R$ such that the projection map $\text{Spec}(A_g) \rightarrow \text{Spec}(R_g)$ is smooth.*

Proof. Since clearly $\text{Spec}(A)$ is locally finitely presented, it is enough to compute $g \in R$ such that

1. The projection map $\text{Spec}(A_g) \rightarrow \text{Spec}(R_g)$ is flat.
2. For any $\mathfrak{p} \in \text{Spec}(R_g)$, the projection map $\text{Spec}(A_g \otimes \overline{k(\mathfrak{p})}) \rightarrow \text{Spec}(\overline{k(\mathfrak{p})})$ is smooth, where $\overline{k(\mathfrak{p})}$ is the algebraic closure of $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

By Lemma 42, we can compute a non-zero element $g_1 \in R$ such that the projection map $\text{Spec}(A_{g_1}) \rightarrow \text{Spec}(R_{g_1})$ is flat and all of its fibers are of a fixed dimension d . On the other hand, since the generic fiber is smooth, by Lemma 44, $J_{m-d}(F) \cap R$ is non-zero. By computing a Gröbner basis of $J_{m-d}(F)$, we can compute a non-zero element $g_2 \in J_{m-d}(F) \cap R$. It is easy to check that $g = g_1 g_2$ gives us the desired property. \square

Lemma 46. *Let R be an infinite computable noetherian integral domain, F a finite subset of $R[\underline{X}]$ and $A = R[\underline{X}]/\langle F \rangle$. Then we can compute a matrix $[a_{ij}] \in \text{GL}_m(R)$, a non-zero element $g \in R$, elements $x_{d+1} \in A_g$, $f \in R_g[X'_1, \dots, X'_d]$ and $p \in R_g[X'_1, \dots, X'_d, T]$, where $X'_i = \sum a_{ij} X_j$, such that the following hold*

1. $R_g[X'_1, \dots, X'_d] \cap \langle F \rangle_g = 0$.
2. A_g is an integral extension of $R_g[X'_1, \dots, X'_d]$.
3. $A_{gf} = (R_g[X'_1, \dots, X'_d])_f[x_{d+1}] \simeq (R_g[X'_1, \dots, X'_d, T]/\langle p \rangle)_f$.

Proof. Let K be the quotient field of R . By [45], we can compute a matrix $[a_{ij}] \in \text{GL}_m(R)$ and elements $r_{d+2}, \dots, r_m \in R$ such that the following hold

1. X'_1, \dots, X'_d are algebraically independent in $A \otimes K$.
2. X'_j are integral over $K[X'_1, \dots, X'_d]$.
3. $S^{-1}A = K(X'_1, \dots, X'_d)[x_{d+1}]$, where $S = K[X'_1, \dots, X'_d] \setminus \{0\}$ and $x_{d+1} = X'_{d+1} + \sum_{i=d+2}^m r_i X'_i$.

where $X'_i = \sum a_{ij} X_j$.

Again computing Gröbner basis of the ideal generated by F in

$$K(X'_1, \dots, X'_d)[X'_{d+1}, \dots, X'_m]$$

with respect to various orderings, we can compute the minimal polynomials of X'_i over $K(X'_1, \dots, X'_d)$. Since X'_i are integral over $K[X'_1, \dots, X'_d]$ and the

ring of polynomials over a field is integrally closed, all the minimal polynomials are monic polynomials with coefficients in $K[X'_1, \dots, X'_d]$. Hence we can compute a non-zero element $g \in R$ such that A_g is an integral extension of $R_g[X'_1, \dots, X'_d]$. Moreover writing X'_i as a polynomial in terms of x_{d+1} with coefficients in $K(X'_1, \dots, X'_d)$, we can find $f_1 \in K[X'_1, \dots, X'_d]$ such that $A_{f_1} \otimes K = K[X'_1, \dots, X'_d]_{f_1}[x_{d+1}]$. We can also compute the minimal polynomial p of x_{d+1} over $K(X'_1, \dots, X'_d)$. Now let f be the product of f_1 by the product of all the denominators of the coefficients of the minimal polynomial. It is clear that these choices satisfy the desired properties. \square

Proof of Theorem 40. By Lemma 45 and Lemma 42, we can compute a non-zero element $g_1 \in R$ such that the projection map $\text{Spec}(A_{g_1}) \rightarrow \text{Spec}(R_{g_1})$ is smooth and A_{g_1} is a free R_{g_1} -module. Let $g_2 \in R$, f and p be the parameters which are given by Lemma 46. Changing R to $R_{g_1 g_2}$ and using the above results, we can and will assume that

1. A is a free R -module.
2. The projection map $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is smooth,
3. A is an integral extension of $R[X_1, \dots, X_d]$ and the latter is the ring of polynomials,
4. $A_f \simeq (R[X_1, \dots, X_d, T]/\langle p \rangle)_f$.

Let $B = R[X_1, \dots, X_d, T]/\langle p \rangle$. Since the generic fiber of $\text{Spec}(A)$ over R is geometrically irreducible, so is the generic fiber of $\text{Spec}(B)$ over R . Hence by virtue of [31, Lemma 9.7.5], we can compute a non-zero element $g_3 \in R$ such that all the fibers of $\text{Spec}(B_{g_3}) \rightarrow \text{Spec}(R_{g_3})$ are geometrically irreducible. In particular, all the fibers of $\text{Spec}(A_{g_3 f}) \simeq \text{Spec}(B_{g_3 f}) \rightarrow \text{Spec}(R_{g_3})$ are geometrically irreducible. This means for any $\mathfrak{p} \in \text{Spec}(R_{g_3})$

$$A_{g_3 f} \otimes \overline{k(\mathfrak{p})}$$

is an integral domain. If it is a non-zero ring, then $A_{g_3} \otimes \overline{k(\mathfrak{p})}$ is also an integral domain. On the other hand, $A_{g_3 f} \otimes \overline{k(\mathfrak{p})} = 0$ if and only if f is either zero or a zero-divisor in $A_{g_3} \otimes k(\mathfrak{p})$.

By a similar argument as in Lemma 42, we can compute a non-zero element $g_4 \in R$ such that $(A/\langle f \rangle)_{g_4}$ is a free R_{g_4} -module. Let $g = g_3 g_4$. We claim that all the fibers of $\text{Spec}(A_g) \rightarrow \text{Spec}(R_g)$ are geometrically irreducible. By the above discussion, it is enough to show that for any $\mathfrak{p} \in \text{Spec}(R_g)$, f is not either zero nor a zero-divisor in $A_g \otimes \overline{k(\mathfrak{p})}$.

Let $\lambda_f(x) = fx$ be the map of multiplication by f in A_g . Since A_g is an integral domain and f is not zero, we have the following short exact sequence of R_g -modules:

$$0 \rightarrow A_g \xrightarrow{\lambda_f} A_g \rightarrow (A/\langle f \rangle)_g \rightarrow 0.$$

Hence for any $\mathfrak{p} \in \text{Spec}(R_g)$ we have the following exact sequence

$$\text{Tor}((A/\langle f \rangle)_g, k(\mathfrak{p})) \rightarrow A_g \otimes k(\mathfrak{p}) \rightarrow A_g \otimes k(\mathfrak{p}).$$

Since $(A/\langle f \rangle)_g$ is a free R_g -module, $\text{Tor}((A/\langle f \rangle)_g, k(\mathfrak{p})) = 0$. Therefore f is neither a zero nor a zero-divisor in $A_g \otimes k(\mathfrak{p})$. Thus by the above discussion, we are done. \square

A.2 Proof of Lemma 38.

By Definition 37, a pair of a Lie ring and an algebraic group scheme is acceptable if and only if it satisfies three properties. In this section, we show how one can use Theorem 40 to guarantee the first property. The second property is achieved using smoothness and the definition of the Lie algebra of a smooth group scheme. The third property is dealt with in Lemma 52.

Lemma 47. *Let \mathbb{G} be an algebraic group and X an irreducible subvariety. If $1 \in X = X^{-1}$, then*

$$\prod_{\dim \mathbb{G}} X = X \cdots X,$$

is the group generated by X .

Proof. This is clear. \square

Lemma 48. *Let \mathbb{G} be an algebraic group and X an irreducible subvariety. Then*

$$\prod_{\dim \mathbb{G}} X \cdot X^{-1} = (X \cdot X^{-1}) \cdots (X \cdot X^{-1})$$

is the group generated by X .

Proof. It is a consequence of Lemma 47. \square

Lemma 49. *Let \mathbb{G} be an algebraic group and X_i irreducible subvarieties which contain 1. Let $\tilde{X} = (X_1 \cdot X_1^{-1}) \cdots (X_k \cdot X_k^{-1})$. Then $\prod_{\dim \mathbb{G}} (\tilde{X} \cdot \tilde{X}^{-1})$ is the group generated by X_i .*

Proof. By Lemma 48, it is enough to observe that $X_i \subseteq \tilde{X}$. \square

Lemma 50. *Let R be a computable integral domain whose characteristic is at least $2n$ and $\mathbf{z} = (z_1, \dots, z_k) \in \mathbb{Y}_{n,k}(R)$. Let K be the quotient field of R , \mathbb{H} be the K -subgroup scheme of $(\mathbb{GL}_n)_K$ which is generated by $\exp(tz_i)$ and \mathcal{H} be the closure of \mathbb{H} in $(\mathbb{GL}_n)_R$. Then we can algorithmically find an element $g \in R$ and a finite subset $F = \{f_1, \dots, f_l\} \subseteq R_g[\mathbb{GL}_n]$ such that*

$$\mathcal{H} \times_{\text{Spec}(R)} \text{Spec}(R_g) \simeq R_g[\mathbb{GL}_n]/\langle F \rangle$$

as closed subschemes of $(\mathbb{GL}_n)_{R_g}$.

Proof. It is clear that, for any i , the image of $a_{z_i} : \mathbb{A}_K^1 \rightarrow (\mathbb{GL}_n)_K$

$$a_{z_i}(t) := \exp(tz_i),$$

is a 1-dimensional irreducible K -algebraic subgroup of $(\mathbb{GL}_n)_K$. Hence by Lemma 49 we can find an algebraic morphism $\Phi : \mathbb{A}^{(2k-1)n^2} \rightarrow (\mathbb{GL}_n)_K$ whose image is exactly \mathbb{H} . Hence by means of the elimination method we can compute a presentation for \mathbb{H} , i.e. $F = \{f_1, \dots, f_l\} \in R[\mathbb{GL}_n]$ such that

$$\mathbb{H} \simeq \text{Spec}(K[\mathbb{GL}_n]/\langle F \rangle_K),$$

as K -varieties. Now by the second part of Lemma 42, we can compute a non-zero element $g \in R$ such that

$$\mathcal{H}_g := \mathcal{H} \times_{\text{Spec}(R)} \text{Spec}(R_g) \simeq \text{Spec}(R_g[\mathbb{GL}_n]/\langle F \rangle_g).$$

□

Lemma 51. *Let R be a computable integral domain, K be the quotient field of R , $\mathbf{z} \in \mathbb{Y}_{n,k}(R)$, $L = L_{\mathbf{z}}$ and \mathbb{H} be a closed subgroup of $(\mathbb{GL}_n)_K$. If $(\mathbb{H}, L \otimes K)$ is an acceptable pair, then we can algorithmically find a non-zero element $g \in R$ such that $\text{Lie}(\mathcal{H})(R_g) = L_g$, where \mathcal{H} is the closure of \mathbb{H} in $(\mathbb{GL}_n)_R$ and $L_g = L \otimes R_g$.*

Proof. By [51, Lemma 2.12], we know that \mathbb{H} is generated by $\exp(tz_i)$. Hence by Lemma 49 and Theorem 40, we can compute a non-zero element $g_1 \in R$ and a finite subset $F = \{f_1, \dots, f_l\} \subseteq R[\mathbb{GL}_n]$ such that

1. The projection map $\mathcal{H}_{g_1} := \mathcal{H} \times_{\text{Spec}(R)} \text{Spec}(R_{g_1}) \rightarrow \text{Spec}(R_{g_1})$ is smooth.
2. As R_{g_1} -schemes,

$$\mathcal{H}_{g_1} \simeq \text{Spec}((R[\mathbb{GL}_n]/\langle F \rangle)_{g_1}) \simeq \text{Spec}(R_{g_1}[\underline{X}]/\langle \tilde{F} \rangle),$$

where $\underline{X} = \{X_1, \dots, X_{n^2+1}\}$, $\tilde{F} = F \cup \{X_{n^2+1}D(X_1, \dots, X_{n^2}) - 1\}$ and D is the determinant of the first n^2 variables.

Since \mathcal{H}_{g_1} is a smooth R_{g_1} -scheme, $\text{Lie}(\mathcal{H}_{g_1}/R_{g_1}) = \text{Ker}(\text{Jac}(\tilde{F}))$, where $\text{Jac}(\tilde{F}) = [\partial \tilde{f}_i / \partial X_j]$ is the Jacobian of

$$(X_1, \dots, X_{n^2+1}) \mapsto (\tilde{f}(X_1, \dots, X_{n^2+1}))_{\tilde{f} \in \tilde{F}}.$$

By Gauss-Jordan process, we can compute a non-zero element $g_2 \in R$ such that $\text{Ker}(\text{Jac}(\tilde{F}))_{g_2}$ is a free R_{g_2} -module. We can also compute an R_{g_2} -basis. Since we know that $L \otimes K = \text{Ker}(\text{Jac}(\tilde{F}))_{g_2} \otimes_{R_{g_2}} K$ and we have R_{g_2} -basis for both of them, we can compute a non-zero element g such that $L_g = \text{Ker}(\text{Jac}(\tilde{F}))_g$, which finishes the proof. □

Lemma 52. *Let R , K , \mathbf{z} , L , \mathbb{H} and \mathcal{H} be as in Lemma 51. If $(\mathbb{H}, L \otimes K)$ is an acceptable pair, then we can algorithmically find a non-zero element $g \in R$ such that*

$$(e(\mathbb{L}^{(n)}) \times_{\text{Spec}(R)} \text{Spec}(R_g))_{\text{red}} = (\mathcal{H}^{(u)} \times_{\text{Spec}(R)} \text{Spec}(R_g))_{\text{red}}.$$

Proof. Since L is given through an R -basis, we can compute a non-zero element $g_1 \in R$ and an R_{g_1} -basis for the dual of L . Hence we can compute a presentation for \mathbb{L} . Thus using elimination method we can compute a presentation of $e(\mathbb{L}^{(n)})_{g_1} := e(\mathbb{L}^{(n)}) \times_{\text{Spec}(R)} \text{Spec}(R_{g_1})$. We can also compute a presentation of $\mathcal{H}^{(u)}$.

Since $(\mathbb{H}, L \otimes K)$ is an acceptable pair, we have

$$(e(\mathbb{L}^{(n)}) \times_{\text{Spec}(R)} \text{Spec}(K))_{\text{red}} = (\mathcal{H}^{(u)} \times_{\text{Spec}(R)} \text{Spec}(K))_{\text{red}}.$$

So having a presentation of both sides over R_{g_1} , one can easily compute $g_2 \in R$ such that

$$(e(\mathbb{L}^{(n)}) \times_{\text{Spec}(R)} \text{Spec}(R_g))_{\text{red}} = (\mathcal{H}^{(u)} \times_{\text{Spec}(R)} \text{Spec}(R_g))_{\text{red}}$$

holds for $g = g_1 g_2$. □

Proof of Lemma 38. One can repeat Nori's argument [51, Lemma 2.8] and get the effective version using Theorem 40, Lemma 51 and Lemma 52. □

A.3 Proof of Lemma 39.

In this section, first we give a precise presentation of $\mathbb{Y}_{n,k}$. Then using Lemma 38 by an inductive argument we get the desired result.

Definition 53. Let $F = \{f_1, \dots, f_l\}$ and $F' = \{f'_1, \dots, f'_{l'}\}$ be two subsets of $R[\underline{X}]$, where $\underline{X} = \{X_1, \dots, X_m\}$. Then let $V(F)$ denote the closed subscheme of \mathbb{A}_R^m defined by the relations F , and

$$W(\mathbb{A}_R^m; F, F') := V(F) \setminus V(F').$$

If \mathfrak{a} and \mathfrak{b} are two ideals of $R[\underline{X}]$, then $V(\mathfrak{a})$ denotes the closed subscheme of \mathbb{A}_R^m defined by \mathfrak{a} and

$$W(\mathbb{A}_R^m; \mathfrak{a}, \mathfrak{b}) := V(\mathfrak{a}) \setminus V(\mathfrak{b}).$$

Definition 54. For any $\mathbf{z} = (z_1, \dots, z_k) \in M_n(R)^k$, fix the standard R -basis of $M_n(R)$ and view z_i as column vectors in this basis. Let $F_{\mathbf{z}}$ be the set of all the maximum dimension minors of the matrix $\begin{bmatrix} z_1 & \cdots & z_k \end{bmatrix}$ and $\mathfrak{a}_{\mathbf{z}}$ be the ideal generated by $F_{\mathbf{z}}$.

We also consider the case $R = \mathbb{Z}[\underline{X}]$, where $\underline{X} = \{X_{ij}^{i'} | 1 \leq i, j \leq n, 1 \leq i' \leq k\}$ and set $\mathbf{x} = (x_1, \dots, x_k)$, where the ij -th entry of $x_{i'}$ is $X_{ij}^{i'}$. In this case, $F_{n,k} := F_{\mathbf{x}}$ and $\mathfrak{a}_{n,k} := \mathfrak{a}_{\mathbf{x}}$.

Remark 55. We sometimes identify the functor M_n with $\mathbb{A}_{\mathbb{Z}}^{n^2}$. This way, any $\mathbf{z} \in M_n(R)$ gives rise to a ring homomorphism $\phi_{\mathbf{z}}$ from $\mathbb{Z}[\mathbb{A}^{kn^2}]$ to R and it is clear that $\phi_{\mathbf{z}}(\mathfrak{a}_{n,k}) = \mathfrak{a}_{\mathbf{z}}$.

Lemma 56. *Let (A, \mathfrak{m}) be a pair of a local ring and its maximal ideal. Let $\phi : A^n \rightarrow A^n$ be an A -linear map. Then the following statements are equivalent:*

1. ϕ is surjective.
2. $\bar{\phi} : (A/\mathfrak{m})^n \rightarrow (A/\mathfrak{m})^n$ is invertible.
3. ϕ is invertible.

Proof. This is clear. □

Lemma 57. *Let R be any commutative ring. Then*

$$\mathbf{z} = (z_1, \dots, z_k) \in W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; 0, \mathfrak{a}_{n,k})(R)$$

if and only if $M_n(R)/L$ is locally of dimension $n^2 - k$, where $L = Rz_1 + \dots + Rz_k$.

Proof. By the definition of $W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; 0, \mathfrak{a}_{n,k})(R)$, it is straightforward to check that $\mathbf{z} \in W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; 0, \mathfrak{a}_{n,k})(R)$ if and only if $\mathfrak{a}_{\mathbf{z}} = R$.

On the other hand, $M_n(R)/L$ is locally of dimension $n^2 - k$ if and only if for any $\mathfrak{p} \in \text{Spec}(R)$ there are $z_{k+1}^{(\mathfrak{p})}, \dots, z_{n^2}^{(\mathfrak{p})} \in M_n(R)$ such that

$$M_n(R_{\mathfrak{p}}) = \left(\sum_{i=1}^k R_{\mathfrak{p}} z_i \right) + \bigoplus_{i=k+1}^{n^2} R_{\mathfrak{p}} z_i^{(\mathfrak{p})}. \quad (27)$$

Let $\phi_{\mathfrak{p}} : R_{\mathfrak{p}}^{n^2} \rightarrow M_n(R_{\mathfrak{p}})$ be the following $R_{\mathfrak{p}}$ -linear map

$$\phi_{\mathfrak{p}}(r_1, \dots, r_{n^2}) := \sum_i r_i z_i^{(\mathfrak{p})},$$

where $R_{\mathfrak{p}}^{n^2}$ is the direct sum of n^2 copies of $R_{\mathfrak{p}}$ and $z_i^{(\mathfrak{p})} = z_i$ for any $i \leq k$. By Lemma 56, it is clear that (27) holds if and only if $\bar{\phi}_{\mathfrak{p}}$ is invertible. It is easy to see that the latter is equivalent to $\mathfrak{a}_{\bar{\mathbf{z}}} = k(\mathfrak{p})$, where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_k) \in M_n(k(\mathfrak{p}))^k$. Let $S_{\mathfrak{p}} = R \setminus \mathfrak{p}$. By the definition, it is clear that $\mathfrak{a}_{\bar{\mathbf{z}}} = k(\mathfrak{p})$ if and only if $S_{\mathfrak{p}}^{-1} \mathfrak{a}_{\mathbf{z}} = R_{\mathfrak{p}}$. The latter holds for any $\mathfrak{p} \in \text{Spec}(R)$ if and only if $\mathfrak{a}_{\mathbf{z}} = R$, which completes the proof. □

Definition 58. Let $\mathbf{z} = (z_1, \dots, z_k) \in (R^{n^2})^k$. We sometimes view such a vector in two other ways: as an $n^2 \times k$ matrix whose i -th column is z_i ; or a k -tuple of $n \times n$ matrices whose i -th entry is z_i written in matrix form with respect to the standard basis.

Let $J \subseteq \{1, \dots, n^2\}$ be of order k . Then \mathbf{z}_J denotes the $k \times k$ submatrix of \mathbf{z} whose rows are determined by J . For a vector $v \in R^{n^2}$, v_J denotes the subvector of size k determined by J .

For a given $a \in \text{Mor}(\mathbb{A}_{\mathbb{Z}}^{kn^2}, \mathbb{A}_{\mathbb{Z}}^{n^2})$ and any subsets $J, J' \subseteq \{1, \dots, n^2\}$ of order k , we define $f_{J,J'}^{(a)} \in \text{Mor}(\mathbb{A}_{\mathbb{Z}}^{kn^2}, \mathbb{A}_{\mathbb{Z}}^k)$ as follows:

$$f_{J,J'}^{(a)}(\mathbf{z}) = \mathbf{z}_{J'} \text{adj}(\mathbf{z}_J) a(\mathbf{z})_J - \det(\mathbf{z}_J) a(\mathbf{z})_{J'}.$$

Also let $F_{n,k}^{(a)}$ be the set consisting of all the entries of $f_{J,J'}^{(a)}$ for all the possible J and J' .

Lemma 59. *Let R be any commutative ring and $a \in \text{Mor}(\mathbb{A}_{\mathbb{Z}}^{kn^2}, \mathbb{A}_{\mathbb{Z}}^{n^2})$. Then*

$$\mathbf{z} = (z_1, \dots, z_k) \in W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; F_{n,k}^{(a)}, F_{n,k})(R)$$

if and only if

1. $M_n(R)/L_{\mathbf{z}}$ is locally of dimension $n^2 - k$, where $L_{\mathbf{z}} = Rz_1 + \dots + Rz_k$,
2. $a(\mathbf{z}) \in L_{\mathbf{z}}$.

Proof. Let $\mathcal{Y}_{n,k} = W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; \emptyset, F_{n,k})$ and let $\mathcal{Y}_{n,k}^{(a)}$ be the functor from commutative rings to sets such that

$$\mathcal{Y}_{n,k}^{(a)}(R) = \{\mathbf{z} \in \mathcal{Y}_{n,k}(R) \mid a(\mathbf{z}) \in L_{\mathbf{z}}\}.$$

By Lemma 57, it is enough to show that

$$\mathcal{Y}_{n,k}^{(a)}(R) = W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; F_{n,k}^{(a)}, F_{n,k})(R).$$

Let us view \mathbf{z} as an $n^2 \times k$ matrix. Then if $\mathbf{z} \in \mathcal{Y}_{n,k}$, then it belongs to $\mathcal{Y}_{n,k}^{(a)}(R)$ if and only if there is $\vec{r} = (r_1, \dots, r_k)$ such that $\mathbf{z}\vec{r} = a(\mathbf{z})$. The latter holds if and only if for any $J \subseteq \{1, \dots, n^2\}$ of order k we have $\mathbf{z}_J \vec{r} = a(\mathbf{z})_J$.

We claim that if $\mathbf{z} \in \mathcal{Y}_{n,k}^{(a)}(R)$ then there is a unique \vec{r} which satisfies the equations $\mathbf{z}_J \vec{r} = a(\mathbf{z})_J$ for all the subsets J of order k in $\{1, \dots, n^2\}$. To show this claim it is enough to notice that $\det(\mathbf{z}_J) \vec{r} = \text{adj}(\mathbf{z}_J) a(\mathbf{z})_J$ and the ideal generated by $\det(\mathbf{z}_J)$ as J runs through all the subsets of order k contains 1 as $\mathbf{z} \in \mathcal{Y}_{n,k}(R)$.

We also observe that if $\mathbf{z} \in \mathcal{Y}_{n,k}^{(a)}(R)$, then for any J and J' we have

$$\mathbf{z}_{J'} \text{adj}(\mathbf{z}_J) a(\mathbf{z})_J = \det(\mathbf{z}_J) \mathbf{z}_{J'} \vec{r} = \det(\mathbf{z}_J) a(\mathbf{z})_{J'}.$$

Hence $\mathbf{z} \in W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; F_{n,k}^{(a)}, F_{n,k})(R)$, i.e. $\mathcal{Y}_{n,k}^{(a)}(R) \subseteq W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; F_{n,k}^{(a)}, F_{n,k})(R)$.

Let $\mathbf{z} \in W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; F_{n,k}^{(a)}, F_{n,k})(R)$. We claim that if $\det(\mathbf{z}_J)$ is a unit in R for some J , then $\mathbf{z} \in \mathcal{Y}_{n,k}^{(a)}(R)$. To see this it is enough to check that

$$\vec{r} = \det(\mathbf{z}_J)^{-1} \text{adj}(\mathbf{z}_J) a(\mathbf{z})_J$$

satisfies all the equations $\mathbf{z}_{J'} \vec{r} = a(\mathbf{z})_{J'}$. In particular, for any local ring R , we have

$$\mathcal{Y}_{n,k}^{(a)}(R) = W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; F_{n,k}^{(a)}, F_{n,k})(R).$$

For an arbitrary commutative ring R , let again $\mathbf{z} \in W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; F_{n,k}^{(a)}, F_{n,k})(R)$. By the above discussion, for any $\mathfrak{p} \in \text{Spec}(R)$, we have that $\mathbf{z} \in \mathcal{Y}_{n,k}^{(a)}(R_{\mathfrak{p}})$, i.e. there is a unique $\vec{r}_{\mathfrak{p}} \in R_{\mathfrak{p}}^k$ such that $\mathbf{z}_J \vec{r}_{\mathfrak{p}} = a(\mathbf{z})_J$ for any J . On the other hand, by the uniqueness argument, since the ideal generated by $\det(\mathbf{z}_J)$ is equal to R , there is $\vec{r} \in R^k$ such that for any \mathfrak{p} and any J we have $\mathbf{z}_J \vec{r} = a(\mathbf{z})_J$ in $R_{\mathfrak{p}}^k$. Now one can easily deduce that $\mathbf{z}_J \vec{r} = a(\mathbf{z})_J$ in R^k , which means $\mathbf{z} \in \mathcal{Y}_{n,k}^{(a)}(R)$ and we are done. \square

Definition 60. Let $a_{ij} \in \text{Mor}(\mathbb{A}_{\mathbb{Z}}^{kn^2}, \mathbb{A}_{\mathbb{Z}}^{n^2})$ be the following morphism

$$a_{ij}(\mathbf{z}) := [z_i, z_j] = z_i z_j - z_j z_i,$$

for any $1 \leq i, j \leq k$. Let $\tilde{F}_{n,k} := \bigcup_{i,j=1}^k F_{n,k}^{(a_{ij})}$.

Corollary 61. For any commutative ring R , we have

$$\mathbb{Y}_{n,k}(R) = W(\mathbb{A}_{\mathbb{Z}}^{kn^2}; \tilde{F}_{n,k}, F_{n,k})(R).$$

Proof. This is a direct consequence of Lemma 59 □

Lemma 62. Let F and $F' = \{f'_1, \dots, f'_l\}$ be two subsets of $\mathbb{Z}[\underline{X}]$, where $\underline{X} = \{X_1, \dots, X_m\}$. Assume that $\langle F \rangle$ is a radical ideal. Then we can computationally determine if $W(\mathbb{A}_{\mathbb{Z}}^m; F, F')$ is nonempty, and if it is, then we can give a presentation of an integral domain R and $z \in W(\mathbb{A}_{\mathbb{Z}}^m; F, F')(R)$ such that

1. $z : \text{Spec}(R) \rightarrow W(\mathbb{A}_{\mathbb{Z}}^m; F, F')$ is an open immersion.
2. For any given $d \in R$, we can computationally describe the complement of $z(\text{Spec}(R[\frac{1}{d}]))$ in $W(\mathbb{A}_{\mathbb{Z}}^m; F, F')$.

Proof. It is clear that $W(\mathbb{A}_{\mathbb{Z}}^m; F, F')$ is empty if and only if $\langle F' \rangle \subseteq \sqrt{\langle F \rangle}$, which can be computationally determined. To show the rest, first we claim that we can assume that $F' = \emptyset$. To show this claim, we start with the following open affine covering:

$$W(\mathbb{A}_{\mathbb{Z}}^m; F, F') = \cup_i W(\mathbb{A}_{\mathbb{Z}}^m; F, \{f'_i\}).$$

And, we notice that $W(\mathbb{A}_{\mathbb{Z}}^m; F, \{f'_i\}) \simeq W(\mathbb{A}_{\mathbb{Z}}^{m+1}; F \cup \{f'_i X_{m+1} - 1\}, \emptyset)$. Now if we find R and z for $F \cup \{f'_i X_{m+1} - 1\}$ and $F' = \emptyset$, then one can see that the first assertion still holds and the complement of $z(\text{Spec}(R[\frac{1}{d}]))$ in $W(\mathbb{A}_{\mathbb{Z}}^m; F, F')$ is equal to the union of its complement in $W(\mathbb{A}_{\mathbb{Z}}^m; F, \{f'_i\})$ and $W(\mathbb{A}_{\mathbb{Z}}^m; F \cup \{f'_i\}, F')$.

So without loss of generality, we can and will assume that $F' = \emptyset$. By [5, Chapter 8.5], we can compute a primary decomposition $\cap_i \mathfrak{p}_i$ of $\langle F \rangle$. Since $\langle F \rangle = \sqrt{\langle F \rangle}$, \mathfrak{p}_i is a prime ideal for any i . If $\langle F \rangle$ is a prime ideal, let $c = 1$ and $R = \mathbb{Z}[\underline{X}]/\langle F \rangle$; otherwise, let $c \in \cap_{i \leq 2} \mathfrak{p}_i \setminus \mathfrak{p}_1$ (we can computationally find such c) and $R = (\mathbb{Z}[\underline{X}]/\mathfrak{p}_1)[\frac{1}{c}]$. Clearly this choice of R satisfies the first assertion in the statement of Lemma. Now let $d \in R$ be a given element. Then one can easily check that the complement of the natural open immersion of $\text{Spec}(R[\frac{1}{d}])$ in $W(\mathbb{A}_{\mathbb{Z}}^m; F, \emptyset)$ is isomorphic to

$$W(\mathbb{A}_{\mathbb{Z}}^m; F \cup \{c\}, \emptyset) \cup W(\mathbb{A}_{\mathbb{Z}}^m; F \cup \{d\}, \emptyset).$$

□

Proof of Lemma 39. Following Nori's proof of [51, Proposition 2.7] and using Lemma 38, Corollary 61, and Lemma 62 whenever needed, one can easily prove this lemma. □

A.4 Proof of Theorem 41.

Lemma 63. *Let $S \subseteq \mathrm{GL}_n(\mathbb{Q})$ be a finite set of matrices. Let Γ be the group generated by S . Assume that the Zariski-closure of Γ in $(\mathrm{GL}_n)_{\mathbb{Q}}$ is Zariski-connected. Then we can compute a square-free integer q_0 and a finite subset $F = \{f_1, \dots, f_l\} \subseteq \mathbb{Z}[1/q_0][\mathrm{GL}_n]$ such that $\Gamma \subseteq \mathrm{GL}_n(\mathbb{Z}[1/q_0])$ and its Zariski-closure in $(\mathrm{GL}_n)_{\mathbb{Z}[1/q_0]}$ is isomorphic to $\mathbb{Z}[1/q_0][\mathrm{GL}_n]/\langle F \rangle$.*

Proof. Since S is a finite set of matrices, we can find an odd prime p such that $\Gamma \subseteq \mathrm{GL}_n(\mathbb{Z}_p)$. Hence by changing Γ to $\Gamma \cap \mathrm{GL}_n^{(1)}(\mathbb{Z}_p)$ (the first congruence subgroup is denoted by $\mathrm{GL}_n^{(1)}(\mathbb{Z}_p)$), we can and will assume that Γ is torsion free. It is worth mentioning that we are allowed to make such a change because of the following:

1. We can compute representatives for the cosets of $\Gamma \cap \mathrm{GL}_n^{(1)}(\mathbb{Z}_p)$ in Γ . Thus we can compute a generating set for $\Gamma \cap \mathrm{GL}_n^{(1)}(\mathbb{Z}_p)$.
2. Since we have assumed that the Zariski-closure \mathbb{H} of Γ in $(\mathrm{GL}_n)_{\mathbb{Q}}$ is Zariski-connected, the Zariski-closure of $\Gamma \cap \mathrm{GL}_n^{(1)}(\mathbb{Z}_p)$ in $(\mathrm{GL}_n)_{\mathbb{Q}}$ is also \mathbb{H} .

We find a presentation for $\mathbb{Q}[\mathbb{H}]$ and then similar to the proof of Lemma 50 we can finish the argument.

Since Γ is torsion-free, the Zariski-closure of the cyclic group generated by any element of Γ is of dimension at least one. Hence by the virtue of Lemma 49 it is enough to find a presentation of the Zariski-closure \mathbb{G} of the cyclic group generated by $\gamma \in S$. We can compute the Jordan-Chevalley decomposition $\gamma_u \cdot \gamma_s$ of γ . Let \mathbb{G}_u (resp. \mathbb{G}_s) be the Zariski-closure of the group generated by γ_u (resp. γ_s). Then $\mathbb{G} \simeq \mathbb{G}_u \times \mathbb{G}_s$ as \mathbb{Q} -groups [6, Theorem 4.7]. Using the logarithmic and exponential maps, one can easily find a presentation of \mathbb{G}_u . So it is enough to find a presentation of \mathbb{G}_s . We can compute all the eigenvalues $\lambda_1, \dots, \lambda_n$ of γ_s . By [6, Proposition 8.2], in order to find a presentation of $k[\mathbb{G}]$, where k is the number field generated by λ_i , it is enough to find a basis for the following subgroup of \mathbb{Z}^n

$$\{(m_1, \dots, m_n) \in \mathbb{Z}^n \mid \prod_i \lambda_i^{m_i} = 1\},$$

i.e. all the character equations, which is essentially done in [53]. So far we found a finite subset F' of $k[\mathrm{GL}_n]$ such that $k[\mathbb{G}_s] \simeq k[\mathrm{GL}_n]/\langle F' \rangle$. Since \mathbb{G}_s is defined over \mathbb{Q} , we have that $\mathbb{Q}[\mathbb{G}_s] \simeq \mathbb{Q}[\mathrm{GL}_n]/\langle F' \rangle \cap \mathbb{Q}[\mathrm{GL}_n]$. On the other hand, using Gröbner basis we can find a generating set F_s for $\langle F' \rangle \cap \mathbb{Q}[\mathrm{GL}_n]$, which finishes our proof. \square

Lemma 64. *Let \mathcal{L} be a smooth $\mathbb{Z}[1/q_0]$ -subgroup scheme of $(\mathrm{GL}_n)_{\mathbb{Z}[1/q_0]}$. Let $F \subseteq \mathbb{Z}[1/q_0][\mathrm{GL}_n]$ such that $\mathbb{Z}[1/q_0][\mathrm{GL}_n]/\langle F \rangle$. If the generic fiber \mathbb{L} of \mathcal{L} is a simply-connected semisimple \mathbb{Q} -group, then*

1. *we can algorithmically find a positive integer p_0 such that for any prime $p > p_0$, the special fiber $\mathcal{L}_p := \mathcal{L} \times_{\mathrm{Spec}(\mathbb{Z}[1/q_0])} \mathrm{Spec}(\mathbb{F}_p)$ of \mathcal{L} over p is a semisimple \mathbb{F}_p -group.*

2. we can algorithmically find a positive integer p_0 such that, for any $p > p_0$, $\mathcal{L}(\mathbb{Z}_p)$ is a hyper-special parahoric in $\mathbb{L}(\mathbb{Q}_p)$.

Proof. The second part is a consequence of the first part as it is explained in [60, Section 3.9.1]. Here we only prove the first part.

We can compute the Lie algebra \mathfrak{l} of \mathbb{L} . Since \mathbb{L} is not a nilpotent Lie algebra, not all the elements of a basis of \mathfrak{l} can be ad-nilpotent. Hence we can find an ad-semisimple element x of \mathfrak{l} . Since \mathfrak{l} is a semisimple Lie algebra and x is a semisimple element, the centralizer $c_{\mathfrak{l}}(x)$ of x in \mathfrak{l} is a reductive algebra and $c_{\mathfrak{l}}(x)/\mathfrak{z}(c_{\mathfrak{l}}(x))$ is a semisimple Lie algebra (if not trivial), where $\mathfrak{z}(c_{\mathfrak{l}}(x))$ is the center of $c_{\mathfrak{l}}(x)$. If $c_{\mathfrak{l}}(x)$ is not commutative, then repeating the above argument we can find $x' \in c_{\mathfrak{l}}(x) \setminus \mathfrak{z}(c_{\mathfrak{l}}(x))$. We can compute the eigenvalues λ_i (resp. λ'_i) of $\text{ad}(x)$ (resp. $\text{ad}(x')$) and find

$$\lambda \neq \frac{\lambda_i - \lambda_j}{\lambda'_{i'} - \lambda'_{j'}},$$

for any i, j, i', j' , then $c_{\mathfrak{l}}(x + \lambda x') = c_{\mathfrak{l}}(x) \cap c_{\mathfrak{l}}(x')$. By repeating this process, we can compute a number field k over which \mathfrak{l} splits and we can also compute a Cartan subalgebra. Hence we can compute a Chevalley basis x_i for $\mathfrak{l} \otimes_{\mathbb{Q}} k$. Looking at the commutator relations, we can compute an element a of k such that $\sum_i \mathcal{O}_k[1/a]x_i$ form a Lie subring of $\mathfrak{l} \otimes_{\mathbb{Q}} k$, where \mathcal{O}_k is the ring of integers in k . Thus for any p which does not divide $N_{k/\mathbb{Q}}(a)$ the special fiber \mathcal{L}_p is a semisimple \mathbb{F}_p -group, as we wished. \square

Lemma 65. *Let \mathbb{H} be a perfect Zariski-connected \mathbb{Q} -subgroup of GL_n . Let F be a finite subset of $\mathbb{Q}[\text{GL}_n]$ such that $\mathbb{Q}[\mathbb{H}] \simeq \mathbb{Q}[\text{GL}_n]/\langle F \rangle$. Then one can compute a square-free integer q_1 and a finite subset F' of $\mathbb{Z}[1/q_1][\text{GL}_n]$ such that*

1. The Zariski-closure \mathcal{H} of \mathbb{H} in $(\text{GL}_n)_{\mathbb{Z}[1/q_1]}$ is defined by F' .
2. The projection map $\mathcal{H} \rightarrow \text{Spec}(\mathbb{Z}[1/q_1])$ is smooth.
3. We can compute a generating set for $\mathcal{H}(\mathbb{Z}[1/q_1])$.
4. $\pi_p(\mathcal{H}(\mathbb{Z}[1/q_1]))$ is a perfect group if $p \nmid q_1$.

Proof. By [32, Algorithm 3.5.3], we can compute the unipotent radical and a Levi subgroup of \mathbb{H} . Therefore we can effectively write \mathbb{H} as the semidirect product of a semisimple \mathbb{Q} -group \mathbb{L} and a unipotent \mathbb{Q} -group \mathbb{U} . We can compute a square-free integer q_2 and $\mathbb{Z}[1/q_2]$ -group schemes \mathcal{L} and \mathcal{U} such that:

1. The projection maps to $\text{Spec}(\mathbb{Z}[1/q_2])$ are smooth.
2. All the fibers are geometrically irreducible.
3. \mathcal{L} acts on \mathcal{U} .
4. The generic fiber of \mathcal{L} (resp. $\mathcal{U}, \mathcal{H} := \mathcal{L} \ltimes \mathcal{U}$) is isomorphic to \mathbb{L} (resp. \mathbb{U}, \mathbb{H}).

It is worth mentioning that the first and the second items are consequences of Theorem 40 and the rest are easy. Using logarithmic and exponential maps, we can effectively enlarge q_2 , if necessary, and assume that for any $p \nmid q_2$ we have $[\mathcal{U}_p(\mathbb{F}_p), \mathcal{U}_p(\mathbb{F}_p)] = [\mathcal{U}, \mathcal{U}]_p(\mathbb{F}_p)$, where $\mathcal{U}_p = \mathbb{U} \times_{\text{Spec}(\mathbb{Z}[1/q_2])} \text{Spec}(\mathbb{F}_p)$ ($[\mathcal{U}, \mathcal{U}]_p$ is defined in a similar way). We can also get a generating set for $\mathcal{U}(\mathbb{Z}[1/q_2])$. By Lemma 64, we can enlarge q_2 and assume that $\mathcal{L}(\mathbb{Z}_p)$ is a hyper-special parahoric subgroup of $\mathbb{L}(\mathbb{Q}_p)$ for any $p \nmid q_2$. In particular, by further enlarging q_2 , we have that $\mathcal{L}(\mathbb{Z}[1/q_2])$ is an arithmetic lattice in (the non-compact semisimple group) $\mathbb{L}(\mathbb{R}) \cdot \prod_{p|q_2} \mathbb{L}(\mathbb{Q}_p)$. Thus we have

1. By the classical strong approximation theorem, we have that

$$\pi_p(\mathcal{L}(\mathbb{Z}[1/q_2])) = \mathcal{L}_p(\mathbb{F}_p)$$

is a product of quasi-simple groups.

2. By [33], we can compute a generating set Ω for $\mathcal{L}(\mathbb{Z}[1/q_2])$. Thus we get a generating set for $\mathcal{H}(\mathbb{Z}[1/q_2])$.

On the other hand, since \mathbb{H} is perfect, the action of \mathbb{L} on $\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ has no non-trivial fixed vector, where $\mathfrak{u} = \text{Lie}(\mathbb{U})$. This is equivalent to say that the elements of Ω do not have a common non-zero fixed vector. Fix a basis \mathfrak{B} of $\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ and let $X_\gamma := [\gamma]_{\mathfrak{B}} - I$, where the $[\gamma]_{\mathfrak{B}}$ is the matrix associated with the action of γ and I is the identity matrix. Let X be a column blocked-matrix whose blocked-entries are X_γ for $\gamma \in \Omega$. By our assumption, X is of full rank, i.e. the product of its minors of maximum dimension is a non-zero element of $\mathbb{Z}[1/q_2]$. Hence by enlarging q_2 , if necessary, we can assume that the elements of $\pi_p(\Omega)$ do not fix any non-trivial element of $\mathcal{U}_p(\mathbb{F}_p)/[\mathcal{U}_p(\mathbb{F}_p), \mathcal{U}_p(\mathbb{F}_p)]$. Hence by the above discussion, for any prime $p \nmid q_2$, we have

$$\pi_p(\mathcal{H}(\mathbb{Z}[1/q_2])) = \mathcal{L}_p(\mathbb{F}_p) \times \mathcal{U}_p(\mathbb{F}_p)$$

is a perfect group, which finishes our proof. \square

Proof of Theorem 41. Let q_1 be a square-free integer given by Lemma 65. Let \mathcal{G} be the Zariski-closure of \mathbb{G} in $(\mathbb{G}\mathbb{L}_n)_{\mathbb{Z}[1/q_1]}$. Lemma 65 provides us with an effective version of Theorem A for the group $\mathcal{G}(\mathbb{Z}[1/q_1])$. On the other hand, we have already proved the effective versions of [51, Theorem B and C]. Hence following the proof of Proposition 16, one can effectively compute a positive number δ such that: for any proper subgroup $H = H^+$ of $\mathcal{G}_p(\mathbb{F}_p)^+$, one has that

$$\{\gamma \in \mathcal{G}(\mathbb{Z}[1/q_1]) \mid \|\gamma\|_S \leq [\pi_p(\mathcal{G}(\mathbb{Z}[1/q_1])) : H]^\delta\}$$

is in a proper algebraic subgroup. In particular, if Ω generates a Zariski-dense subgroup of \mathbb{G} , then $\pi_p(\Gamma)^+ = \mathcal{G}_p(\mathbb{F}_p)^+$ for any $p > \max_{\gamma \in \Omega} \{\|\gamma\|_S^{1/\delta}\}$, where $S = \{p \mid p \text{ is a prime divisor of } q_1\}$. \square

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