AN INVITATION TO RIGIDITY THEORY

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Dedicated to Anatole Katok on the occasion of his 60th birthday.

INTRODUCTION

This survey is dedicated to Professor Anatole Katok on the occasion of his sixtieth birthday. He has made numerous important contributions to dynamics and ergodic theory proper. During the last two decades, he was one of the key researchers in the field of rigidity or geometric rigidity. My goal here is to give a bird's eye view of this subject with particular attention to the many ideas and topics Professor Katok has been directly involved with.

Rigidity theory is now in its fifth decade. It started with conjectures of A. Selberg and the early local rigidity theorems of A. Weil as well as E. Calabi and E. Vesentini in the early sixties. Mostow's celebrated Strong Rigidity Theorem in 1968 was a spectacular breakthrough and inspired whole new areas of research for rigidity phenomena in geometry, group theory and dynamics. After a brief review of this early history, we will discuss these new areas and topics.

Rigidity theory by now is a large field with many branches and connections, and it is impossible to even just mention all important developments in one short survey. We refer to [91, 145, 146, 160, 190] for more intensive introductions to this field, and will refer to other more specialized surveys as we go along.

EARLY RIGIDITY THEORY

In 1960, A. Selberg made the beautiful discovery that up to conjugation the fundamental groups of certain compact locally symmetric spaces are always defined over the algebraic numbers. Thus it is implausible that such groups can be deformed. More precisely, we say that a subgroup Γ of another group G is *deformation rigid* in G if for any continuous path ρ_t of embeddings of Γ into G starting with $\rho_0 = id$, ρ_t is conjugate to ρ_0. E. Calabi and E. Vesentini, and then A. Weil in full generality found the following local rigidity theorem in the early sixties [29, 183, 184]. Here we call a discrete subgroup Γ of a group G *cocompact* if G/Γ is compact.

LOCAL RIGIDITY THEOREM *Cocompact discrete subgroups Γ in semisimple Lie groups without compact nor $SL(2,\mathbb{R})$ nor $SL(2,\mathbb{C})$ local factors is deformation rigid.*

These results left open the possibility of embedding the same abstract group as a cocompact lattice in more than one way. To get a global theorem, G. D. Mostow in 1968 invented a completely new scheme of proof, using ideas and tools from topology, differential and conformal geometry,

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group theory, ergodic theory and harmonic analysis. Mostow's theorem can be stated entirely both in geometric and group theoretic terms, which are easily seen to be equivalent [150].

STRONG RIGIDITY - GEOMETRIC FORM *If two closed manifolds of constant negative curvature and dimension at least 3 have isomorphic fundamental group, then they are isometric.*

STRONG RIGIDITY - ALGEBRAIC FORM *Let* Γ *be a cocompact discrete subgroup of* $PSO(1,n)$, $n \geq 3$. *Suppose* $\phi : \Gamma \to PSO(1,n)$ *is a homomorphism whose image is again a cocompact discrete subgroup of* $PSO(1,n)$. *Then* ϕ *extends to an automorphism from* $PSO(1,n)$.

A crucial tool in Mostow's analysis are quasi-isometries. Given two metric spaces X and Y, we call a map $\phi : X \mapsto Y$ a *quasi-isometry* if there are positive constants A and B such that for all $p,q \in X$

$$A\,d(x,y) - B < d(\phi x, \phi y) < A\,d(x,y) + B.$$

The proof of Mostow's theorem is roughly accomplished in the following steps.

Step 1: The fundamental group Γ acts on the universal covers of M and N by deck translations. There is a Γ-equivariant quasi-isometry ϕ between the universal covers $\tilde{M}$ and $\tilde{N}$. Essentially this quasi-isometry just moves the γ-translate of a fundamental domain of the Γ-action on $\tilde{M}$ to the γ-translate of a fundamental domain of Γ on $\tilde{N}$ where $\gamma \in \Gamma$.

Step 2: ϕ extends to a homeomorphism ψ of the boundaries S^{n-1} of $\tilde{M}$ and $\tilde{N}$. This uses a fundamental property of spaces of negative curvature going back to M. Morse: quasi-geodesics i.e. quasi-isometric maps of $\mathbb{R}$ into such a space are a bounded distance from a geodesic.

Step 3: ψ is Γ-equivariant and quasi-conformal as follows from negative curvature and ϕ being a quasi-isometry.

Step 4: ψ is conformal. This follows from ergodicity of the Γ-action on the boundary and the invariance under Γ of the distortion of ψ.

Step 5: ψ extends to a Γ-equivariant isometry of $\tilde{M}$ and $\tilde{N}$.

In essence, Mostow showed that Γ-equivariant quasi-isometries are a finite distance from an isometry.

Surfaces: Both strong and local rigidity fail in dimension 2 as compact surfaces of genus g at least 2 have a $6g-6$-dimensional moduli space of metrics of constant curvature -1. This is most easily seen by the pair of pants construction [180].

Symmetric Spaces: Locally symmetric spaces are a special class of Riemannian manifolds characterized by the vanishing of the covariant derivative of the curvature tensor. Equivalently, the geodesic symmetries about every point are isometries (locally). They can all be described as double cosets of suitable Lie groups, e.g. $SO(2) \setminus SL(2,\mathbb{R})/\Gamma$ or $SO(n) \setminus SL(n,\mathbb{R})/\Gamma$, Γ a discrete subgroup. Mostow generalized his Strong Rigidity Theorem in 1974 to arbitrary compact locally symmetric spaces of of nonpositive curvature that do not split off one or two-dimensional factors metrically [151].

Finite volume: Call a discrete subgroup Γ of a Lie group G a *lattice* if G/Γ has finite Haar measure. G. Prasad generalized Mostow's arguments for the Strong Rigidity Theorem in 1973 by analyzing the manifold with the cusps cut off [164].

STRONG RIGIDITY IN FINITE VOLUME Strong rigidity holds for finite volume locally symmetric spaces of negative curvature or equivalently for lattices in the corresponding isometry groups.

SUPERRIGIDITY

Mostow's Strong Rigidity Theorem shows that a homomorphism from a lattice Γ into the ambient Lie group G extends to a homomorphism of G provided that the image of Γ is also a lattice. In a spectacular breakthrough in 1973, G. A. Margulis classified all finite dimensional representations of irreducible lattices in higher rank groups [141, 142, 143]. These are semisimple groups G which contain a 2-dimensional abelian subgroup diagonalizable over $\mathbb{R}$. A typical example is $SL(n,\mathbb{R}), n \geq 3$. A lattice is caled *irreducible* if no finite index subgroup is a product.

SUPERRIGIDITY *Let Γ be an irreducible lattice in a connected semisimple Lie group G of $\mathbb{R}$-rank at least 2, trivial center, and without compact factors. Suppose k is a local field. Then any homomorphism π of Γ into a noncompact k-simple group over k with Zariski dense image either has precompact image or π extends to a homomorphism of the ambient groups.*

Margulis' motivation and first major application was the arithmeticity of lattices in higher rank semisimple groups. Here we call a group Γ in a linear group G *arithmetic* if (up to subgroups of finite index) it is the integer points of an algebraic group defined over $\mathbb{Q}$. A typical example is $SL(n,\mathbb{Z})$ in $SL(n,\mathbb{R})$.

ARITHMETICITY *Irreducible lattices in connected higher rank semisimple Lie groups are arithmetic.*

In subsequent work, Margulis showed that homomorphisms of higher rank lattices into compact groups essentially come from the restriction of scalars construction for arithmetic groups [143]. This completes our understanding of the finite dimensional representations of these groups.

The proof of the superrigidity theorem was inspired in part by Mostow's work. In the first step, Margulis finds a measurable Γ-equivariant map between suitable Furstenberg boundaries of the groups. This first step is similar to Mostow's except that the resulting map is only measurable. Also, the maps do not go between spheres at infinity of the associated symmetric spaces but rather between suitable Furstenberg boundaries G/P, P a parabolic subgroup of G. These boundaries however correspond to subsets of the geometric sphere at infinity. Margulis used the multiplicative ergodic theorem to construct this map. Later, R. Zimmer found an alternative construction using the fact that Γ acts amenably on a minimal boundary of G. This first step does not require higher rank. In the second step, Margulis shows regularity of the map between the boundaries using strongly that the real rank is at least 2.

Harmonic maps method: There is now an alternative proof of superrigidity that also works for $Sp(n,1)$ and F^4_{-20}, the isometry groups of quaternionic hyperbolic space and the Cayley plane respectively. Eells and Sampson showed in the 1960's that each homotopy class between compact nonpositively curved spaces contains a unique harmonic map [52]. Sometimes properties of the curvature tensor of the domain force the harmonic map to be an isometry. Examples of such results are due to Y. T. Siu in 1980 in Kähler geometry [176] and K. Corlette for quaternionic hyperbolic spaces in 1990 [36]:

THEOREM: Strong rigidity holds for compact Kähler manifolds of "strongly negative curvature".

THEOREM: (Archimedean) superrigidity holds for cocompact lattices in $Sp(n,1)$.

In 1992, Gromov and Schoen extended Corlette's argument to groups over local fields by generalizing the notion of harmonic map to non-manifold targets.

THEOREM: Cocompact lattices in $Sp(n,1)$ are arithmetic.

Finally, Mok-Siu-Yeung, Jost-Yau and Jost-Zuo extended the harmonic maps approach to superrigidity to higher rank groups and quasiprojective varieties in the early 1990's [147, 116, 115].

Non-arithmeticity: There are few constructions of non-arithmetic lattices for the remaining rank one semisimple Lie groups except for surfaces where non-arithmetic lattices are abundant as there are only countably many arithmetic groups and a $6g-6$-dimensional moduli space. Makarov and Vinberg managed special constructions with reflection groups for other low-dimensional real hyperbolic spaces. On the other hand, Vinberg proved non-existence of cocompact reflection groups in dimensions larger than 30 [182]. In arbitrary dimension, one can sometimes combine different arithmetic pieces with isometric totally geodesic boundaries. Gromov and I. Piatetski-Shapiro used this idea in 1988 [92]:

THEOREM: There exist non-arithmetic lattices for real hyperbolic spaces of all dimensions.

In complex hyperbolic space, the only known non-arithmetic examples were constructed first by Mostow in 1980 and later by P. Deligne and Mostow in 1986 in complex dimensions 2 and 3 [153, 46, 47]. They arise either as complex reflection groups or as

THEOREM: There exist non-arithmetic lattices in $PU(2,1)$ and $PU(3,1)$.

Naturally the question arises for criteria which ensure arithmeticity of lattices in $PU(2,1)$. Some progress on this problem has been made by B. Klingler [133]. In particular he proves that every fake $\mathbf{P}^2\mathbb{C}$ is arithmetic. Superrigidity and existence of non-arithmetic lattices in high dimensional complex hyperbolic spaces remain outstanding problems. Y. Shalom used representation theoretic tools to prove some restrictions on homomorphisms of lattices in the non-rigid rank 1 groups [174] and irreducible subgroups of products of groups [175]. The latter extends results on trees by Burger and Mozes [26, 27].

The results above indicate increasing rigidity as one goes from amenable groups to free groups and $Sl(2,\mathbb{R})$, then $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$ and finally higher rank semisimple groups.

GEOMETRY AND TOPOLOGY

The rigidity results we have encountered so far concerned the very special class of locally symmetric spaces. Clearly, one cannot hope for a Mostow type rigidity theorem in variable negative curvature as one can simply perturb a metric of constant negative curvature. However, several interesting questions arise.

- Are there structures weaker than the metric structure which are rigid?
- Characterize locally symmetric spaces within a wider class of spaces.
- Find additional invariants that determine the metric structure.

Differential topology

Most fundamentally, one can ask if the the topological structure of a manifold of nonpositive curvature is determined by the fundamental group. This was resolved in a remarkable series of papers in the 80's and 90's by T. Farrell and L. Jones [60]. It is a special case of Borel 's conjecture which makes the same assertion for any compact aspherical manifold.

TOPOLOGICAL RIGIDITY THEOREM *Two nonpositively curved compact manifolds of dimension at least 5 with isomorphic fundamental group are always homeomorphic but not necessarily diffeomorphic.*

Farrell and Jones explicitly constructed examples of homeomorphic compact manifolds with metrics of negative sectional curvature which are not diffeomorphic. They accomplished this for both real and complex hyperbolic spaces. The idea is to glue in exotic spheres while maintaining negativity of the curvature. Interestingly, it is not clear if such constructions can be done for the remaining rank one symmetric spaces. Such examples are not possible for irreducible higher rank symmetric spaces as follows from the rank rigidity theorem below.

Characterizing symmetric spaces

Symmetric spaces are very special Riemannian manifolds that are at the crossroads of geometry and Lie theory. They are defined via the stringent requirements that either the curvature tensor is parallel or that all local geodesic symmetries are isometries. In particular the isometry groups of the universal cover are highly transitive. Naturally one hopes for characterizations of these spaces using milder properties.

Remarkably, symmetric spaces X contain lots of totally geodesic two-dimensional subspaces of constant curvature equal to the maximal curvature of X. We will see in this section that one can indeed characterize them in terms of these extremal curvature properties. The first instance of this phenomenon was for nonpositive curvature where every geodesic is contained in a totally geodesic flat surface. It was partially motivated by the higher rank rigidity phenomena described above.

Euclidean Rank: A Riemannian manifold has *higher (Euclidean) rank* if for every geodesic there is a parallel Jacobi field perpendicular to the geodesic. This condition is satisfied in particular if every geodesic is contained in a 2-dimensional totally geodesic flat subspace, and one should think of it as the infinitesimal analogue of the latter. Basic examples are products or locally symmetric spaces of higher rank.

In 1983 in [7, 8] W. Ballmann, M. Brin, P. Eberlein and the author laid the ground for the rank rigidity theorem proved later by Ballmann [4, 5, 10] and independently K. Burns and the author [28].

RANK RIGIDITY THEOREM *Compact irreducible manifolds of higher rank are locally symmetric.*

There are a number of natural extensions of this result. In particular, J. Heber found a similar result for homogeneous spaces of non-positive curvature [106]. S. Adams and L. Hernandez proved such results for leaves of foliations of compact manifolds if these leaves are endowed with non-positively curved metrics [1]. Eberlein and Heber relax the compactness condition in the theorem above, replacing it with a milder recurrence assumption [49, 51]. To date, no example of a complete manifold of non-positive curvature of higher rank is known which is not a product or a locally symmetric space. There are also some results for singular spaces of higher rank, namely three-dimensional Euclidean polyhedra [6]. B. Leeb gave a characterization of symmetric spaces and Euclidean buildings in terms of the Tits geometry at infinity [135], Kleiner (unpublished) and R. Charney and A. Lytchak in terms of the local geometry [30].

Ballmann and Eberlein also showed that the rank of a nonpositively curved metric is an invariant of the fundamental group [9]. In particular, we immediately get a rigidity result of Gromov [10] and Eberlein (in the case of local products) [49].

COROLLARY *Nonpositively curved metrics on irreducible locally symmetric space of higher rank are unique up to scaling.*

Hyperbolic rank: In 1991, U. Hamenstädt coined the notion of hyperbolic rank of a Riemannian manifold M in analogy with Euclidean rank: M has *higher hyperbolic rank* if its curvature is bounded above by -1, and for every geodesic c there is a Jacobi field which makes sectional curvature -1 with c [99]. Note that this is actually a slightly weaker condition than in the Euclidean case. Hamenstädt also established rigidity.

Hyperbolic Rank Rigidity Theorem If a closed manifold M has higher hyperbolic rank, then M is locally symmetric.

Surprisingly, the analogue of this theorem fails for homogeneous manifolds of negative curvature as was found by C. Connell. He also found a slight additional assumption under which the statement holds [32].

Connell recently weakened the geometric assumption on Jacobi fields in Hamenstädt's theorem to the dynamical condition that there is a set of positive measure of tangent vectors for which some Lyapunov exponent is exactly -1 [33]. He used this to characterize symmetric spaces in terms of the quasi-conformal structure and Hausdorff dimension of the boundary of the universal cover: Let M be a compact manifold, and $\partial\tilde{M}$ the sphere at infinity of the universal cover of M. Then $\partial\tilde{M}$ carries a natural bi-Lipschitz class of metrics, and $\partial\tilde{M}$ has a natural Hausdorff dimension.

THEOREM *Suppose $\partial\tilde{M}$ has the same Hausdorff dimension as some boundary ∂S of a negatively curved symmetric space S. If $\partial\tilde{M}$ is quasi-conformally homeomorphic to ∂S, then M is locally symmetric.*

Using totally different methods, M. Bonk and B. Kleiner proved the following related result in 2002 [22].

THEOREM *Consider a properly discontinuous, quasi-convex cocompact and isometric action of a group Γ on a CAT(-1)-space X. If the Hausdorff and topological dimensions of the limit set agree, and are equal to $n \geq 2$, then X contains real hyperbolic n-space as a convex Γ-invariant subset.*

They have other such characterizations, e.g. in terms of actions on Ahlfors n-regular spaces of topological dimension n [22, 23, 24].

Spherical rank: Finally let us consider complete Riemannian manifolds M with upper sectional curvature bound 1. The natural analogue to Euclidean and hyperbolic rank is the existence of a Jacobi field of the form $\sin(t)E$ along every geodesic where E is a parallel field. Using Rauch's comparison theorem, it is not hard to see that this condition is equivalent to the following: M has *positive spherical rank* if the conjugate radius of M is π. Recall that the conjugate radius is always at least π by Rauch's comparison theorem.

Using Morse theoretic methods, K. Shankar, B. Wilking and the author recently found the following characterization.

SPHERICAL RANK RIGIDITY THEOREM *Let M^n be a complete, simply connected Riemannian manifold with sectional curvature at most 1 and positive spherical rank. Then M is isometric to a compact, rank one symmetric space i.e., M is isometric to $\mathbf{S}^n$, $\mathbf{CP}^{\frac{n}{2}}$, $\mathbf{HP}^{\frac{n}{4}}$ or $\mathrm{Ca}\mathbf{P}^2$.*

Minimizing curvature: Let us mention that the analogous situation with lower curvature bounds and extremal Jacobi fields is not understood except for nonnegatively curved manifolds. For these, Heintze found homogeneous spaces for which every geodesic is contained in a totally geodesic flat subspace [107]. Later Strake and the author found many such metrics, and in particular ones for which the manifold may not even be homotopy equivalent to a homogeneous space [177].

Pinching Rigidity: The earliest rigidity theorem for positively curved manifolds is the sphere theorem of M. Berger and W. Klingenberg: Call a manifold $1/4$-*pinched* if all sectional curvatures lie between 1 and 4 or -1 and -4. If the inequalities are strict, call the manifold *strictly* $1/4$-*pinched*. Klingenberg proved that a complete strictly $1/4$-pinched simply connected manifold is homeomorphic to a sphere. Berger extended this theorem to $1/4$-pinched manifolds: either the underlying manifold is a sphere or the manifold is isometric to a rank one symmetric space.

There are analagous questions in negative curvature. The topology of complete simply connected manifolds of nonpositive curvature is trivial by Hadamard's theorem as they are always diffeomorphic to Euclidean space. In particular, these are $K(\pi,1)$-spaces and the algebraic topology is determined by the fundamental group. What fundamental grops are possible, and if there are further restrictions under pinching assumptions is unclear. For example, strict $1/4$-pinching may prevent Kazhdan groups as fundamental groups. Also, it is natural to ask when a $1/4$-pinched space carries a locally symmetric structure. Gromov and Thurston found a construction of many new spaces of negative curvature in 1987 [93]. They used branched covers of totally geodesic submanifolds of a constant curvature space. This is reminiscent of the first example of a negatively curved Kähler manifold that does not admit a locally symmetric metric by Mostow and Siu [154]. This latter example was obtained via a very special construction involving complex reflection groups.

THEOREM There are many $1/4$-pinched Riemannian manifolds of negative curvature which do not carry a locally symmetric metric.

Amazingly locally symmetric metrics of negative but not constant curvature exhibit strong rigidity properties as was shown with the harmonic maps approach by Hernandez, Yau and Zheng and Gromov [108, 187, 90].

$1/4$-PINCHING RIGIDITY THEOREM *Any* $1/4$*-pinched metric on a compact complex or quaternionic hyperbolic space is locally symmetric.*

Katok asked for a dynamical analogue replacing the $\frac{1}{4}$ pinching assumption on the curvature by a $\frac{1}{2}$-pinching assumption on the Lyapunov exponents. J. Boland found counterexamples to the most optimistic statement that $\frac{1}{2}$-pinched contact perturbations of the geodesic flow of a compact complex or quaternionic hyperbolic are smoothly conjugate to this geodesic [21]. However, his counterexamples are certain smooth time changes of the geodesic flow with smooth stable foliations and as such are classified by Benoist, Foulon and Labourie [16]. This leaves open the possibility of dynamical rigidity in this context.

Minimal Volume and Entropy: A common theme in Riemannian geometry is the analysis of metrics extremal or critical for some functional. Einstein metrics for example are critical points for the total scalar curvature. Given a differentiable manifold M, one can try and minimize the volume over all metrics with sectional curvature bounded by -1 and 1. This may or may not be possible. The infimum of such volumes is called the *minimal volume* of M, a notion coined by Gromov [88]. J. Cheeger, K. Fukaya and Gromov analyzed when the minimal volume is 0 in terms of flat and more generally nilpotent structures [31]. In general, one can ask when there is a metric with minimal volume. Compact surfaces of constant negative curvature have minimal volume by Gauss-Bonnet. Little is known in higher dimension except for the following breakthrough by G. Besson, G. Courtois and S. Gallot in 1995 [19, 78].

THEOREM *Let M be a closed manifold of dimension at least 3 which admits a metric g_0 of constant sectional curvature -1. Then g_0 is the unique metric with minimal volume.*

Conjecturally, all (suitably normalized) locally symmetric manifolds of nonpositive curvature minimize the volume. Note however that compact manifolds with a metric of negative curvature need not carry a metric of minimal volume. For example, one can change the differentiable structure on a closed manifold M of constant negative curvature by taking connected sums with an exotic sphere. L. Bessières showed that the resulting closed manifold does not admit a metric with minimal volume [17]. His results generalize to closed manifolds which map into M with nonzero degree.

Bessières' work also implies that the minimal volume is only a differential and not a topological invariant.

Besson, Courtois and Gallot actually proved their result on minimal volume via their celebrated work on minimal entropy which we will now discuss. A natural geometric invariant closely linked to dynamics is the *volume growth entropy h*. It is the exponential growth rate of the volume of balls in the universal cover:

$$h = \lim_{r \to \infty} \frac{1}{r} \log volB_r(p)$$

where $p \in \tilde{M}$ is some point in the universal cover $\tilde{M}$ of M. The volume growth entropy always is a lower bound for the topological entropy of the geodesic flow, and equals it if the metric does not have conjugate points [140, 71]. Now define the *minimal entropy* of a compact manifold as the infimum of the volume growth entropies of all metrics with volume 1. Remarkably, the minimal entropy is a homotopy invariant as was shown by I. Babenko [3]. Just like the minimal volume, the minimal entropy is often 0, for example for manifolds with nontrivial flat structures, as was shown by G. Paternain and J. Petean [162]. They also analyze in detail which 4- and 5-manifolds have minimal entropy 0.

A topological invariant closely related to the minimal volume is Gromov's *simplicial volume* [88]. Given a compact manifold M, it is the infimum of $\sum_i \|r_i\|$ where $\sum_i r_i \sigma_i$ is a singular chain representing the fundamental class. The simplicial volume of a closed negatively curved manifold is never 0. Gromov used this invariant to give a completely new proof of Mostow's rigidity theorem for constant curvature manifolds [91, 180]. Minimal volume, minimal entropy and also the also simplical volume $\| M \|$ of Gromov are closely related by the following inequalities [88, 18].

$$MinVol(M) \geq \frac{1}{(n-1)^n} MinEnt(M) \geq \frac{n^{n/2}}{(n-1)^n \cdot n!} \| M \| .$$

Metrics which minimize the volume growth entropy are of particular interest. Natural candidates are the locally symmetric metrics of nonpositive curvature, as was conjectured by A. Katok and M. Gromov. The case of negative curvature is now fairly well understood due to Besson, Courtois, and Gallot's groundbreaking work in the early 90's [19].

MINIMAL ENTROPY RIGIDITY *On a compact locally symmetric space M of negative curvature, precisely the symmetric metrics minimize the volume growth entropy (amongst metrics with fixed volume).*

As before, this result generalizes to maps $f : N \mapsto M$ of nonzero degree into M [19]. Let us describe the barycenter method, the main idea of the proof, in the simplest case when both metrics are negatively curved. First lift f to the universal covers $\tilde{N}$ and $\tilde{M}$ and extend this lift to a map ϕ between the spheres at infinity of N and M. Each $x \in \tilde{N}$ determines special measures μ_x on the sphere at infinity called the *Patterson-Sullivan measures*. Roughly they measure how the orbit of x under the fundamental group equidistributes at infinity. Then $\phi_*(\mu_x)$ is a probability measure on the sphere at infinity of M. One can show that $\phi_*(\mu_x)$ does not have atoms. In this case, one can define the barycenter of $\phi_*(\mu_x)$, and get a map $F : \tilde{N} \mapsto \tilde{M}$ given by $x \mapsto$ barycenter $\phi_*(\mu_x)$. Then F

is differentiable and amazingly, one can estimate the Jacobian of F. This allows one to compare the volume growth entropy of N and M.

The case of nonpositively curved closed locally symmetric spaces M without Euclidean factors is open. C. Connell and B. Farb generalized the barycenter method to this case. However, this approach does not lead to the optimal constants needed to prove that the minimal volume of all closed locally symmetric spaces is achieved by the locally symmetric metric [34, 35]. It does show however that the minimal volume of such spaces is positive, and bounds the degree of maps into M in terms of the ratio of the volumes. The stronger property that the simplical volume of these spaces is nonzero remains open at this time except for certain special cases [171].

The barycenter method of Besson, Courtois and Gallot has been applied and extended to numerous settings, from 3-manifold topology to Hamiltonian systems and Einstein manifolds. We refer to [20, 78, 35, 161] for more extensive surveys.

More on Entropy: There are other flavors of entropy besides the volume growth and topological entropy: given any invariant measure μ for the geodesic flow, we can consider the measure-theoretic entropy h_μ of μ. Entropy essentially is a measure of the speed of mixing of the system [104]. If the manifold M has negative curvature, the most interesting measures are geometrically defined: the Liouville measure λ, the harmonic measure ν and the Bowen-Margulis measure μ. The Liouville measure is the volume determined by the canonical invariant contact structure on the unit tangent bundle of M. The harmonic measure corresponds to the hitting probability of the Brownian motion on the sphere at infinity of the universal cover of M. Finally, the Bowen-Margulis measure is just the unique measure of maximal entropy. By the maximum principle, its entropy is equal to the topological entropy, and majorizes all other measure-theoretic entropies. For closed surfaces of genus at least 2, A. Katok showed that $h = h_\mu > h_\lambda$ unless the curvature is constant [123]. If M is a closed locally symmetric space of negative curvature, then all three measures and their entropies are equal. This lead to the following conjecture due to Katok, D. Sullivan and V. Kaimanovich [123, 178, 118]:

ENTROPY CONJECTURE *Let M be a closed manifold of negative curvature. If at least two of λ, ν or μ are equal, then M is locally symmetric.*

This conjecture has been established for deformations of constant curvature metrics by L. Flaminio using representation theory [68]. He also showed that h_λ does not achieve a maximum at the constant curvature metric, unlike in the 2-dimensional case [123]. Furthermore, Katok, Knieper and Weiss showed that the negatively curved locally symmetric metrics are always critical points for both topological and Liouville entropy [125].

Regularity of the Anosov splitting: For any Anosov diffeomorphism or flow, the stable and unstable distributions are always Hölder, and in general not better than that [105, 166, 79]. For closed surfaces or strictly $\frac{1}{4}$-pinched closed manifolds of negative curvature, the geodesic flow always has C^1 stable and unstable distributions [109, 102, 103]. Stronger regularity is expected to force rigidity. In dimension 3, S. Hurder and Katok and later Ghys obtained the following rigidity result with almost optimal regularity [82, 114, 113]. This is based on Ghys' work in the $\mathbb{C}^\infty$-case [80].

THEOREM *Suppose an Anosov flow ϕ_t on a closed three-dimensional manifold has $C^{1+Lipschitz}$ stable and weak unstable distributions, then ϕ_t is a time change of an algebraic Anosov flow. Moreover, if ϕ_t is a geodesic flow then the curvature of the underlying surface is constant*

In higher dimension, the best known result is due to Y. Benoist, P. Foulon and F. Labourie [16]. This followed earlier work of Kanai and later Katok and R. Feres who classified such systems under pinching or dimension assumptions. Kanai introduced a natural connection invariant under the flow, now called the Kanai connection, which proved crucial for all these developments.

THEOREM Let ϕ_t be a contact Anosov flow on a closed manifold M with C^∞-stable and unstable foliation. Then ϕ_t is C^∞-conjugate to either a suspension of an algebraic Anosov diffeomorphism or a time change of a geodesic flow of a locally symmetric space.

The time changes in question are of a very special nature, given in terms of suitable 1-forms. Benoist and Labourie later proved the analogous result for symplectic or connection preserving diffeomorphisms [12]. For geodesic flows, one can combine Benoist, Foulon and Labourie's theorem with the work of Besson, Courtois and Gallot [19].

THEOREM If the geodesic flow of a closed manifold M of negative curvature has smooth stable distribution, then M is locally symmetric.

V. Sadovskaya recently applied the work of Benoist, Foulon and Labourie to classify uniformly quasi-conformal contact Anosov flows [170] with stable distribution of dimension at least 3. Up to smooth conjugacy, they are time changes of the geodesic flow of a constant curvature manifold. This generalized earlier results by Sullivan and C. Yue for geodesic flows and R. de la Llave for conformal flows [178, 188, 45]. She obtained similar results for diffeomorphisms with B. Kalinin [120]. The key idea in these works is to prove that the stable foliation is smooth by analysing the holonomy along the transverse foliation. The holonomy is conformal, and hence defines a C^∞-map on stable leaves by Liouville's theorem on conformal maps on $\mathbb{R}^l$.

Harmonic manifolds: Call a manifold *harmonic* if the distance spheres of the universal cover have constant mean curvature. Foulon and Labourie combined minimal entropy rigidity with their previous joint work with Benoist in 1992 to prove the following special case of the Lichnerowicz' conjecture [69].

THEOREM *If a closed manifold M of negative curvature is harmonic, then M is locally symmetric.*

They first show that the horospheres have constant mean curvature. In turn this forces regularity of the Anosov splitting. Compactness of the manifold proves crucial here due to the examples of Damek and Ricci from 1992 [44].

THEOREM *There are non-harmonic negatively curved homogeneous manifolds.*

Harmonic solvmanifolds of negative curvature have not yet been classified. Some partial progress in this direction was recently obtained by Benson, Payne and Ratcliff for three-step solvmanifolds [11].

Spectral Rigidity: Kac' famous problem whether the spectrum of the Laplacian of a Riemannian manifold determines the metric has stirred up a storm of counterexamples, even for closed manifolds with constant negative curvature [181, 179]. There are now plenty of isospectral manifolds with different topology, different local Riemannian structures and deformations [85]. In negative curvature however, Guillemin and Kazhdan for surfaces and very pinched metrics, and later Croke and Sharafutdinov in general showed [95, 94, 41]:

ISOSPECTRAL DEFORMATION RIGIDITY *If g_t is a path of negatively curved metrics with the same spectrum of the Laplacian on a compact manifold, then the g_t are all isometric.*

It is crucial here that the manifolds do not have boundary. Gordon and Szabo recently constructed examples of negatively curved manifolds with boundary which admit isospectral deformations [86]. An important dynamical ingredient in the proof is Livsic' theorem that a function is a Lie derivative of an Anosov flow if all averages over closed orbits vanish.

The spectrum of the Laplacian is closely related to the length of the closed geodesics in the manifold via the trace formula [48]. In negative curvature moreover, each free homotopy class of loops contains a unique closed geodesic. Thus one can define the *marked length spectrum* as this length function on the fundamental group. Remarkably, both C. Croke and J.-P. Otal showed that this function sometimes determines the manifold [38, 158].

THEOREM *The marked length spectrum determines a non-positively curved compact surface up to isometry.*

In higher dimension, much less is known. If two negatively curved closed manifolds M and N have the same marked length spectrum, then their geodesic flows are C^0-conjugate [98]. Hence the topological and thus volume growth entropies coincide. If M also has C^1-Anosov splitting as in the case of locally symmetric spaces, Hamenstädt proves in [100] that the volumes are also equal. Combining this with Besson, Courtois and Gallot's work on volume growth entropy, we get

THEOREM *A negatively curved closed manifold with the same marked length spectrum as a negatively curved closed locally symmetric space M is isometric to M.*

Croke, Eberlein and Kleiner generalized this result to nonpositively curved symmetric spaces whose geodesic flows are conjugate to that of another manifold of nonpositive curvature [39]. For arbitrary curvature, homotopic closed geodesics may have different lengths,and one may just want to consider the minimal or maximal lengths. Eberlein and recently Gornet and Mast have results for 2-step nilmanifolds [50, 87]. Alternately, consider manifolds M and N whose geodesic flows are conjugate, and therefore have the same marked length spectra. Amazingly, if N say admits a parallel vector field, then Croke and Kleiner showed that M and N are isometric [40].

QUASI-ISOMETRIES AND GROUP THEORY

Quasi-isometries played a crucial role in Mostow's proof of his Strong Rigidity Theorem. He considered a symmetric space X of non-positive curvature which does not have Euclidean or hyperbolic plane factors. Then a quasi-isometry of X equivariant for the action of a lattice has to be a bounded distance from an isometry. For the real and complex hyperbolic spases, the equivariance condition is needed: one can simply extend a diffeomorphism or contact diffeomorphism f respectively from the sphere at infinity to a quasi-isometry $\tilde{f}$ of the interior by $\tilde{f}(r,\theta) = (r, f(\theta))$ in terms of suitable polar coordinates [91]. In higher rank however, Margulis conjectured in the 1970's that the equivariance assumption is not needed. Actually, this was first shown by P. Pansu for the (rank one !) quaternionic hyperbolic spaces and the Cayley hyperbolic plane [159]. Then first B. Kleiner and B. Leeb, and independently A. Eskin and B. Farb affirmed the Margulis conjecture in the mid 1990's [132, 56]:

CLASSIFICATION OF QUASI-ISOMETRIES *Any quasi-isometry of an irredicible higher rank symmetric space or quaternionic hyperbolic space or the Cayley hyperbolic plane is a finite distance from an isometry.*

Kleiner and Leeb argue via ultralimits: rescale the locally symmetric metric by $\varepsilon \mapsto 0$. Then the symmetric space converges to a Euclidean building, at least with respect an ultrafilter - a technical device to insure convergence. The quasi-isometry limits to a bi-Lipschitz map. Kleiner and Leeb then classify the homeomorphisms of an irreducible building as essentially isometries. Translating this back to the approximating symmetric space shows that the quasi-isometry is a finite distance from an isometry.

Inspired by the Morse Lemma in negative curvature, Eskin and Farb study quasi-isometric maps of $\mathbb{R}^k$ into a symmetric space or real rank k. Unlike in negative curvature, such a map is not in general a bounded distance from a flat. Rather one needs finitely many Weyl chambers in flats to get within a bounded distance of a quasi-isometry.

This classification of quasi-isometries of symmetric spaces has important consequences in geometric group theory. The idea is to make a finitely generated group Γ into a geometric object endowing it with a word metric. Given a finite generating set S, the distance between two group elements $\gamma_1, \gamma a_2 \in \Gamma$ is simply the length of the shortest word for $\gamma_2^{-1}\gamma_1$ in S. The word metrics for two different finite generating sets are quasi-isometric. Thus the "geometry in the large" of Γ is well determined. A fundamental problem, posed by Gromov, is the classification of quasi-isometry classes of groups. This program has proved enormously successful in the case of lattices in Lie groups where we have gained an almost complete understanding over the last decade. Combining works of Casson, Chow, Drutu, Eskin, Farb, Gabai, Gromov, Jungreis, Kleiner, Leeb, Pansu, Schwartz, Sullivan and Tukia one gets two main results, quasi-isometric rigidity and classification.

QUASI-ISOMETRIC RIGIDITY OF LATTICES If a finitely generated group Γ is quasi-isometric to an irreducible lattice in a semisimple Lie group G, then there is a finite subgroup $F \subset \Gamma$ such that Γ/F is isomorphic to a lattice in G.

Call two lattices *commensurable* if they contain isomorphic subgroups of finite index.

QUASI-ISOMETRIC CLASSIFICATION OF LATTICES There is one quasi-isometry class of cocompact lattices for each semisimple group G. Further, there is one quasi-isometry class for each commensurability class of irreducible non-cocompact lattices, except for $G = SL(2,\mathbb{R})$ where there is precisely one quasi-isometry class of non-cocompact lattices.

Gromov's program has also been investigated for groups acting on trees, and in particular for Baumslag-Solitar groups [58, 149, 185]. We refer to [57] for a much more detailed survey.

Gromov also introduced a measure theoretic analogue of quasi-isometry. Call two groups Γ and Λ *measure equivalent* if they admit commuting actions on a Borel space Ω by measure preserving transformations such that both groups have finite measure fundamental domains. A. Furman studied the case of a countable group Λ measure equivalent to a lattice Γ in a higher rank noncompact simple Lie group [72]. He showed that then Λ itself is essentially a lattice in a higher rank simple Lie group. The proof crucially uses Zimmer's superrigidity for cocycles. Furman applied this to orbit equivalences of actions of such groups [73]. Very recently, D. Gaboriau showed that l^2-Betti numbers of groups are essentially invariant under measure equivalence [76]. This allowed him in particular to distinguish free groups and their products under measure equivalence (cf. also [77]). Recently, N. Monod and Y. Shalom used techniques from bounded cohomology to prove measure equivalence rigidity of products of groups acting on CAT(-1)-spaces [148].

DYNAMICS AND GROUP ACTIONS

Dynamics and ergodic theory furnished important tools for rigidity theory, and in particular for Mostow's and Margulis' theorems. In turn, ideas and tools from rigidity theory proved important in dynamics. We will mostly discuss two developments here, namely the Zimmer program on actions of large groups and the recent work on hyperbolic actions of higher rank abelian groups. We refer to other surveys for other important developments and in particular for Ratner's work on measure rigidity of unipotent actions and applications, e.g. [167, 81, 144, 145, 146, 131] and hyperbolic dynamical systems [101].

Anosov actions

The recent work on Anosov actions of higher rank abelian groups drew its inspiration from several sources: the Zimmer program studying actions of semisimple groups and their lattices, Furstenberg's conjecture on scarcity of measures jointly invariant for $\times 2$ and $\times 3$, Ratner's work on unipotent flows and measures, and finally the higher rank rigidity results for Riemannian manifolds. In fact, the geometry of negatively and nonpositively curved manifolds has always been tied closely to dynamics via the geodesic flow. Negativity of the curvature naturally corresponds to hyperbolicity for the flow. Thus the geometric rigidity results suggest rigidity for actions of higher rank abelian groups (the analogues of flats) with transverse hyperbolic behaviour.

Suppose a group A acts C^∞ and locally freely on a manifold M with a Riemannian norm $\| \, \|$. Call the action *Anosov* if there is an an element $g \in A$ and there exist real numbers $\lambda > \mu > 0$, $C, C' > 0$ and a continuous splitting of the tangent bundle

$$TM = E_g^+ + E^0 + E_g^-$$

such that E^0 is the tangent distribution of the A-orbits and for all $p \in M$, for all $v \in E_g^+(p)$ ($v \in E_g^-(p)$ respectively) and $n > 0$ ($n < 0$ respectively) we have for the differential $g_* : TM \to TM$

$$\| g_*^n(v) \| \leq Ce^{-\lambda |n|} \| v \| .$$

Anosov actions of $\mathbb{Z}$ and $\mathbb{R}$ are called Anosov diffeomorphisms and flows.

Structural stability, cocycles and local rigidity: The first rigidity result goes back to D. Anosov in the 1960's who proved for Anosov diffeomorphisms and flows that the orbit structure is rigid under small perturbations [2]. M. Hirsch. C. Pugh and M. Shub generalized this to [110]:

STRUCTURAL STABILITY [Anosov, Hirsch-Pugh-Shub]] Any C^1-small perturbation of an Anosov action is orbit equivalent to the original action, i.e. there is a homeomorphism taken orbits to orbits.

This theorem often serves as a starting point to prove other stronger rigidity results in that one can try to improve upon the orbit equivalence coming from structural stability. In the best possible cases one wants to improve the orbit equivalence to an isomorphism, ideally of higher regularity. Straightening an orbit equivalence to an isomorphism can be thought of as a cohomology problem. Let us first recall the notion of cocycle. Suppose a group G acts on a space X. We call $\alpha : G \times X \mapsto H$ into a group H a *cocycle* if

$$\alpha(g_1 g_2, x) = \alpha(g_1, g_2 x)\, \alpha(g_2, x).$$

Given a map $\beta : X \mapsto H$, we call the cocycle $\alpha'(g,x) = \beta(gx)^{-1}\, \alpha(g,x)\, \beta(x)$ *cohomologous* to α. If $\alpha \cong 1$, then α' is called a *coboundary* of β. We call the cohomology or coboundary measurable, continuous or smooth depending on the regularity of β.

If ϕ is an orbit equivalence between actions of G and H and the H-action is free, then the equation

$$\phi(gx) = \alpha(g,x)\, \phi(x)$$

determines a cocycle. Changing an orbit equivalence along the orbits gives cohomologous cocycles. The problem of straightening an orbit equivalence into an isomorphism is equivalent to showing that the associated cocycle is cohomologous to a constant cocycle $\beta(\gamma, x) \cong \beta(\gamma)$, i.e. a homomorphism $\beta : G \mapsto H$.

In the case of Anosov actions, two early theorems of Livsic [138] from 1972 play an important role. For Anosov flows they give criteria when a Hölder or C^∞ function f is a Lie derivative along the flow, namely either if all integrals of f over closed orbits are 0, or if there is a measurable solution (cf. our discussion of isospectral rigidity). This was generalized to abelian Anosov actions by Katok and the author [128].

LIVSIC' THEOREM Let α be a volume preserving Anosov action of $\mathbb{R}^k$ on a compact manifold M, and β a Hölder cocycle taking values in $\mathbb{R}$. Then β is a coboundary of a Hölder function P if and only if $\beta(a,x) = 0$ whenever $ax = x$. Furthermore, if β is C^∞, so is P.

This provides countably many obstructions to solve the cohomology equation. Unlike in the case of flows and diffeomorphisms, they mysteriously vanish for higher rank "algebraic" actions as was shown by Katok and the author. We used harmonic analysis and in particular the exponential decay of matric coefficients to do this [128]. In the discrete case, these "algebraic" actions come from commuting automorphisms of tori or nilmanifolds more generally. In the continuous case, they are actions by left translations of the diagonals in $SL(n,\mathbb{R})$ on $SL(n,\mathbb{R})/\Gamma$, Γ a cocompact lattice, and generalizations of this. We will call such actions *affine*.

COCYCLE RIGIDITY THEOREM *Any $\mathbb{R}$-valued Hölder or C^∞ cocycle over an R^n or $\mathbb{Z}^n$ affine Anosov action is Hölder or C^∞ cohomologous to a constant cocycle provided that all non-trivial elements of some $\mathbb{Z}^2$ subgroup act ergodically.*

Katok, Nitica and Török gave a more geometric treatment for a special class of toral actions [127]. We refer to Nitica and Török's sruvey [157] for this and other generalizations as well as some instructive examples.

The cocycle rigidity theorem allowed us to straighten out orbit equivalences of affine actions. Combining this with structural stability yields Hölder local rigidity. Here we call a C^∞-action ρ of a compactly generated group Γ on a compact manifold M C^∞respectively *Hölder locally rigid* if any C^∞-action $\tilde{\rho}$ of Γ on M C^1-close to ρ on a fixed compact set of generators is C^∞ respectively Hölder conjugate to a composite of ρ with an automorphism of Γ. We improved the Hölder regularity of the conjugacy using the nonstationary normal forms of M. Guysinsky and Katok [97, 96, 63].

LOCAL RIGIDITY THEOREM *If $n \geq 2$, then the R^n or $\mathbb{Z}^n$ affine Anosov actions with semisimple linear part are C^∞ locally rigid.*

This result together with the scarcity of the known actions suggests the

Problem: Are all "irreducible" higher rank Anosov actions algebraic?

In the discrete case, a closely related old conjecture asks if all Anosov diffeomorphisms are topologically conjugate to an affine Anosov automorphism. This would reduce the problem of classifying higher rank Anosov actions to one about the regularity of the conjugacy. The methods of the local rigidity theorem may prove helpful. Let us note though that Farrell and Jones constructed Anosov diffeomorphisms on exotic tori [61]. In the continuous case, there are Anosov flows on non-homogeneous spaces, e.g. the Handel-Thurston examples, adding to the potential difficulty.

The following phenomenon lies at the core of the stronger rigidity properties of higher rank abelian Anosov actions. For simplicity let's consider one of the homogeneous model actions. Then a singular (i.e. non - Anosov element) a acts via isometries in certain directions. This is clear in the homogeneous model situations where lack of hyperbolicity corresponds to having eigenvalues of modulus 1. Then a acts via isometries w.r.t. a suitable metric on the leaves of the foliation $\mathcal{F}$ corresponding to the eigenvectors with eigenvalues of modulus 1. In particular, if the orbit of a of a point p recurs to a point q in the leaf $\mathcal{F}(p)$ then a limit of suitable powers a^{n_k} will limit to an isometry of $\mathcal{F}(p)$ which takes p to q. Thus if the orbit of p is dense on $\mathcal{F}(p)$ then $\mathcal{F}(p)$ admits a transitive group of isometries constructed canonically from the dynamics of a. This isometry group will also preserve additional structures invariant under the given action such as conditional measures with respect to $\mathcal{F}$ of an invariant measure. This argument plays a central role in the local rigidity results and the classification mentioned above as well as the measure rigidity results below where it was first introduced by Katok.

Measure rigidity: The core phenomenon for higher rank rigidity is closely connected with measure rigidity. In a nutshell, this is the assertion that invariant measures for higher rank Anosov and more generally weakly hyperbolic actions are scarce. This took its first manifestation in Furstenberg's conjecture that any probability measure μ jointly invariant under $\times 2$ and $\times 3$ on $[0,1]$ is a convex combination of Lebesgue measure and Dirac measures at periodic points [74]. Under the additional strong assumption that one of the elements is K for μ, R. Lyons proved this in 1988 [139]. In 1990 D. Rudolph could weaken the hypotheses so that only the entropy of the measure has to be positive with respect to at least one of the elements [168]. Other proofs and generalizations were given by A. Johnson, J. Feldman and B. Host [169, 62, 111, 112].

Measure rigidity for higher rank affine abelian actions in higher dimensions was first considered by Katok and the author in 1994 [128]. Let us call a measure on a homogeneous space H/Λ *algebraic* if it is a Haar measure on some closed homogeneous subspace. Margulis and Katok and the author made the following conjecture [128, 145]. We refer to [145] for a more technical and general statement as well as the topological analogue.

CONJECTURE: Any invariant ergodic Borel probability measure μ of an affine Anosov action is either Haar measure on a homogeneous algebraic subspace or the support of μ is contained in an invariant homogeneous subspace that has a rank one factor for the action.

This is similar in spirit to Ratner's theorems on measure rigidity of unipotent groups. Katok and the author used the core phenomenon discussed above to prove this conjecture under additional strong assumptions [128]. Most importantly, we needed to assume positivity of entropy for some element in the action. This implies that conditional measures on certain stable manifolds are not atomic. Suitable ergodicity assumptions combined with the core phenomenon yield non-trivial groups of isometries which fix the conditional measures. This allows to apply Ratner's results or similar more elementary uniform ergodicity facts in the toral case. Recently, Einsiedler, Katok and Lindenstrauss introduced new ideas to remove extraneous ergodicity assumptions [53, 55]. This leads to much stronger measure rigidity results and the wonderful applications mentioned below. No progress has been made however in overcoming the assumption on positivity of entropy.

Measure rigidity has already found several exciting applications. It was used in ergodic theory to show algebraicity of measurable isomorphisms and disjointness of higher rank affine actions. In the toral case this was achieved by A. and S. Katok and K. Schmidt , disjointness by A. Katok and B. Kalinin, and for general affine actions by Kalinin and the author [124, 119, 121]. We refer to Schmidt's survey in this Festschrift for generalizations to automorphisms of compact abelian groups [172]. In number theory, full strength measure rigidity implies Littlewood's conjecture on simultaneous Diophantine approximations. Using the state of the art on measure rigidity, Einsiedler, Katok and Lindenstrauss have shown that the set of exceptions has Hausdorff dimension 0 [54]. Finally, in spectral theory, Lindenstrauss established unique quantum ergodicity for certain arithmetic surface groups [136]. We refer to Lindenstrauss' forthcoming survey for a more extensive discussion of these exciting developments [137].

Actions of Large Groups

We will finish this invitation with a glimpse on R. Zimmer's program of studying actions of "large" groups on manifolds. While the classical superrigidity results concern the classification of the finite dimensional representations of a lattice, their natural nonlinear analogues are actions of lattices or their ambient Lie groups on finite dimensional manifolds. Mostly we will consider connected semisimple Lie groups G without compact factors and of real rank at least 2 or a lattice Γ in such a G. Even for just finite measure preserving actions, Zimmer found a generalization of superrigidity, the cocycle superrigidty theorem, to this setting in 1980 [189, 190]. It has become an indispensable tool for analyzing these actions.

COCYCLE SUPERRIGIDITY THEOREM Measurable cocycles over such actions with values in semisimple groups without center are either measurably cohomologous to a constant cocycle, i.e. a homomorphism, or cohomologous to a cocycle taking values in a compact subgroup.

Smooth, volume preserving actions of Γ or G seem to be very special. In fact, most known actions are affine actions on a homogeneous manifold. However, Katok and J. Lewis (and later J. Benveniste) constructed non-algebraic examples of such actions "blowing up" the standard action of $SL(n,\mathbb{Z})$ on the n-torus by replacing the fixed point by a projective space [126, 13]. This opens up a multitude of problems. For one, the Katok-Lewis and also the Benveniste examples are affine on an open dense and conull subset. Is this always true? Is there always an invariant "rigid" geometric structure such as an affine connection on an open dense conull set? By Zimmer's superrigidity theorem, there are always suitable G-invariant measurable affine structures. Further properties such as dynamical features or an invariant Gromov rigid geometric structure also may force algebraicity. The latter are geometric structures with finite dimensional automorphism groups, introduced by Gromov in 1988 [89, 43]. Recently, Benveniste and Fisher showed that the Katok-Lewis and Benveniste examples do not admit an invariant Gromov rigid structure [15]. An example of a classification of actions with invariant geometric structures is Gromov's theorem from 1988 [89].

THEOREM Let M be a compact pseudo-Riemannian metric of type (n_+, n_-). Let G act on M by isometries, and let (m_+, m_-) be the type of the Cartan-Killing form of G. Suppose the rank of G is at least $n_0 - m_0$ where $n_0 = \min(n_+, n_-)$ and $m_0 = \min(m_+, m_-)$. Then the G-action is locally free, and some covering splits equivariantly as $S_0 \times G$ where the fibers project to totally geodesic leaves normal to the G-orbits.

From a dual point of view, all this amounts to the study of geometric structures with large automorphism groups. These have been studied closely in in differential geometry using geometric tools. The novelty here lies in the introduction of superrigidity and tools from ergodic theory and dynamics.

On the dynamical side, let us mention the author's work with E. Goetze [84]. For simplicity we stick to just connected groups.

THEOREM Suppose G has real rank at least 3 and assume that the action is Anosov, volume preserving and multiplicity-free, i.e. we assume that the Lyapunov spaces of the non-zero Lyapunov exponents of a regular element g of the group are all one-dimensional. Then some finite cover of such an action is C^∞-conjugate to an affine action.

Let us finally turn to local rigidity. One cannot expect a completely general result here as the Katok-Lewis and the Benveniste examples are not locally rigid, the latter not even in the class of volume preserving actions [126, 13]. The story is quite different however for the affine and projective actions. It started with J. Lewis who proved infinitesimal rigidity of the standard action of $SL(n,Z)$ on the n-torus, continued with Hurder and deformation rigidity for the same action. Katok and Lewis proved local rigidity. We now have the culmination in the following recent result of Fisher and Margulis [67] which follows a long series of papers by Benveniste, Hurder, Katok, Lewis, Margulis, Qian, Yue, Zimmer and the author.

LOCAL RIGIDITY THEOREM Let G be a connected semisimple Lie group with all factors of real rank at least 2. Then the affine actions of G and higher rank lattices Γ in G and semisimple groups are all locally rigid with C^∞-conjugacies in the category of $C^{\infty,\infty}$-perturbations.

The $C^{\infty,\infty}$-regularity here refers to C^∞perturbations which are sufficiently C^∞-close to the original action on a generating set. We refer to [67] for more precise versions of the regularity properties.

$SL(n,\mathbb{R})$ and its subgroups naturally act on real projective space. This is a special case of the action of G on G/P, P a parabolic subgroup. Call all the latter actions *projective*. The action of a lattice Γ of g on G/P can be interpreted as the holonomy of a weak stable foliation of the action of the Cartan subgroup on G/Γ. Katok and the author used this duality to relate local rigidity of higher rank Anosov actions and projective actions generalizing a partial result by M. Kanai [122, 130].

LOCAL RIGIDITY THEOREM The projective actions of a cocompact lattice Γ in G are locally rigid.

Zimmer also conjectured that higher rank lattices cannot act faithfully on manifolds of low dimension relative to the size of the ambient Lie group. This problem is difficult even for the circle, and was resolved only in the last few years in the works of D. Witte, Ghys, Burger and Monod, and Navas [186, 83, 25, 155]. Burger and Monod's proof calculated the bounded cohomology of the lattice which is closely connected with circle actions. Recently Polterovich showed that non-uniform irreducible higher rank lattices cannot act by area preserving diffeomorphisms on a surface of genus at least 1 [163]. This is part of a general result about growth properties of groups preserving a symplectic form. Using the structure theory of area preserving diffeomorphisms, Franks and Handel showed for any closed surface that many lattices cannot act [70]. We refer to [59] for an extensive survey of this topic.

This is but a small sample of important results in this field. Many other developments took place in recent years, in particular concerning actions on geometric structures, orbit equivalence, isotropy and others. We refer to [89, 43, 134, 75, 64, 65, 66, 156] for detailed presentations.

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