LATTICES OF MINIMUM COVOLUME ARE NON-UNIFORM.

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Abstract. In this article, we prove that a lattice of minimum covolume in a simple Lie group over a positive characteristic local field is non-uniform if the Weil's conjecture on Tamagawa numbers [Wei61] holds. This, in part, answers Lubotzky's conjecture [Lub91]

1. Introduction

A. Lubotzky [Lub91] observed that, in contrast to the $\mathrm{SL}_2(\mathbb{R})$ case (see [Sig45]), a lattice of minimum covolume in $\mathrm{SL}_2(\mathfrak{f}_q((t)))$ is non-uniform and he asked if this is typically the case. To be precise, he asked if a lattice of minimum covolume in a semisimple Lie group over a local field is uniform or non-uniform.

In this article this question is answered for most of simple groups over a positive characteristic local field. The results of this article are based on a particular case of a well-established Weil's conjecture; namely I will assume that the Tamagawa number of a special unitary group over a global function field is one (see [Wei61] for more details).

Theorem 1. (1) *Let $\mathbb{G}_0$ be a simply connected $\mathfrak{f}_q((t))$-group of absolute type A. Any lattice of minimum covolume in $G = \mathbb{G}_0(\mathfrak{f}_q((t)))$ is non-uniform if the Weil conjecture holds for groups of type A.*
(2) *Let $\mathbb{G}_0$ be a simply connected absolutely almost simple $\mathfrak{f}_q((t))$-group whose $\mathfrak{f}_q((t))$-rank is larger than 1. Any lattice of minimum covolume in $G = \mathbb{G}_0(\mathfrak{f}_q((t)))$ is non-uniform if the Weil conjecture holds for groups of type A.*

Let us remark that by [Har75], G does not have a uniform arithmetic lattice unless $\mathbb{G}_0$ is of type A. On the other hand, by Margulis's arithmeticity theorem, any lattice of a simple Lie group of higher rank is arithmetic. Hence the second part of Theorem 1 is a consequence of the first part. Thus from this point on, we will assume that $\mathbb{G}_0$ is always of absolute type A.

In order to prove Theorem 1, following the same principle as in [Sal09], first a $\mathfrak{f}_q(t)$-group $\mathbb{G}$ is constructed such that

(1) $\mathbb{G}$ is isomorphic to $\mathbb{G}_0$ over $\mathfrak{f}_q((t))$,
(2) $\mathrm{rank}_{\mathfrak{f}_q(t)}(\mathbb{G}) = \mathrm{rank}_{\mathfrak{f}_q((t))}(\mathbb{G}_0)$.

Notice that $\mathfrak{f}_q((t))$ is viewed as the completion $k_{\mathfrak{p}_0}$ of k at the place $\mathfrak{p}_0$. We also give a family of parahoric subgroups $P_{\mathfrak{p}}$ of $\mathbb{G}(k_{\mathfrak{p}})$, where $k = \mathfrak{f}_q(t)$. This family is chosen in a way that

$$\Lambda_0 = \mathbb{G}(\mathfrak{f}_q(t)) \cap \prod_{\mathfrak{p}\neq\mathfrak{p}_0} P_{\mathfrak{p}} \qquad (1)$$

becomes a large principle congruence lattice. It is worth mentioning that since $\mathbb{G}$ is $\mathfrak{f}_q(t)$-isotropic, Λ_0 is non-uniform. In the next step, the covolume of Λ_0 is computed. In the final step, it is showed that the covolume of any uniform lattice is larger than the covolume of Λ_0. In order to get such a result, by Kazhdan-Margulis

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(we use Raghunathan's result [Rag89] as we are working in the positive characteristic), we show that any uniform lattice does not intersect a "large" neighborhood of identity in G.

2. Constructing large non-uniform lattices.

As it is mentioned in the introduction, we can and will assume that $\mathbb{G}_0$ is of absolute type A. We shall consider the following cases separately.

(1) If $\mathbb{G}_0$ is an inner form, then there is a $\mathfrak{f}_q(\!(t)\!)$-central division algebra D_0, such that $\mathbb{G}_0$ is isomorphic to $\mathbb{SL}_{m,D_0}$ over $\mathfrak{f}_q(\!(t)\!)$, for some m.

(2) If $\mathbb{G}_0$ is an outer form, then it is a unitary group, i.e. there are a quadratic extension L of $\mathfrak{f}_q(\!(t)\!)$ and a hermitian form h_0 on L^n such that $\mathbb{G}_0$ is isomorphic to $\mathbb{SU}_{h_0}$.
   - (O1) $\mathbb{G}_0$ is quasi-split and it splits over the maximal unramified extension of $\mathfrak{f}_q(\!(t)\!)$.
   - (O2) $\mathbb{G}_0$ is quasi-split and it does not split over the maximal unramified extension of $\mathfrak{f}_q(\!(t)\!)$.
   - (O3) $\mathbb{G}_0$ is not quasi-split and it splits over the maximal unramified extension of $\mathfrak{f}_q(\!(t)\!)$.
   - (O4) $\mathbb{G}_0$ is not quasi-split and it does not split over the maximal unramified extension of $\mathfrak{f}_q(\!(t)\!)$.

   By Tits's classification [Tit79], for any given m or n, there is a unique $\mathfrak{f}_q(\!(t)\!)$-group of any of the types (O1), (O2), (O3) and (O4).

2.1. **Inner form.** By Albert-Brauer-Hasse-Noether theorem, there is a $k=\mathfrak{f}_q(t)$-central division algebra D such that

$$\mathrm{inv}_{\mathfrak{p}}(D)=\begin{cases}\mathrm{inv}(D_0) & \text{if } \mathfrak{p}=\mathfrak{p}_0,\\ -\mathrm{inv}(D_0) & \text{if } \mathfrak{p}=\mathfrak{p}_\infty,\\ 0 & \text{otherwise,}\end{cases}$$

where $\mathfrak{p}_0$ is the place associated with the point 0 in the projective plane, $\mathfrak{p}_\infty$ is the place associated with the point infinity in the projective plane and $\mathrm{inv}_{\mathfrak{p}}(D)$ is the Hasse invariant of D at the place $\mathfrak{p}$. Let $\mathbb{G}=\mathbb{SL}_{m,D}$; then $\mathbb{G}$ is isomorphic to $\mathbb{G}_0$ over $\mathfrak{f}_q(\!(t)\!)$ and it is isomorphic to $\mathbb{SL}_n$, where $n=m\cdot d$ and $d=\mathrm{ind}(D_0)$, at every place except at $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$. Let $P_{\mathfrak{p}}$ be a hyperspecial parahoric subgroup of $\mathbb{G}(k_{\mathfrak{p}})=\mathrm{SL}_n(k_{\mathfrak{p}})$ for $\mathfrak{p}$ other than $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$ and a special parahoric otherwise. We also assume that $\prod_{\mathfrak{p}\in V_k}P_{\mathfrak{p}}$ is an open subgroup of $\mathbb{G}(\mathbb{A}_k)$. Let $\Lambda_0=\mathbb{G}(k)\cap\prod_{\mathfrak{p}\neq\mathfrak{p}_0}P_{\mathfrak{p}}$ where $\mathbb{G}(k)$ is embedded diagonally in the product of $\mathbb{G}(k_{\mathfrak{p}})$. We can view Λ_0 as a discrete subgroup of $\mathbb{G}(k_{\mathfrak{p}_0})=\mathrm{SL}_m(D_0)$. By Prasad's volume formula [Pra89] we have

$$\begin{aligned}\mu(G/\Lambda_0)&=q^{-\dim\mathbb{G}}\cdot\prod_{i=2}^{n}\zeta_k(i)\cdot\prod_{d\nmid k,1\le k\le n-1}(q^k-1)^2\\&=q^{1-n^2}\cdot\prod_{i=2}^{n}\frac{1}{(1-q^{-i})(1-q^{1-i})}\cdot\prod_{d\nmid k,1\le k\le n-1}(q^k-1)^2\\&=\frac{(q-1)(q^n-1)}{\prod_{d\mid k,1\le k\le n-1}(q^k-1)^2},\end{aligned}\tag{2}$$

where $\mu(P_{\mathfrak{p}_0})=1$.

2.2. **Quasi-split and split over the maximal unramified extension.** These conditions are equivalent to say that $L\simeq\mathfrak{f}_{q^2}(\!(t)\!)$ and h_0 is the split hermitian form. Let $l=\mathfrak{f}_{q^2}(t)$, $k=\mathfrak{f}_q(t)$ and h be the split hermitian form on l^n. Let $\mathbb{G}=\mathbb{SU}_{h,l^n}$. Thus $\mathbb{G}$ is a k-group which is isomorphic to $\mathbb{G}_0$ over $k_{\mathfrak{p}_0}$; moreover for any $\mathfrak{p}$ there is a hyperspecial parahoric subgroup in $\mathbb{G}(k_{\mathfrak{p}})$. Let $P_{\mathfrak{p}}$ be a hyperspecial parahoric for any $\mathfrak{p}$ such that $\prod_{\mathfrak{p}\in V_k}P_{\mathfrak{p}}$ is an open subgroup in $\mathbb{G}(\mathbb{A}_k)$. We set $\Lambda_0=\mathbb{G}(k)\cap\prod_{\mathfrak{p}\neq\mathfrak{p}_0}P_{\mathfrak{p}}$ and compute its volume:

$$\mu(G/\Lambda_0)=q^{-\dim\mathbb{G}}\cdot\zeta_k(2)\cdot L_{l/k}(3)\cdot\dots\cdot*(n),$$

where the last term is either the zeta or the L-function depending on the parity of n. On the other hand,

$$\begin{aligned} L_{l/k}(s) &= \zeta_l(s)/\zeta_k(s) \\ &= \frac{(1-q^{-s})(1-q^{1-s})}{(1-q^{-2s})(1-q^{2-2s})} \\ &= \frac{1}{(1+q^{-s})(1+q^{1-s})} \end{aligned} \tag{3}$$

Hence we have

$$\mu(G/\Lambda_0) = \prod_{i=2}^{n-1}(q^{2i}-1)^{-1}\cdot(q-1)^{-1}\cdot(q^n-(-1)^n)^{-1}. \tag{4}$$

2.3. **Quasi-split and non-split over the maximal unramified extension.** These conditions are equivalent to say that there is a ramified quadratic extension L of $\mathfrak{f}_q((t))$ such that $\mathbb{G}_0 \simeq \mathbb{SU}_{h_0,L^n}$, where h_0 is the split hermitian form on L^n. Since the characteristic p of L is odd, it has a traceless uniformizer. So without loss of generality, we can assume that $L = \mathfrak{f}_q((t))$, $K = \mathfrak{f}_q((t^2))$ and $\mathbb{G}_0 \simeq \mathbb{SU}_{h_0,L^n}$, where h_0 is the split hermitian form on L^n. Now let $l = \mathfrak{f}_q(t)$, $k = \mathfrak{f}_q(t^2)$ and $\mathbb{G} = \mathbb{SU}_{h,l^n}$, where h is the split hermitian form on l^n. First we notice that any place $\mathfrak{p}$ of k, other than $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$, is unramified over l. Hence $\mathbb{G}$ splits over the maximal unramified field extension $\widehat{k_\mathfrak{p}}$ of $k_\mathfrak{p}$ for any $\mathfrak{p} \neq \mathfrak{p}_0, \mathfrak{p}_\infty$. Therefore $\mathbb{G}(k_\mathfrak{p})$ has a hyperspecial parahoric subgroup $P_\mathfrak{p}$ for any $\mathfrak{p}$ other than $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$. When $\mathfrak{p}$ is either $\mathfrak{p}_0$ or $\mathfrak{p}_\infty$, $\mathbb{G}$ is a quasi-split $k_\mathfrak{p}$-group which does not split over a maximal unramified field. In this case, let $P_\mathfrak{p}$ be one of the special parahoric subgroups. It is worth mentioning that since both of the special parahoric subgroups have the same volume, it does not matter which one is chosen. As before, let $\Lambda_0 = \mathbb{G}(k) \cap \prod_{\mathfrak{p}\neq\mathfrak{p}_0} P_\mathfrak{p}$ and compute its covolume:

$$\mu(G/\Lambda_0) = q^{-\dim\mathbb{G}}\cdot q^{\mathfrak{s}(\mathbb{G})}\cdot\zeta_k(2)\cdot L_{l/k}(3)\cdot\dots\cdot *(n), \tag{5}$$

where the last term is either the zeta or the L-function depending on the parity of n, $\mu(P_{\mathfrak{p}_0}) = 1$ and

$$\mathfrak{s}(\mathbb{G}) = \begin{cases} \frac{(n-2)(n+1)}{2} & \text{if } 2|n \\ \frac{(n-1)(n+2)}{2} & \text{if } 2\nmid n. \end{cases} \tag{6}$$

On the other hand, since both k and l are of genus zero, $L_{l/k}(s) = \zeta_l(s)/\zeta_k(s) = 1$. Altogether we have that

$$\mu(G/\Lambda_0) = \prod_{i=1}^{2\lfloor n/2\rfloor}(q^i-1)^{-1}. \tag{7}$$

2.4. **Non-quasi-split and split over the maximal unramified field extension.** Similar to Section 2.2, we can assume that $L = \mathfrak{f}_{q^2}((t))$. However in this case h_0 is the non-split hermitian form on L^n. We again set $l = \mathfrak{f}_{q^2}(t)$ and $k = \mathfrak{f}_q(t)$. We also note that in this case n is definitely even. Now let the diagonal matrix

$$h = \mathrm{diag}(t, 1, \cdots, 1), \tag{8}$$

represents a hermitian form on l^n and let $\mathbb{G} = \mathbb{SU}_{h,l^n}$. If a place $\mathfrak{p}$ of k splits over l, then $\mathbb{G} \simeq \mathbb{SL}_n$ over $k_\mathfrak{p}$. If $\mathfrak{p}$ is also a prime over l and $\mathfrak{p}$ is neither $\mathfrak{p}_0$ nor $\mathfrak{p}_\infty$, then $\det(h) = t \in N_{l_\mathfrak{p}/k_\mathfrak{p}}(l_\mathfrak{p})$. On the other hand, we know that over non-Archimedean fields a hermitian form is uniquely determined by its dimension and determinant [Sch85, Chapter 10]. Therefore $\mathbb{G}$ is quasi-split over $k_\mathfrak{p}$ for any $\mathfrak{p}$ other than $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$. Thus in these cases, we can choose a hyperspecial parahoric subgroup $P_\mathfrak{p}$. When $\mathfrak{p}$ is either $\mathfrak{p}_0$ or $\mathfrak{p}_\infty$, $\mathbb{G}(k_{pfr})$ is isomorphic to G. In either of these cases, let $P_\mathfrak{p}$ be a special parahoric subgroup. Again let $\Lambda_0 = \mathbb{G}(\mathfrak{f}_q(t)) \cap \prod_{\mathfrak{p}\neq\mathfrak{p}_0} P_\mathfrak{p}$ and compute its covolume:

$$\mu(G/\Lambda_0) = q^{-\dim\mathbb{G}}\cdot\zeta_k(2)\cdot L_{l/k}(3)\cdot\dots\cdot L_{l/k}(n-1)\cdot\zeta_k(n)\cdot e'(P_{\mathfrak{p}_0})\cdot e'(P_{\mathfrak{p}_\infty}), \tag{9}$$

where $\mu(P_{\mathfrak{p}_0}) = 1$ and

$$e'(P_{\mathfrak{p}_0}) = e'(P_{\mathfrak{p}_\infty}) = \frac{q^n - 1}{q + 1} \tag{10}$$

(see [MS, Lemma 2] for more information on the factors $e'(P_{\mathfrak{p}})$). Similar to Section 2.2, we can compute the values of the zeta function and the L-function. Altogether, we get

$$\mu(G/\Lambda_0) = \frac{q^n - 1}{q + 1} \cdot \prod_{i=1}^{n-1} (q^{2i} - 1)^{-1}. \tag{11}$$

2.5. **Non-quasi-split and non-split over the maximal unramified extension.** Similar to Section 2.3, we can assume that $L = \mathfrak{f}_q((t))$ and $K = \mathfrak{f}_q((t^2))$ and let $l = \mathfrak{f}_q(t)$ and $k = \mathfrak{f}_q(t^2)$. Let the diagonal matrix $\mathrm{diag}(\varepsilon, 1, \cdots, 1)$ represent the hermitian form h on l^n, where $\mathfrak{f}_q^\times = \mathfrak{f}_q^{\times 2} \cup \varepsilon \mathfrak{f}_q^{\times 2}$. Let $\mathbb{G} = \mathbb{SU}_{h, l^n}$.

If a place $\mathfrak{p}$ of k is neither $\mathfrak{p}_0$ nor $\mathfrak{p}_\infty$ and it is a prime over l, then $\det(h) = \varepsilon \in N_{l_\mathfrak{p}/k_\mathfrak{p}}(k_\mathfrak{p}^\times)$. Thus by [Sch85, Chapter 10] h is a split hermitian form on $l_\mathfrak{p}^n$ (note that here n is definitely even and so the determinant of the split hermitian form is 1). Therefore for any $\mathfrak{p}$ other than $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$, $\mathbb{G}$ is a quasi-split $k_\mathfrak{p}$-group which splits over the maximal unramified extension. Hence we can choose a hyperspecial parahoric subgroup $P_\mathfrak{p}$. If $\mathfrak{p}$ is either $\mathfrak{p}_0$ or $\mathfrak{p}_\infty$, then again by looking at the determinant of h we can see that h is a non-split hermitian form. Thus in both of these cases $\mathbb{G}(k_\mathfrak{p})$ is isomorphic to G. Let $P_\mathfrak{p}$ be a special parahoric subgroup. Again we set $\Lambda_0 = \mathbb{G}(k) \cap \prod_{\mathfrak{p} \neq \mathfrak{p}_0} P_\mathfrak{p}$ and compute its covolume:

$$\mu(G/\Lambda_0) = q^{-\dim \mathbb{G}} \cdot q^{\mathfrak{s}(\mathbb{G})} \cdot \zeta_k(2) \cdot L_{l/k}(3) \cdot \cdots \cdot L_{l/k}(n-1) \cdot \zeta_k(n) \cdot e'(P_{\mathfrak{p}_0}) \cdot e'(P_{\mathfrak{p}_\infty}), \tag{12}$$

where $\mu(P_{\mathfrak{p}_0}) = 1$, $\mathfrak{s}(\mathbb{G}) = (n-2)(n+1)/2$ (as in Section 2.3) and

$$e'(P_{\mathfrak{p}_0}) = e'(P_{\mathfrak{p}_\infty}) = q^{n/2} - 1 \tag{13}$$

(see [MS, Lemma 2]). Similar to Section 2.3, we can compute the values of the zeta and the L-functions and altogether we get:

$$\mu(G/\Lambda_0) = (q^{n/2} - 1)^2 \cdot \prod_{i=1}^{n} (q^i - 1)^{-1}. \tag{14}$$

3. A lower bound for the covolume of uniform lattices.

In this short section, using Kazhdan-Margulis, we observe that any uniform lattice intersects a large neighborhood of the identity only at the identity and as a consequence we get a lower bound for their covolume.

Raghunathan [Rag89] gave a version of Kazhdan-Margulis theorem for rank one adjoint groups over a positive characteristic local field. However [Rag89] is good enough in order to get Kazhdan-Margulis theorem for all the lattices in a simply connected simple group over a positive characteristic local field for the following reasons:

(1) Raghunathan assumed the characteristic is a good prime as he was working with adjoint groups. However this condition is not needed for simply connected groups as all the unipotent elements of $\mathbb{G}_0(\mathfrak{f}_q((t)))$ are good (see [Sal09, Section 2] for a short survey of the definitions and the results regarding good unipotents and good primes.)
(2) By Margulis's arithmeticity theorem [Mar91, Chapter IX], any lattice in a higher rank simple group is arithmetic and by works of Harder [Har69] and Behr [Beh], one can conclude that any uniform lattice does not have a (good) unipotent.

Let us add that one can avoid arithmeticity and repeat the original proof of Kazhdan and Margulis in the positive characteristic case. This can be done (similar to [Rag89]) using the following results regarding unipotent subgroups of a simply connected simple $\mathfrak{f}_q((t))$-group:

Theorem 2 (Proposition 3.1 in [BT71] and Theorem 2 in [Gil02])**.** *Let $\mathbb{H}$ be a semisimple simply connected $\mathfrak{f}_q(\!(t)\!)$-group. Let U be a finite p-subgroup of $\mathbb{H}(\mathfrak{f}_q(\!(t)\!))$. Then there is a $\mathfrak{f}_q(\!(t)\!)$-parabolic subgroup $\mathbb{P}$ such that*

(1) $U \subseteq R_u(\mathbb{P})(\mathfrak{f}_q(\!(t)\!))$,
(2) $N_{\mathbb{H}(\mathfrak{f}_q(\!(t)\!))}(U) \subseteq \mathbb{P}(\mathfrak{f}_q(\!(t)\!))$.

Let us add that the first congruence subgroup $P^{(1)}$ of any parahoric subgroup P of $\mathbb{H}(\mathfrak{f}_q(\!(t)\!))$ can be treated as the positive characteristic version of the "Margulis-Zassenhaus" neighborhood of the identity (Since $P^{(1)}$ is a pro-p group, the intersection $\Gamma \cap P^{(1)}$ of any lattice Γ with $P^{(1)}$ is a finite p-group). Now one can use Theorem 2 to find a canonical $\mathfrak{f}_q(\!(t)\!)$-parabolic and repeat Kazhdan-Margulis argument. Thus in our setting, we have that

$$\Gamma \cap P_{\mathfrak{p}_0}^{(1)} = 1, \tag{15}$$

where Γ is a uniform lattice and $P_{\mathfrak{p}_0}^{(1)}$ is the first congruence subgroup of the parahoric subgroup $P_{\mathfrak{p}_0}$ of $\mathbb{G}(k_{\mathfrak{p}_0})$ that we chose in Section 2. In particular,

$$\mu(G/\Gamma) \geq \mu(P_{\mathfrak{p}_0}^{(1)}) = \frac{1}{[P_{\mathfrak{p}_0} : P_{\mathfrak{p}_0}^{(1)}]}. \tag{16}$$

In fact, one can get a better lower bound. By Bruhat-Tits theory [Tit79], there is a smooth $\mathfrak{f}_q[\![t]\!]$-group scheme $\mathcal{G}_0$ such that

(1) The generic fiber of $\mathcal{G}_0$ is isomorphic to $\mathbb{G}_0$.
(2) $P_{\mathfrak{p}_0} = \mathcal{G}_0(\mathfrak{f}_q[\![t]\!])$.
(3) The special fiber $\overline{\mathcal{G}}_0$ is a connected $\mathfrak{f}_q$-group and it has a Levi $\mathfrak{f}_q$-subgroup $\overline{\mathbb{M}}$.
(4) The reduction modulo t homomorphism $P_{\mathfrak{p}_0} \to \overline{\mathcal{G}}_0(\mathfrak{f}_q)$ is surjective.
(5) The kernel of the homomorphism $\phi : P_{\mathfrak{p}_0} \to \overline{\mathbb{M}}(\mathfrak{f}_q)$ is a pro-p group.

Hence by Kazhdan-Margulis if Γ is a uniform lattice in G, then

(1) $\Gamma \cap P_{\mathfrak{p}_0}$ embeds into $\overline{\mathbb{M}}(\mathfrak{f}_q)$.
(2) The order of $\Gamma \cap P_{\mathfrak{p}_0}$ is not divisible by p.

In general it is clear that $\mu(G/\Gamma) \geq 1/|\Gamma \cap P_{\mathfrak{p}_0}|$ if $\mu(P_{\mathfrak{p}_0}) = 1$. Hence one gets the following lower bounds for the covolume of uniform lattices.

Theorem 3. *In the setting of Section 2, let μ be a Haar measure on $G = \mathbb{G}_0(\mathfrak{f}_q(\!(t)\!))$ such that $\mu(P_{\mathfrak{p}_0}) = 1$ and let Γ be a uniform lattice in G.*

(1) *If $\mathbb{G}_0 \simeq \mathbb{SL}_{m,D_0}$ where D_0 is a $\mathfrak{f}_q(\!(t)\!)$-central division algebra of degree d, then*

$$\mu(G/\Gamma) \geq \frac{q-1}{\prod_{d|k, 1\leq k\leq md-1}(q^k-1)}.$$

(2) *If $\mathbb{G}_0$ is of type (O1) (see the beginning of Section 2), then*

$$\mu(G/\Gamma) \geq \prod_{i=2}^{n}(q^i - (-1)^i)^{-1}.$$

(3) *If $\mathbb{G}_0$ is of type (O2), then*

$$\mu(G/\Gamma) \geq \prod_{i=1}^{\lfloor n/2 \rfloor}(q^{2i}-1)^{-1}.$$

(4) *If $\mathbb{G}_0$ is of type (O3), then*

$$\mu(G/\Gamma) \geq \frac{q^n-1}{q+1}\prod_{i=2}^{n}(q^i - (-1)^i)^{-1}.$$

(5) *If $\mathbb{G}_0$ is of type (O4), then*

$$\mu(G/\Gamma) \geq (q^{n/2}-1)\prod_{i=1}^{n/2}(q^{2i}-1)^{-1}.$$

4. The final comparison and a few comments.

Proof of Theorem 1. We proceed by a case-by-case consideration.

(1) If $\mathbb{G}_0$ is an inner form, by Equation (2) and Theorem 3 it is enough to check that for any $q \geq 3$ one has

$$\prod_{d|k, 1\leq k\leq md-1}(q^k-1) > 1,$$

which is clear.

(2) If $\mathbb{G}_0$ is of type (O1) (see the beginning of Section 2), then by Equation (4) and Theorem 3 it is enough to check that for any $q \geq 3$ one has

$$(q-1)\prod_{i=2}^{n-1}(q^i-(-1)^i) > 1,$$

which is clear.

(3) If $\mathbb{G}_0$ is of type (O2), then by Equation (7) and Theorem 3 it is enough to check that for any $q \geq 3$ one has

$$\prod_{1\leq i\leq 2\lfloor n/2\rfloor, 2\nmid i}(q^i-1) > 1,$$

which is clear.

(4) If $\mathbb{G}_0$ is of type (O3), then by Equation (11) and Theorem 3 it is enough to check that for any $q \geq 3$ one has

$$(q^2-1)\prod_{i=2}^{n-1}(q^i+(-1)^i) > q^n-1. \tag{17}$$

Now we notice that in this case, n is an even number which is at least 4. Therefore the left hand side of the inequality (17) is at least $(q^2-1)(q^{n-1}-1)$. So in order to get the desired result, it is enough to check that for any $q \geq 3$ and $n \geq 4$ we have

$$q^{n+1}-q^n-q^{n-1}-q^2+2 > 0,$$

which is clear.

(5) If $\mathbb{G}_0$ is of type (O4), then by Equation (14) and Theorem 3 it is enough to check that for any $q \geq 3$ one has

$$\prod_{1\leq i\leq n, 2\nmid i}(q^i-1) > q^{n/2}-1,$$

which is clear as $n > 2$.

□

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