1.(a) Suppose for contradiction that for some integer n d divides n <u>and</u> d divides $n+1$. So there are integers k and k' such that

$$n = dk \quad \text{and} \quad n+1 = dk'.$$

Hence $1 = (n+1) - n = d(k'-k)$ which implies

$$\frac{1}{d} = k'-k$$

is an integer. On the other hand, $0 < \frac{1}{d} < 1$ as $d > 1$. Thus $0 < k'-k < 1$ which is a contradictions as there is no integer larger than 0 and less than 1. ■

(b) Suppose to the contrary that there is an integer $d > 1$ such that $d \mid n_1$ and $d \mid n_2$. Hence for some integers k_1 and k_2 we have

$$n_1 = d k_1 \quad \text{and} \quad n_2 = d k_2.$$

Thus $1 = r_1 n_1 + r_2 n_2 = d r_1 k_1 + d r_2 k_2 = d(r_1 k_1 + r_2 k_2)$.

Similar to part (a), we conclude that

$$0 < \frac{1}{d} = r_1 k_1 + r_2 k_2 < 1,$$

which is a contradiction. ∎

2. Construction of a proof backwards:

$$(ac+bd)^2 \le (a^2+b^2)(c^2+d^2) \iff$$

$$\cancel{a^2c^2}+\cancel{b^2d^2}+2abcd \le \cancel{a^2c^2}+a^2d^2+b^2c^2+\cancel{b^2d^2} \iff$$

$$2\,ad\,bc \le (ad)^2+(bc)^2.$$

Proof. For any real numbers x and y we have

$$2xy \le x^2+y^2.$$

Let $x = ad$ and $y = bc$. So

$$2(ad)(bc) \le (ad)^2+(bc)^2.$$

$$\Longrightarrow a^2c^2+b^2d^2+2acbd \le a^2c^2+b^2d^2+a^2d^2+b^2c^2.$$

$$\Longrightarrow (ac+bd)^2 \le (a^2+b^2)(c^2+d^2).$$ ∎

3. (a) We use the truth-table

P	Q	$\neg P$	$(\neg P) \Rightarrow Q$
T	F	F	T
~~F~~	~~F~~	~~T~~	~~F~~

$(\neg P) \Rightarrow Q$ is true only in the first row. In the first row P is true. ∎

(b) You can use the truth-table <u>or</u> the following direct proof.

$$(P \wedge (\neg Q)) \Rightarrow O \equiv (\neg((\neg P) \vee Q)) \Rightarrow O \quad \text{(De Morgan's law)}$$

$$(\neg((\neg P) \vee Q)) \Rightarrow O \vdash (\neg P) \vee Q \quad \text{(Part (a))}$$

$$(\neg P) \vee Q \equiv P \Rightarrow Q \quad \text{(Discussed (*) in class.)}$$

$\blacksquare$

(c) You can use the truth-table <u>or</u> the following direct proof.

$$(P \Rightarrow R) \wedge (Q \Rightarrow R) \equiv (\neg P \vee R) \wedge (\neg Q \vee R)$$

$$\equiv ((\neg P) \wedge (\neg Q)) \vee R \quad \text{(distribution law)}$$

$$\equiv (\neg (P \vee Q)) \vee R \quad \text{(De Morgan's law)}$$

$$\equiv (P \vee Q) \Rightarrow R. \quad \text{(Similar to (*))}$$

$\blacksquare$

4. (a) $a \mid b_1 - b_2 \Rightarrow$ for some integer k, $b_1 - b_2 = ak$ (I)

$a \mid c_1 - c_2 \Rightarrow$ for some integer k', $c_1 - c_2 = ak'$ (II)

Add (I) and (II) : $(b_1 + c_1) - (b_2 + c_2) = a(k + k')$.

So $a \mid (b_1 + c_1) - (b_2 + c_2)$. $\blacksquare$

(b) $b_1 c_1 - b_2 c_2 = (b_1 - b_2) c_1 + b_2 (c_1 - c_2)$ (Hint)

$= a k c_1 + a k' b_2$ ((I) and (II))

$= a(k c_1 + k' b_2)$.

So $a \mid b_1c_1 - b_2c_2$ (note that $kc_1 + k'b_2$ is an integer.). $\blacksquare$

5.(a) ($\Rightarrow$) By the definition $\lfloor x \rfloor = n$ is largest integer less than or equal to x. Since $n+1 > n$, it cannot be less than or equal to x. Therefore $x < n+1$, i.e.

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

($\Leftarrow$) Assume n is an integer such that

$$n \leq x < n+1.$$

Then since $n \leq x$ and $\lfloor x \rfloor$ is the largest integer less than or equal to x, we have that $n \leq \lfloor x \rfloor$.

Since $x < n+1$, $\lfloor x \rfloor < n+1$. If not, then

$$\left.\begin{array}{l} x < n+1 \\ n+1 \leq \lfloor x \rfloor \end{array}\right\} \Rightarrow x < \lfloor x \rfloor \quad \text{which is a contradiction.}$$

Hence $\lfloor x \rfloor$ is an integer such that

$$n \leq \lfloor x \rfloor < n+1.$$

Therefore $n = \lfloor x \rfloor$ (there is no integer m such that $n < m < n+1$.). $\blacksquare$

(b) By part (a), we have

$$\lfloor -x \rfloor \le -x < \lfloor -x \rfloor + 1.$$

Since x is NOT an integer, we have

$$\lfloor -x \rfloor < -x < \lfloor -x \rfloor + 1.$$

Thus $$-\lfloor -x \rfloor > x > -\lfloor -x \rfloor - 1.$$

Again by part (a) we have $\lfloor x \rfloor = -\lfloor -x \rfloor - 1$. Hence

$$\lfloor x \rfloor + \lfloor -x \rfloor = -1. \qquad \blacksquare$$

(c) Let $y = x - \lfloor x \rfloor$. So $x = \lfloor x \rfloor + y$, $x + \frac{1}{2} = \lfloor x \rfloor + y + \frac{1}{2}$ and

$$2x = 2\lfloor x \rfloor + 2y.$$

Case $0 \le y < 1/2$.

$$0 \le y < 1/2 \Rightarrow 0 \le y + 1/2 < 1$$

$$\Rightarrow \lfloor x \rfloor \le x + 1/2 < \lfloor x \rfloor + 1$$

$$\Rightarrow \lfloor x + 1/2 \rfloor = \lfloor x \rfloor \quad \text{I} \qquad \text{(By part (a))}.$$

$$0 \le y < 1/2 \Rightarrow 0 \le 2y < 1$$

$$\Rightarrow 2\lfloor x \rfloor \le 2x < 2\lfloor x \rfloor + 1$$

$$\Rightarrow \lfloor 2x \rfloor = 2\lfloor x \rfloor \quad \text{II} \qquad \text{(By part (a))}$$

I and II $\Rightarrow \lfloor 2x \rfloor = 2\lfloor x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor.$

Case $\frac{1}{2} \leq y < 1$.

$$\tfrac{1}{2} \leq y < 1 \implies 1 \leq y + \tfrac{1}{2} < 2$$

$$\implies \lfloor x \rfloor + 1 \leq x + \tfrac{1}{2} < \lfloor x \rfloor + 2$$

$$\implies \lfloor x + \tfrac{1}{2} \rfloor = \lfloor x \rfloor + 1 \quad \text{III} \qquad \text{(By part (a))}$$

$$\tfrac{1}{2} \leq y < 1 \implies 1 \leq 2y < 2$$

$$\implies 2\lfloor x \rfloor + 1 \leq 2x < 2\lfloor x \rfloor + 2$$

$$\implies \lfloor 2x \rfloor = 2\lfloor x \rfloor + 1 \quad \text{IV} \qquad \text{(By part (a))}$$

$$\text{III and IV} \implies \lfloor 2x \rfloor = 2\lfloor x \rfloor + 1$$

$$= \lfloor x \rfloor + (\lfloor x \rfloor + 1)$$

$$= \lfloor x \rfloor + \lfloor x + \tfrac{1}{2} \rfloor . \quad \blacksquare$$

6. Construction of a proof backwards:

$$a^2 + b^2 + c^2 \geq \tfrac{1}{3}(a+b+c)^2 \iff$$

$$3(a^2+b^2+c^2) \geq (a+b+c)^2 \iff$$

$$3a^2 + 3b^2 + 3c^2 \geq a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \iff$$

$$2(a^2+b^2+c^2) \geq 2(ab+ac+bc).$$

__Proof.__ In the class we proved that for any real numbers a, b and c we have

$$ab+ac+bc \leq a^2+b^2+c^2.$$

$$\Rightarrow 2ab+2ac+2bc \leq 2a^2+2b^2+2c^2$$

$$\Rightarrow a^2+b^2+c^2+2ab+2ac+2bc \leq 3(a^2+b^2+c^2)$$

$$\Rightarrow (a+b+c)^2 \leq 3(a^2+b^2+c^2)$$

$$\Rightarrow \frac{(a+b+c)^2}{3} \leq a^2+b^2+c^2. \quad \blacksquare$$