1.(a) <u>f is NOT 1-1</u>: $(0,0) \neq (2,3)$ and

$$f(0,0) = 0 = f(2,3)$$

<u>f is onto</u>: $\forall n \in \mathbb{Z}$, $f(n,n) = 3n - 2n = n$.

(b) $(f \circ f)(B) = f(A \triangle B)$
$= A \triangle (A \triangle B)$
$= (A \triangle A) \triangle B$
$= \emptyset \triangle B = B$

$f \circ f : P(X) \to P(X)$

$\Rightarrow f \circ f = I_{P(X)} \Rightarrow$

f is invertible $\Rightarrow$

f is a bijection $\Rightarrow$

f is <u>1-1</u> and <u>onto</u>.

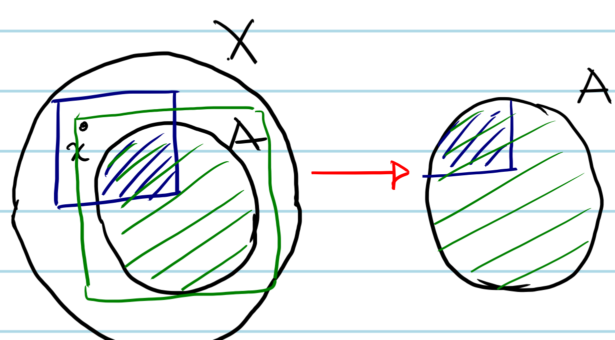
(c) [**Remark:**
<u>f is well-defined</u>: $A \cap B \subseteq A \Rightarrow A \cap B \in P(A)$.]

<u>f is NOT 1-1</u>: Since A is a proper subset of X,

$X \setminus A$ is NOT empty. Let $x \in X \setminus A$. Then

$f(A \cup \{x\}) = (A \cup \{x\}) \cap A = A = f(A)$

and $A \cup \{x\} \neq A$.

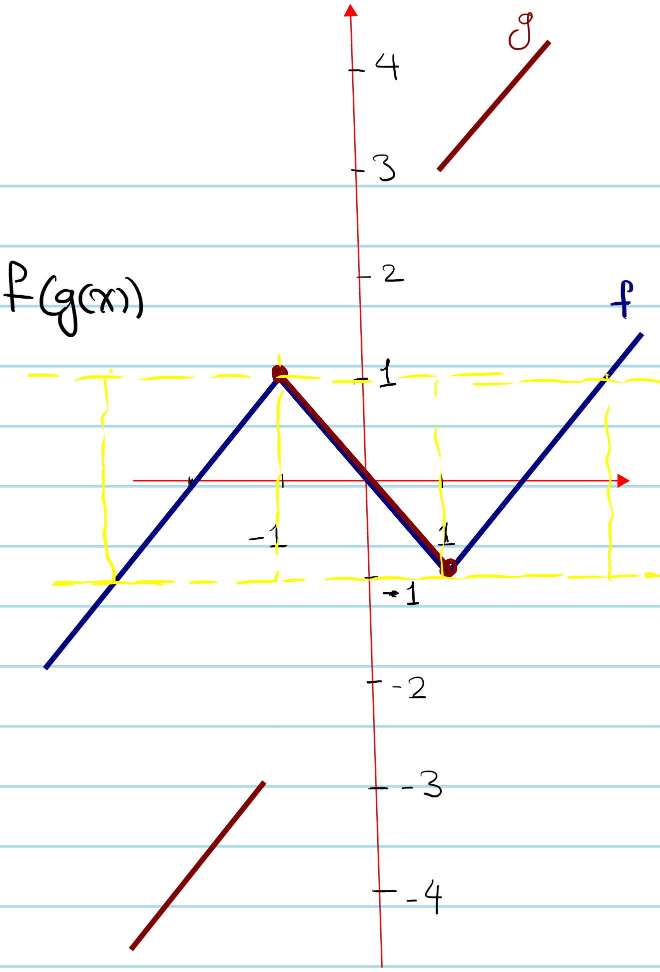


<u>f is onto</u>:

$\forall A' \subseteq A$, (A' is also a subset of X.)

$f(A') = A' \cap A = A'$.

2. $f \circ g(x) = f(g(x))$



In order to find the value of $f(g(x))$ we have to find out, for what values of x, $g(x) < -1$ and $-1 \le g(x) \le 1$ and $1 < g(x)$, i.e.

$\overleftarrow{g}((-\infty, -1))$,

$\overleftarrow{g}([-1, 1])$ and

$\overleftarrow{g}((1, \infty))$. Looking at the graph of g, we have $\overleftarrow{g}((-\infty,-1)) = (-\infty, -1)$, $\overleftarrow{g}([-1,1]) = [-1,1]$ and $\overleftarrow{g}((1,\infty)) = (1,\infty)$. So

$$f \circ g(x) = f(g(x)) = \begin{cases} f(x-2) & \text{if } x < -1 \\ f(-x) & \text{if } -1 \le x \le 1 \\ f(x+2) & \text{if } x > 1 \end{cases}$$

$$= \begin{cases} f(x-2) & \text{if } x-2 < -3 < -1 \\ f(-x) & \text{if } -1 \le -x \le 1 \\ f(x+2) & \text{if } x+2 > 3 > 1 \end{cases}$$

$$= \begin{cases} (x-2)+2 & \text{if} \quad x<-1 \\ -(-x) & \text{if} \quad -1 \leq x \leq 1 \\ (x+2)-2 & \text{if} \quad x>1 \end{cases}$$

$$= x.$$

So $f \circ g : \mathbb{R} \to \mathbb{R}$, $f \circ g = I_{\mathbb{R}}$.

$g \circ f(x) = g(f(x))$ as above we need to find $\overleftarrow{f}((-\infty,-1))$, $\overleftarrow{f}([-1,1])$ and $\overleftarrow{f}((1,\infty))$.

Looking at the graph of f, we have

$\overleftarrow{f}((-\infty,-1)) = (-\infty,-3)$,

$\overleftarrow{f}([-1,1]) = [-3,3]$ and

$\overleftarrow{f}((1,\infty)) = (3,\infty)$.

So we will consider these intervals separately:

$(-\infty,-3)$, $[-3,-1)$, $[-1,1]$, $(1,3]$, $(3,\infty)$

$x<-3$

$$\begin{aligned} g \circ f(x) &= g(x+2) && (x<-3<-1) \\ &= (x+2)-2 && (x+2<-1) \\ &= x. \end{aligned}$$

$-3 \leq x < -1$

$$g \circ f(x) = g(x+2) \quad (x<-1)$$
$$= -x-2 \quad (-1 \leq x+2 \leq -1)$$

$-1 \leq x \leq 1$

$$g \circ f(x) = g(-x) \quad (-1 \leq x \leq 1)$$
$$= -(-x) \quad (-1 \leq -x \leq 1)$$
$$= x$$

$1 < x \leq 3$

$$g \circ f(x) = g(x-2) \quad (1<x)$$
$$= -x+2 \quad (-1 \leq x-2 \leq 1)$$

$3 < x$

$$g \circ f(x) = g(x-2) \quad (1<x)$$
$$= (x-2)+2 \quad (1<x-2)$$
$$= x$$

So overall we have

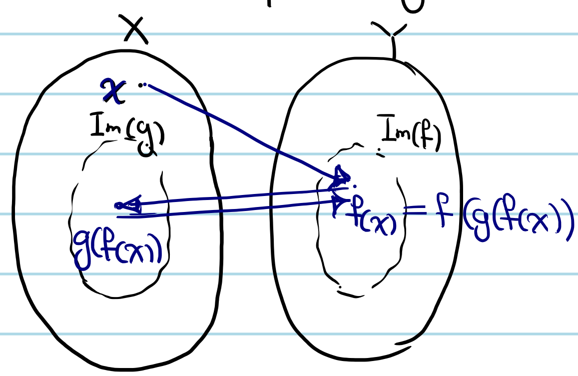
$$g \circ f(x) = \begin{cases} x & \text{if } x<-3 \\ -x-2 & \text{if } -3 \leq x < -1 \\ x & \text{if } -1 \leq x \leq 1 \\ -x+2 & \text{if } 1<x \leq 3 \\ x & \text{if } 3<x. \end{cases}$$

No, g is NOT the inverse of f as $g \circ f \neq I_{\mathbb{R}}$.

- $f \circ g = I_{\mathbb{R}} \implies g$ is 1-1 and f is onto.
- <u>f is NOT 1-1</u>. $f(-2) = f(0)$.
- <u>g is NOT onto</u> as $-2 \notin \text{Im}(g)$ (Check it yourself.)

3. As we discussed in class, f_1 and g_1 are bijections.

($\Rightarrow$) Suppose f is 1-1, we would like to prove g is onto.



If not, there is $x \in X \setminus \text{Im}(g)$.

Then $f(x) = f(g(f(x)))$

$\implies x = g(f(x))$ as f is 1-1

$\implies x \in \text{Im}(g)$ which is a contradiction.

($\Leftarrow$) Suppose g is onto, we would like to prove f is 1-1.

Since $\text{Im}(g) = X$, the domain of f_1 is the entire X.

$f(x_1) = f(x_2) \implies f_1(x_1) = f_1(x_2) \xrightarrow[\text{a bijection}]{f_1 \text{ is}} x_1 = x_2$.

Therefore f is 1-1.

4. (a) True. $x \in \overleftarrow{f}(B_1) \Rightarrow f(x) \in B_1$ (by definition)

$\Rightarrow f(x) \in B_2$ $(B_1 \subseteq B_2)$

$\Rightarrow x \in \overleftarrow{f}(B_2)$ (by definition)

$\Rightarrow \overleftarrow{f}(B_1) \subseteq \overleftarrow{f}(B_2)$.

(b) True. $x \in \overleftarrow{f}(B_1 \cap B_2) \Leftrightarrow f(x) \in B_1 \cap B_2$

$\Leftrightarrow f(x) \in B_1 \wedge f(x) \in B_2$

$\Leftrightarrow x \in \overleftarrow{f}(B_1) \wedge x \in \overleftarrow{f}(B_2)$

$\Leftrightarrow x \in \overleftarrow{f}(B_1) \cap \overleftarrow{f}(B_2)$

$\Rightarrow \overleftarrow{f}(B_1 \cap B_2) = \overleftarrow{f}(B_1) \cap \overleftarrow{f}(B_2)$

(c) False. Let $X = \{1,2,3\}$ and $Y = \{4,5\}$. Let

$f: X \to Y$, $f(1) = 4$, $f(2) = 5$, $f(3) = 4$.

Let $A_1 = \{1,2\}$ and $A_2 = \{2,3\}$

$\Rightarrow A_1 \cap A_2 = \{2\}$

$\overrightarrow{f}(A_1) = \overrightarrow{f}(A_2) = \{4,5\}$ and $\overrightarrow{f}(A_1 \cap A_2) = \{5\}$

$\Rightarrow \overrightarrow{f}(A_1 \cap A_2) = \{5\} \neq \{4,5\} = \overrightarrow{f}(A_1) \cap \overrightarrow{f}(A_2)$.

5. $\forall A \in P(X),\ \overleftarrow{f}(\overrightarrow{f}(A)) \supseteq A$ and

$\forall B \in P(Y),\ \overrightarrow{f}(\overleftarrow{f}(B)) \subseteq B$.

Pf. (a) $x \in A \Rightarrow f(x) \in \overrightarrow{f}(A) \Rightarrow x \in \overleftarrow{f}(\overrightarrow{f}(A))$

So $A \subseteq \overleftarrow{f}(\overrightarrow{f}(A))$.

(b). $y \in \overrightarrow{f}(\overleftarrow{f}(B)) \Rightarrow y = f(x)$ for some $x \in \overleftarrow{f}(B)$

$\Rightarrow y = f(x) \wedge f(x) \in B$

$\Rightarrow y \in B$.

So $\overrightarrow{f}(\overleftarrow{f}(B)) \subseteq B$.

6. • $\overleftarrow{f}(\overrightarrow{f}(A)) \supseteq A \Rightarrow \overrightarrow{f}(\overleftarrow{f}(\overrightarrow{f}(A))) \supseteq \overrightarrow{f}(A)$ (I)

• For $B = \overrightarrow{f}(A)$, we have $\overrightarrow{f}(\overleftarrow{f}(B)) \subseteq B \Rightarrow$

$\overrightarrow{f}(\overleftarrow{f}(\overrightarrow{f}(A))) \subseteq \overrightarrow{f}(A)$ (II)

(I), (II) $\Rightarrow \overrightarrow{f}(\overleftarrow{f}(\overrightarrow{f}(A))) = \overrightarrow{f}(A)$.

• For $A = \overleftarrow{f}(B)$, we have $\overleftarrow{f}(\overrightarrow{f}(A)) \supseteq A$. Hence

$\overleftarrow{f}(\overrightarrow{f}(\overleftarrow{f}(B))) \supseteq \overleftarrow{f}(B)$. (III)

• $\overrightarrow{f}(\overleftarrow{f}(B)) \subseteq B \Rightarrow \overleftarrow{f}(\overrightarrow{f}(\overleftarrow{f}(B))) \subseteq \overleftarrow{f}(B)$ (IV)

(III), (IV) $\Rightarrow \overleftarrow{f}(\overrightarrow{f}(\overleftarrow{f}(B)))$.

7. f injective $\iff \vec{f}$ injective.

($\Rightarrow$) Suppose to the contrary $\exists A_1, A_2 \in \mathcal{P}(X)$, $A_1 \neq A_2 \wedge$ $\vec{f}(A_1) = \vec{f}(A_2)$.

$$A_1 \neq A_2 \Rightarrow \exists x_0, (x_0 \in A_1 \wedge x_0 \notin A_2) \vee (x_0 \in A_2 \wedge x_0 \notin A_1) \quad (\mathrm{I})$$

$\Rightarrow f(x_0) \in \vec{f}(A_1) \cup \vec{f}(A_2) = \vec{f}(A_1) = \vec{f}(A_2)$

$\Rightarrow \exists x_1 \in A_1, \exists x_2 \in A_2, \ f(x_0) = f(x_1) = f(x_2)$

$\xRightarrow[\text{injective}]{f \text{ is}} \quad x_0 = x_1 = x_2 \quad \Rightarrow x_0 \in A_1 \cap A_2$ which contradicts (I)

($\Leftarrow$) Suppose to the contrary that f is NOT injective.

So $\exists x_1, x_2 \in X$, $x_1 \neq x_2 \wedge f(x_1) = f(x_2)$.

$\Rightarrow \vec{f}(\{x_1\}) = \{f(x_1)\} = \{f(x_2)\} = \vec{f}(\{x_2\})$

$\Rightarrow \{x_1\} = \{x_2\}$ as $\vec{f}$ is injective.

$\Rightarrow x_1 = x_2$, which is a contradiction.

$\vec{f}$ injective $\iff \overleftarrow{f}$ surjective.

($\Rightarrow$) This is a corollary of Problem $\underline{\underline{2}}$ and $\underline{\underline{3.a}}$.

<u>Remark</u>. Using Problem 2 and 3.a, we also have

$$\vec{f} \text{ surjective} \iff \overleftarrow{f} \text{ injective.}$$

$\forall A \subseteq X$, we have

8. (a) $z \in \overrightarrow{g \circ f}(A) \Rightarrow \exists a \in A,\ z = g \circ f(a)$

$\Rightarrow \exists a \in A,\ z = g(f(a))$

$\Rightarrow f(a) \in \vec{f}(A) \wedge z = g(f(a))$

$\Rightarrow z \in \vec{g}(\vec{f}(A)) = \vec{g} \circ \vec{f}(A).$

$z \in \vec{g} \circ \vec{f}(A) = \vec{g}(\vec{f}(A)) \Rightarrow \exists\ y \in \vec{f}(A),\ z = g(y)$

$\Rightarrow \exists a \in A,\ y = f(a) \wedge z = g(y)$

$\Rightarrow \exists a \in A,\ z = g(f(a))$

$= (g \circ f)(a)$

$\Rightarrow z \in \overrightarrow{g \circ f}(A).$

Thus $\overrightarrow{g \circ f}(A) = \vec{g} \circ \vec{f}(A)$.

(b) $x \in \overleftarrow{g \circ f}(C) \Leftrightarrow (g \circ f)(x) \in C$

$\Leftrightarrow g(f(x)) \in C$

$\Leftrightarrow f(x) \in \overleftarrow{g}(C)$

$\Leftrightarrow x \in \overleftarrow{f}(\overleftarrow{g}(C)) = \overleftarrow{f} \circ \overleftarrow{g}(C).$