Practice Problems

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1 Notation

Unless otherwise stated, $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$ denote the additive groups of the complex, real, rational numbers and integers respectively. $\mathbb{C}^\times, \mathbb{R}^\times, \mathbb{Q}^\times$ denotes the group of nonzero complex, real, rational numbers under multiplication respectively. $GL_n(\mathbb{R})$ denotes the multiplicative group of $n \times n$ matrices with nonzero determinant while $SL_n(\mathbb{R})$ denotes the subgroup of $n \times n$ matrices with determinant 1. Lastly, For a set A of a group G, let $C_G(A) := \{g \in G \mid ga = ag \ \forall a \in A\}$ be its commutator subgroup.

2 Exercises

2.1 Dihedral groups

From here on, we will use the usual presentation of the Dihedral group of order $2n$: $D_n = \langle R, F \mid R^n = F^2 = 1, FR = R^{-1}F \rangle$. Recall here that R represents a roation (counterclockwise) by an angle of $2\pi/n$ and F is a reflection about a line through one of the vertices of the regular n-gon.

1. Let $x \in D_n$, $x \notin \langle R \rangle$. Show that $Rx = R^{-1}x$.

2. Let G be the group generated by two elements a and b, such that $a^2 = b^2 = (ab)^4 = e$. Show that this group is finite. Show that $G \cong D_4$.

3. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle in the complex plane. You proved in the homework that S^1 is a multiplicative subgroup of $\mathbb{C}^\times$. Describe the cosets of S^1. Prove that $\mathbb{C}^\times / S^1 \cong \mathbb{R}$.

4. Show that D_5 is isomorphic to the subgroup of $GL_2(\mathbb{R})$ generated by the matrices

$$\begin{bmatrix} cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $\theta = 2\pi/5$.

2.2 Symmetric groups

Let X be a set. Recall that the group of permutations, S_X, on X is defined to be the group of all bijective functions from X to itself where the group operation is given by function composition.

1. Let

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{bmatrix}, \tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{bmatrix}$$

   Find the cycle decompositions of the following permutations: $\sigma, \tau, \sigma^2, \sigma\tau, \tau\sigma$ and $\tau^2\sigma$.

2. Find the order of (1 12 8 10 4)(2 13)(5 11 7)(6 9).

3. Let $\Omega = \{1, 2, 3, \ldots\}$. Prove that $|S_\Omega|$ is infinite. Hint: $\infty! = \infty$ is not a valid solution.

4. (a) Let σ be the 12-cycle (1 2 3 4 5 6 7 8 9 10 11 12). For which positive integers i is σ^i also a 12-cycle?
   (b) Let τ be the 8-cycle (1 2 3 4 5 6 7 8). For which positive integers i is τ^i also an 8-cycle?
   (c) Let ω be the 14-cycle (1 2 3 4 5 6 7 8 9 10 11 12 13 14). For which positive integers i is ω^i also a 14-cycle?

2.3 Homomorphisms and Isomorphisms

1. Prove $\mathbb{R}^\times \ncong \mathbb{C}^\times$.

2. Prove $\mathbb{Z} \ncong \mathbb{Q}$

3. Prove $\mathbb{R} \ncong \mathbb{Q}$.

4. Let $\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$. $\mathbb{R}^2$ is a group under componentwise addition. Show that the function $\pi : \mathbb{R}^2 \to \mathbb{R}$ given by $\pi(x, y) = x$ is a group homomorphism. What are the cosets of $\ker(\pi)$ in $\mathbb{R}^2$?

5. Let $T \subset GL_n(\mathbb{R})$ be the set of invertible diagonal matrices. Prove that $T \cong (\mathbb{R}^\times)^n$. (If you are stuck, try the small case when $n = 2$ and then generalize.)

6. Show that $\mathbb{R} \cong \mathbb{R}^\times_{>0}$ where the first group is the additive group of real numbers and the latter is the multiplicative group of positive real numbers.

7. Recall that for a group G, we define the group $Aut(G)$ to be the group of isomorphisms from G to itself, where the group operation is given by function composition. Find $Aut(\mathbb{Z})$.

8. Let $f, g : \mathbb{R} \to \mathbb{R}$ be real valued functions defined by $f(x) = 1/x$ and $g(x) = (x-1)/x$. f and g generate a group G with the operation given by function composition. Prove that $G \cong S_3$.

9. (**Direct Products**) Let G and G' be two groups. Define their *direct product* $G \times G'$ to be the group of all pairs $(g, g') \in G \times G'$ where the group operation is defined by $(g_1, g_1')(g_2, g_2') = (g_1 g_2, g_1 g_2')$.

   (a) Prove that $G \times G'$ is a group.

   (b) Prove that $H_1 = G \times \{e\}$ and $H_2 = e \times G'$ are both subgroups of $G \times G'$.

   (c) Prove that the H_1 and H_2 are both normal in $G \times G'$.

   (d) Prove that if $h_1 \in H_1, h_2 \in H_2$ then $h_1 h_2 = h_2 h_1$.

10. Is $S_3 \cong H_1 \times H_2$ for any two subgroups H_1, H_2 of S_3?

2.4 Subgroups

1. Find an example of a group G and an infinite subset H of G such that H is closed under multiplication but not inversion.

2. Let H and K be two subgroups of a group G. Show that $H \cup K$ is a subgroup of G if and only if $H \subset K$ or $K \subset H$.

3. Let $A \subset B$ be two subsets of a group G. Show that $C_G(B) \leq C_G(A)$.

4. Let H be a subgroup of a group G. Show that $H \leq C_G(H)$ if and only if H is abelian.

5. Show that $GL_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{C})$.

6. Show that if a group G has exactly one element a of order 2, then $a \in Z(G)$.

7. Let

$$H(\mathbb{R}) := \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

   Show that $H(\mathbb{R})$ is a subgroup of $SL_3(\mathbb{R})$.[1] Is $H(\mathbb{R})$ normal in $SL_3(\mathbb{R})$? Find $Z(H(\mathbb{R}))$.

8. Let A and B be two subsets of a group G. Define their product AB to be the set $\{ab \in G \mid a \in A, b \in B\}$. Let H and N be two subgroups of a group G and suppose that N is normal in G. Show that HN is a subgroup of G. (It actually suffices to assume that H normalizes N. That is, for every $h \in H$ and $n \in N$, $hnh^{-1} \in N$).

[1] $H(\mathbb{R})$ is called the *Heisenberg group* of $\mathbb{R}$

9. Show that if H and K are subgroups of a group G such that HK is again a subgroup of G. Then $HK = KH$.

10. (**Derived subgroups**) Let G be a group and let $a, b \in G$. The *commutator* of a and b is defined to be the element $[a, b] := aba^{-1}b^{-1} \in G$. The *first derived subgroup* of G, $[G, G]$ is the subgroup of G generated by all elements of the form $[a, b]$ for $a, b \in G$.

    (a) Prove that $[G, G]$ is a subgroup of G. Hint: it suffices to check that the product of two commutators is a commutator, and the inverse of a commutator is a commutator.

    (b) Prove that $[G, G]$ is normal in G. Hint: show that $g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}]$ for $a, b, g \in G$.

    (c) Prove that the factor group $G/[G, G]$ is abelian.

    (d) Prove that if $\phi : G \to G'$ is a group homomorphism from G to an abelian group G', then $[G, G] \leq ker(\phi)$.

    (e) Prove that $[S_n, S_n] = A_n$. Hint: use part (d) to show that $[S_n, S_n] \subset A_n$. For the other inclusion, show that any 3-cycle can be written as a commutator, and then use the fact that A_n is generated by 3-cycles.

    (f) Use the previous part to show that there is only one homomorphism from S_n onto ± 1.

11. (a) Show that the relation "$a \sim b$ is and only if $a = gbg^{-1}$ for some $g \in G$" is an equivalency relation.

    (b) Let $\mathcal{O}_G(a) = \{b \in G \mid a \sim b\}$. Use part ($a$) to deduce that the sets $\{\mathcal{O}_G(g)\}_{g \in G}$ partition G. $\mathcal{O}_G(a)$ is called the *conjugacy class of a*.

    (c) Let $a \in G$. Prove that the function

    $$\phi : G/C_G(a) \longrightarrow \mathcal{O}_G(a)$$
    $$\phi(xC_G(a)) = xax^{-1}$$

    is a well defined bijection. Warning: $G/C_G(a)$ is not a group necessarily.

    (d) Assume G is finite. Deduce the **class equation**:

    $$|G| = |Z(G)| + \sum_{a \in \Omega} [G : C_G(a)]$$

    , where Ω is a set of representatives of conjugacy classes of order greater than 1.

12. p**-groups**: Let G be a group of order p^n where p is a prime number and n is a positive integer. Use LaGrange's theorem and the class equation to prove that $Z(G) \neq \{e\}$. Show that any group of order p^2 is abelian. Give an example of a nonabelian group of order p^3.

2.5 The first isomorphism theorem

1. Let m, n be coprime. Show that there is no nontrivial homomorphism from Z_m to Z_n.

2. For which natural numbers m is there surjective homomorphism from D_{17} to Z_m. What if the homomorphism is not required to be surjective?

3. Show that $GL_2(\mathbb{R})/SL_2(\mathbb{R}) \cong \mathbb{R}^\times$.

4. (**The second isomorphism theorem**) Let G be a group, and let A and B be normal subgroups[2]. Then AB is a subgroup of G. Prove that B is normal in AB, $A \cap B$ is normal in A, and that

$$A/A \cap B \cong AB/B$$

   Hint: Find a homomorphism from A to AB/B with kernel $A \cap B$ and use the first isomorphism theorem.

5. (**The third isomorphism theorem**) Let G be a group and let H and K be two normal subgroups. Suppose $H \leq K$. Prove that K/H is a normal subgroup of G/H and that

$$(G/H)/(K/H) \cong G/K$$

   .

   Hint: Find a homomorphism from G/H to G/K whose kernel is K/H and use the first isomorphism theorem.

3 Hints for the exercises

3.1 Dihedral groups

1. If $x \in D_n$ is not a rotation, then $x = R^i F$ for some $i \in \{1, \ldots, n-1\}$. Use the relation $FR = R^{-1}F$ to complete the problem.

2. Notice that the elements F and R^3F satisfy the relations $F^2 = e, (R^3F)^2 = e$, and $(FR^3F)^4 = R^4 = e$. Show that the function $\phi : G \to D_4, \phi(a) = F, \phi(b) = R^3F$ is an isomorphism.

3. The function $\phi : \mathbb{C}^\times \to \mathbb{R}^\times_{>0}, \phi(z) = |z|$ is a surjective homomorphism with kernel S^1. The first isomorphism theorem finishes the proof.

[2] You actually only need that A normalizes B.

4. The first matrix rotates the plane $\mathbb{R}^2$ by an angle of $2\pi/5$ while the second matrix reflects the plane about the line $y = x$. Show that the function $\phi(R) = \begin{bmatrix} cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \phi(F) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an isomorphsim

3.2 Symmetric groups

1. Simple calculation. See textbook.
2. The order of disjoint cycles is the least common multiple of their cycle lengths.
3. Show that for every n, S_n is a subgroup of S_Ω. Conclude that $|S_\Omega| \geq |S_n|$ for every positive integer n. Therefore $|S_\Omega| = \infty$.
4. This is essentially a question about cyclic groups. If σ is an n-cycle, σ^i is an n-cycle if and only if $gcd(i, n) = 1$.

3.3 Homomorphisms and Isomorphisms

1. $\mathbb{R}^\times$ has only two elements of finite order whereas $\mathbb{C}^\times$ has infinitely many.
2. $\mathbb{Q}$ is not cyclic. (Prove this!).
3.
4. The cosets of $\ker(\pi)$ are the lines parallel to the x-axis.
5. $\phi(x) = e^x$ is an isomorphism.
6. $Aut(\mathbb{Z}) \cong Z_2$. Any isomorphism must take a generator of $\mathbb{Z}$ to another generator. The only generators of $\mathbb{Z}$ are 1 and -1. Therefore the only automorphisms are the identity map, and the function that takes n to $-n$.
7. The function that takes f to (1 2) and g to (1/2/3) is an isomorphism.
8. everything should follow straight from the definitions.
9. No. If H is a subgroup of S_3, then $|H| \mid 6$. Therefore $|H| = 1, 2, 3$ or 6. If $|H| = 1$, then $H = \{e\}$. If $|H|$ is 6, then $H = S_3$. Therefore the only possibilities for H_1 and H_2 is that H_1 an order 2 cyclic group and H_2 s an order 3 cyclic group. In this case, $H_1 \times H_2 \cong Z_6 \ncong S_3$.

3.4 Subgroups

1. Take $G = \mathbb{C}^\times$ and $H = \{z \in \mathbb{C}^\times \mid |z| > 1\}$.

2. If say $H \subset K$, the $H \cup K = K$, which is a subgroup by assumption. Suppose $H \cup K$ is a subgroup of G and that $H \not\subset K$. Choose an element $h \in H \setminus K$, and $k \in K$. Since $H \cup K$ is a subgroup, $hk \in H \cup K$. That is, either $hk \in H$ or $hk \in K$. If $hk \in K$, then $hkk^{-1} = h \in K$ which is a contradiction. Therefore for every $k \in K$, $hk \in H$. But then $h^{-1}hk = k \in H$ which shows $K \subset H$.

3. If $g \in G$ satusfies $gbg^{-1} = b$ for all $b \in B$, then $gag^{-1} = a$ for all $a \in A$ since $A \subset B$.

4. Quickly follows from the definition of centralizer.

5. This is just a quick calculation.

6. If a is an element of order 2, then for any $g \in G$, gag^{-1} also has order 2.

7. A quick calculation shows that $H(\mathbb{R})$ is a subgroup. Conjugate by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

to see that $H(\mathbb{R})$ is not normal. The center is matrices of the form

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. The proof of this is actually in the book.

9. If $kh \in KH$, then $kh = (h^{-1}k^{-1})^{-1} \in HK$ since HK is a subgroup. Do a similar trick for the reverse inclusion.

10. Most of this exercise was in your homework. To show part (f), note that $[S_n, S_n] = A_n$ is in the kernel of any such homomorphism since $\{\pm 1\}$ is an abelian group. By the first isomorphism theorem, the size of the kernel of such a homomorphism is $n!/2$. Therefore the kernel is exactly A_n, which tells us that the only homomorphism is the sign homomorphism.

11. Will add solutions to these later.

3.5 The first isomorphism theorem

1. If $\phi : Z_n \to Z_m$ were a homomorphism, then by the first isomorphism theorem, $|Z_n|/|\ker(\phi)| = |im(\phi)|$. In particular, $|im(\phi)|$ divides $|Z_n| = n$. On the other hand, $im(\phi)$ is a subgroup of Z_m and so $|im(\phi)|$ divides $|Z_m| = m$. Since m and n are coprime, $|im(\phi)| = 1$.

2. If $\phi : D_{17} \to Z_m$ were a surjective homomorphism, then $|im(\phi)| = |Z_m| = m$ divides $|D_{17}| = 34$ by the first isomorphism theorem. So we can narrow m down to $1, 2, 17, 34$. If $m = 34$, then ϕ would actually be injective, and hence ϕ would be an isomorphism. However, D_{17} is not cyclic. So the only possibilities are $1, 2, 17$. Try to find an example for each.

3. The determinant function $det : GL_n(\mathbb{R}) \to \mathbb{R}^{\times}$ is a surjective homomorphism with kernel $SL_n(\mathbb{R})$.

4. The function $\phi : A \to AB/B$ such that $\phi(a) = aB$ is a surjective homomorphism with kernel $A \cap B$.