Homework 1: Due Oct. 15.

① In class we saw that $|S_3| = 6$. Here are its elements:

$f_1(1) = 1$, $f_1(2) = 2$, $f_1(3) = 3$

$f_2(1) = 1$, $f_2(2) = 3$, $f_2(3) = 2$

$f_3(1) = 2$, $f_3(2) = 1$, $f_3(3) = 3$

$f_4(1) = 2$, $f_4(2) = 3$, $f_4(3) = 1$

$f_5(1) = 3$, $f_5(2) = 1$, $f_5(3) = 2$

$f_6(1) = 3$, $f_6(2) = 2$, $f_6(3) = 1$

Complete the following table

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_5			
f_3	f_3					
f_4	f_4					
f_5	f_5					
f_6	f_6					

[**Notice**: $f_2 \circ f_3 = f_5$]

[**Remark**: This is called the Cayley table of S_3.]

(2) Suppose (as above) $S_3 = \{f_1, f_2, \dots, f_6\}$. Show that there is a positive integer n s.t.

$$f_i^{(n)} = \underbrace{f_i \circ \dots \circ f_i}_{n \text{ times}} = f_1$$

(f_1 is the identity).

[For each $\underline{f_i}$ find such integer, then try to find an integer which works for all of them at the same time.]

(3) Let G be the group of symmetries of a square.

(a) How many elements does G have? Justify your answer.

(b) Are there two symmetries T and S of a square such that $T \circ S \neq S \circ T$?

④ Suppose k and n are two positive integers. Prove that $k \mid n$ or $k \mid n+1$ or $\cdots$ or $k \mid n+k-1$.

[Hint. Use divison algorithm for n and k to get: $\exists\, q, r \in \mathbb{Z}$ s.t. (1) $n = kq + r$
(2) $0 \le r < k$.]

⑤ Suppose $a, b \in \mathbb{Z}^{>0}$ and $n_0 \in \mathbb{Z}$. Prove that

$$\gcd(a + n_0 b, b) = \gcd(a, b).$$

[Hint: Think about $(a+nb)\mathbb{Z} + b\mathbb{Z}$ and compare it with $a\mathbb{Z} + b\mathbb{Z}$.]

⑥ Show that for any integer n we have

ⓐ $\gcd(n, n+1) = 1$.

ⓑ $\gcd(10n+3, 5n+2) = 1$

⑦ Do the following equations have an integer solution?

ⓐ $4x - 10y = 7$

ⓑ $21x + 35y = 20$

Justify your answer.

8. Find all integers x such that $3x+7$ is divisible by 11.

9. Prove that $\left.\begin{array}{l} n \mid x_1 - x_2 \\ n \mid y_1 - y_2 \end{array}\right\} \Rightarrow n \mid x_1 y_1 - x_2 y_2$.

10. (a) Prove that $\left.\begin{array}{l} \gcd(a,b) = 1 \\ d \mid b \end{array}\right\} \Rightarrow \gcd(a,d) = 1$

(b) Prove that $\left.\begin{array}{l} \gcd(a,n) = 1 \\ \gcd(b,n) = 1 \end{array}\right\} \Rightarrow \gcd(ab,n) = 1$.

[You are NOT allowed to use the unique factorization of integers as product of prime numbers.

<u>Hint</u> (a) Think about integer linear combinations.

(b) $d = \gcd(ab,n) \Rightarrow \left.\begin{array}{l} d \mid n \\ \gcd(b,n) = 1 \end{array}\right\} \Rightarrow \gcd(b,d) = 1$

Continue using $d \mid ab$.

(b)' Alternative approach: think about

$$\begin{cases} a\mathbb{Z} + n\mathbb{Z} = \mathbb{Z} \\ b\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}. \end{cases}$$

(b)'' Easiest way: $(ax_1+ny_1)(bx_2+ny_2)$

$$= ab\,x_1x_2 + n\mathbb{Z}. \;]$$