The sixth problem set.

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1. Let G be a group and X be a set.

(a) Suppose $(g,x) \longmapsto g\cdot x$ is a left action.

Prove that $x \odot g := g^{-1}\cdot x$ defines a right action.

(b) Let $F(X) := \{ f : X \longrightarrow \mathbb{C} \}$. Suppose $G \curvearrowright X$.

$\forall f \in F(X)$, $g \in G$, $x \in X$, let $(g * f)(x) := f(g^{-1}.x)$

Prove that $*$ defines an action of G on $F(X)$.

2. Let $\mathbb{H}^2 := \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$.

(a) Show that $\operatorname{Im}\left(\dfrac{az+b}{cz+d}\right) = \dfrac{(ad-bc)\operatorname{Im}(z)}{|cz+d|^2}$.

for any $a,b,c,d \in \mathbb{R}$ and $z \in \mathbb{C}$.

[<u>Hint.</u> $\operatorname{Im}(\omega) = \frac{1}{2i}(\omega - \overline{\omega})$.

$\Rightarrow \operatorname{Im}\left(\dfrac{az+b}{cz+d}\right) = \dfrac{1}{2i}\left(\dfrac{az+b}{cz+d} - \dfrac{a\overline{z}+b}{c\overline{z}+d}\right)$.]

In particular, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathbb{H}^2$,

then $\dfrac{az+b}{cz+d} \in \mathbb{H}^2$.

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(b) Show that the following defines an action of $SL_2(\mathbb{R})$ on $\mathbb{H}^2$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}.$$

3. Let G be a group and X be a set. Suppose $\theta: G \times X \to X$ defines a group action, $\theta(g,x) = g \cdot x$.

(a) Prove that $\rho_\theta(g_1) \circ \rho_\theta(g_2) = \rho_\theta(g_1 \circ g_2)$

where $\rho_\theta(g): X \to X$, $\rho_\theta(g)(x) := g \cdot x$.

(b) Prove that $\rho_\theta(e) = \mathrm{id}_X$ and

$$\rho_\theta(g) \circ \rho_\theta(g^{-1}) = \rho_\theta(g^{-1}) \circ \rho_\theta(g) = \mathrm{id}_X.$$

In particular, $\rho_\theta(g) \in S_X$ and $\rho_\theta(g^{-1}) = \rho_\theta(g)^{-1}$.

[So $\rho_\theta: G \to S_X$ is a well-defined group homomorphism.]

4. Let G be a group and X be a set. Suppose

$$\rho: G \to S_X$$

be a group homomorphism, i.e.

$$① \ \rho(e) = \mathrm{id}_X \qquad ② \ \rho(g_1 \cdot g_2) = \rho(g_1) \circ \rho(g_2).$$

Let's define $g \cdot x := \rho(g)(x)$. Prove that $\cdot$ defines a group action.

5. Let H_1, H_2 be two subgroups of G. Let

$$H_1 \cdot H_2 := \{ h_1 h_2 \mid h_1 \in H_1 , h_2 \in H_2 \}.$$

Prove that $H_1 H_2$ is a subgroup if and only if

$$H_1 H_2 = H_2 H_1 .$$

[$(\Rightarrow)$ Use inverse.

$(\Leftarrow)$ Use <u>subgroup criteria</u>.]

6. (a) Show that in general the following is NOT a well-defined function:

$$\theta : {}_H\backslash^G \longrightarrow G/_H , \quad \theta(Hg) = gH .$$

(b) Show that for any $H \leq G$ the following is a well-defined bijection:

$$\theta : H\backslash G \to G/H, \quad \theta(Hg) = g^{-1}H.$$

(Hint. For part (a) consider

$G = S_3$ and $H = \{id., f\}$ where

$$1 \overset{f}{\longmapsto} 2, \quad 2 \longmapsto 1, \quad 3 \longmapsto 3.)$$

7. Let $H \leq G$. Show that

$$\theta : H\backslash G \longrightarrow G/H, \quad \theta(Hg) = gH$$

is well-defined if and only if for any $g \in G$ we have $Hg = gH$.

(Remark. This kind of subgroup is called a normal subgroup. Alternatively we can write $g^{-1}Hg = H$ for any $g \in G$.)