The seventh problem set.

Monday, November 17, 2014
3:30 PM

1. Let $\sigma = c_1 \circ c_2 \circ \cdots \circ c_s$ and c_i's be disjoint cycles. Suppose $c_i = (a_{i1}, a_{i2}, \ldots, a_{ik_i})$ and $k_i \geq 2$. Prove that $o(\sigma) = \operatorname{lcm}(k_1, \ldots, k_s)$.

(Hint ① In class we considered the natural action of $\langle \sigma \rangle$ on $\{1, 2, \ldots, n\}$ and proved that the orbit of a_{i1} under this action is $\{a_{i1}, \ldots, a_{ik_i}\}$.

② $c_i \circ c_j = c_j \circ c_i \Rightarrow \sigma^\ell = c_1^\ell \circ \cdots \circ c_s^\ell$ for any integer ℓ.)

2. Prove that $\langle (1,2), (2,3), \ldots, (n-1,n) \rangle = S_n$.

(Hint. ① We have proved that any permutation is a product of transpositions. i.e.

$$\langle (i,j) \mid 1 \leq i < j \leq n \rangle = S_n.$$

② We proved $(i+1, i+2)(i+2, i+3) \cdots (j-1, j)$

$$= (i+1, i+2, \ldots, j)$$

③ Consider $(j, j-1, \ldots, i+1)\,(i, i+1)\,(j, j-1, \ldots, i+1)^{-1}$.)

3. Prove that $\langle (1,2), (1,2,\ldots,n)\rangle = S_n$

(Hint. Consider $(1,2,\ldots,n)^{i}\,(1,2)\,(1,2,\ldots,n)^{-i}$

and use problem 2.)

Use problem 1, to answer the following questions.

4. (a) Show that an element of order 5 in S_9 is a 5-cycle. Conclude that S_9 has $9\times 8\times 7\times 6$ many elements of order 5.

(b) Show that an element of order 5 in S_{10} is either a 5-cycle or a product of two disjoint cycles. Find the number elements of order 5 in S_{10}.

5. Show that $o(aba^{-1}) = o(b)$ and $o(ab) = o(ba)$.

CONJUGACY CLASSES OF S_n.

Any permutation $\sigma \in S_n$, as we proved in the class, can be uniquely written as a product of disjoint cycles c_i. Suppose length of c_i is k_i and

$$\sigma = c_1 \circ c_2 \circ \cdots \circ c_\ell \ , \quad 2 \le k_1 \le k_2 \le \cdots \le k_\ell .$$

The cyclic type of σ is $\underbrace{1, \ldots, 1}_{n-(k_1+\cdots+k_\ell)}, k_1, k_2, \ldots, k_\ell$.

For instance the cyclic type of (1) is

$$\underbrace{1, 1, \ldots, 1}_{n \text{ times}} .$$

- The cyclic type of $\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 2 & 6 & 7 & 6 & 5 & 1 \end{array}$

$1 \rightarrow 4 \rightarrow 7 \qquad 2 \qquad 3 \rightarrow 6 \rightarrow 5$

So the cyclic type is $1, 3, 3$.

- The cyclic type of $(1,2,3) \in S_7$ is

$$1, 1, 1, 1, 3$$

$1 \rightarrow 2 \rightarrow 3 \qquad 4 \qquad 5 \qquad 6 \qquad 7$

6. Let $\tau \in S_{2015}$ and $\sigma = (1,2,3)(3,4)(6,7)$.

Find the cyclic types of σ and $\tau\sigma\tau^{-1}$.

(Justify your answer.)

Hint. $\tau\sigma_1\sigma_2\tau^{-1} = (\tau\sigma_1\tau^{-1})(\tau\sigma_2\tau^{-1})$.

Remark. You can see that the same argument shows that σ and $\tau\sigma\tau^{-1}$ have the same cyclic type for any $\sigma, \tau \in S_n$.

7. Show that $\exists\, \tau \in S_{12}$ s.t. $\sigma_2 = \tau\sigma_1\tau^{-1}$

where $\sigma_1 = (1,2)(3,4,5)(6,7)$

and $\sigma_2 = (1,3)(6,10,12)(8,9)$.

Remark. You can see that the same argument shows that, if σ_1 and σ_2 have the same cyclic type, then $\exists\, \tau \in S_n$, $\sigma_2 = \tau\sigma_1\tau^{-1}$.

Remark. By definition, you can see that

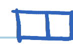
$$1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$$

is a cyclic type of an element of S_n if and only if $m_1 + m_2 + \cdots + m_k = n$.

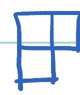
The number of ways $\underline{n}$ can be written as a sum of increasing positive integers is denoted by $p(n)$. For instance,

$p(1) = 1$

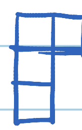
$p(2) = 2$ $1+1$ and 2

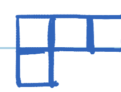


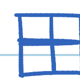
$p(3) = 3$ $1+1+1$, $1+2$, 3



$p(4) = 5$ $1+1+1+1$, $1+1+2$, $1+3$, $2+2$,







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By the above remarks, you can see that

> the number of conjugacy classes of S_n is equal to $p(n)$.