The eighth problem set

Wednesday, November 26, 2014

3:21 PM

1. Suppose $N \trianglelefteq S_n$ and $(1,2) \in N$. Prove that $N = S_n$.

(<u>Hint</u>. For any $i<j$, let $\sigma \in S_n$ be s.t.

$$\sigma(1)=i \quad \text{and} \quad \sigma(2)=j.$$

And consider $\sigma\,(1,2)\,\sigma^{-1}$. Then use the fact that any permutation is a product of transpositions.)

2. Let $\phi : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$, $\phi([x]_n) = a\,[x]_n$.

   (i) Prove that ϕ is a homomorphism.

   (ii) Find $|\operatorname{Im}\phi|$ and $|\ker\phi|$.

3. Let $H \leq G$. Suppose $[G:H]=2$. Prove that

$$H \trianglelefteq G.$$

(<u>Hint</u>. $G/H = \{H, g_0 H\}$. Proceed by contradiction to show ① $g_0^{-1}H = g_0 H$

② $\forall h \in H$, $hg\,H = g\,H$.

Conclude $g_0^{-1} H g_0 = H$.)

4. Let $H \leq G$. Show that $m : G/H \times G/H \to G/H$,

$$m(g_1 H, g_2 H) = g_1 g_2 H$$

is well-defined if and only if $H \trianglelefteq G$.

(Hint. ($\Longrightarrow$) $\forall h \in H$, $H = hH \Rightarrow m(H, gH) = m(hH, gH)$

$$\Rightarrow gH = hgH \quad \cdots$$

($\Longleftarrow$) $\left.\begin{array}{l} g_1 H = g_1' H \\ g_2 H = g_2' H \end{array}\right\} \overset{?}{\Longrightarrow} g_1 g_2 H = g_1' g_2' H$.

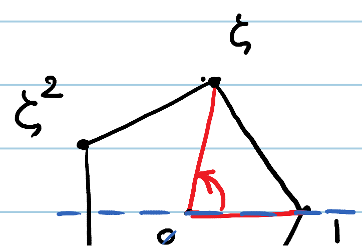
Use the usual criteria: $gH = g'H \Longleftrightarrow g^{-1} g' = h \in H$)

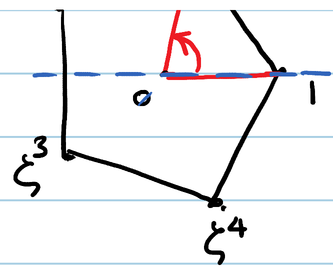
5. Let $Z(G) = \{ g \in G \mid \forall g' \in G,\ gg' = g'g \}$.

Show that $Z(G) \trianglelefteq G$.

6. Let G be the group of symmetries of the regular pentagon:

$\zeta = e^{\frac{2\pi i}{5}}$.





Let a be the reflection about the x-axis and b be the $\frac{2\pi}{5}$-rotation about the origin.

Let $H = \langle b \rangle$.

(i) List all the elements of G in terms of a and b. (And, in particular, conclude $G = \langle a, b \rangle$.)

(ii) Show that $H \lhd G$.

(<u>Hint</u>: Let σ be a symmetry of the (given) regular pentagon $\Rightarrow$ $\sigma(1)$ is one of the vertices

$\Rightarrow$ after rotating by a multiple of $\frac{2\pi}{5}$ we can bring $\sigma(1)$ back to 1, i.e.

$$b^i \sigma(1) = 1$$

Since $b^i \sigma$ is a symmetry and fixes 1, ...

If a symmetry fixes three non-linear points, it fixes the entire plane.)

7. (i) Show that $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ is a group with complex multiplication.

(ii) Show that $f : \mathbb{Z} \to S^1$, $f(m) = \zeta_n^m$ is a group homomorphism where $\zeta_n = e^{\frac{2\pi i}{n}}$.

(iii) Find $\ker f$ and $|\operatorname{Im} f|$.