Abstract algebra : Group Theory

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. APM 7230

. Office hour WF 4-5.

. Course webpage: Go through my webpage.

. Please go to the course webpage for the needed information about all aspects of this class including homework, grades, and exams.

. There will be two in-class exams and your weighted score is the best of

HW 20% + Exam I 20% + Exam II 20% + Final 40%

HW 20% + best of midterms 20% + Final 60%

If more than 90% of the students fill out the CAPE questioner, all the students get ONE ADDITIONAL POINT towards their weighted score.

. You should be completely comfortable with the topics of

Math 109 (B and more):

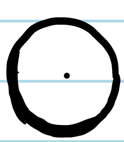
function, bijection, injection, surjection; quantifiers; induction, etc.

- The focus of this course is Group Theory.
- Groups are all over mathematics, physics, computer science
- How do groups appear in various parts of mathematics or other sciences?

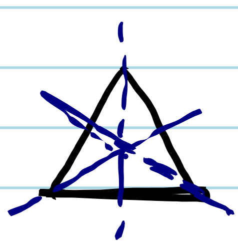
One of the key methods to study and identify any object or structure is to investigate its symmetries.

- What do we mean by "symmetries"?

It depends on our point of view. Here are a few examples:

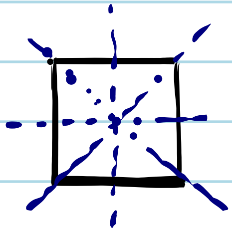
(1)  (unit circle): • rotations about its center

• reflection about a line which passes through its center. [infinite]

(2)  (equilatral triangle): $\pm \frac{\pi}{3}$ rotation

. reflection about three axis.

. Identity [finite]

(3)  (square): • $\pm \pi/2$, π rotation.

• reflections about 4 axis.

. Identity.

In the above examples, what do we mean by symmetry?

"Symm"$(X) = \{ f : X \to X \mid$ ① f is a bijection
② f preseres the "structure" of X $\}$

For instance in the above example $\underline{\underline{f}}$ should preserve distance (and angle).

(4) In the above sense, what do you think the "group" of symmetries of a set with $\underline{\underline{n}}$ elements should be?

$X = \{ x_1, \dots, x_n \}$

$\mathrm{Sym}(X) := \{ f : X \to X \mid f \text{ is a bijection} \}.$

<u>**Def.**</u> $\mathrm{Sym}(\{1, 2, \dots, n\})$ is called the <u>symmetric group</u> and it is denoted by S_n.

(5) What is the order of S_3?

It is the number of permutations of 1, 2, 3.

As you have learned in <u>109</u>, it is $3! = 6$.

In general $|S_n| = n!$.

(6) If we view $\mathbb{R}^2$ as a <u>vector space</u>, what are the "Symmetries" of $\mathbb{R}^2$?

We are looking for functions

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

s.t. (1) f is a bijection.

(2) f is <u>linear</u>.

In linear algebra, you have learned that any linear map can be represented by a matrix, and a square matrix is <u>invertible</u> if and only if its determinant is not zero. So

$$GL_2(\mathbb{R}) = \{ A \in M_2(\mathbb{R}) \mid A \text{ is invertible} \}$$

$$= \{ A \in M_2(\mathbb{R}) \mid \exists x \in \mathbb{R},\ \det(A)\, x = 1 \}$$

Let's think about "$\underset{\text{Symm}}{\overset{G}{\shortparallel}}$"$(X)$ in a bit more abstract way.

Q No matter what X is what can we say about "Symm"(X)?

ⓐ $Id._X \in G$.

ⓑ $f \in G \Rightarrow f^{-1} \in G$.

ⓒ $f, g \in G \Rightarrow f \circ g \in G$

<u>Def.</u> $(G, *)$ is called a group if

ⓐ $\forall g_1, g_2 \in G$, $g_1 * g_2 \in G$

(multiplication is a function $* : G \times G \longrightarrow G$.)

ⓑ $\exists e \in G$, $\forall g \in G$, $e * g = g * e = g$

(a neutral element)

ⓒ $\forall g \in G, \exists g' \in G$, $g * g' = g' * g = e$

ⓓ $\forall g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$.