Lecture 15 : group action.

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9:14 AM

<u>Define</u>. Let X be a set, and G be a group.

We say G acts on X (from left) via $\theta : G \times X \to X$

when θ has the following properties:

① $\theta(e, x) = x$

② $\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x)$.

<u>Remark</u> ① Similar to the operation of a group,

θ is usually written as a "multiplication":

$$g \cdot x := \theta(g, x).$$

② We denote it by $G \curvearrowright X$ and then explain how it acts. A group G can act on a set X is various ways.

③ G can act from <u>right</u> $\longmapsto$ $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$

There is a bijection between the right actions of G on X and the left actions of G on X.

(Since it is easy, I put it in the next week's HW assignment.)

Examples of group actions

① Let $H \le G$. Then $H \curvearrowright G$ by left multiplication

I.e. $h \cdot g := hg$ $\qquad \theta(h,g) := hg$.

(i) $e \cdot g = eg = g$

(ii) $h_1 \cdot (h_2 \cdot g) = h_1 \cdot (h_2 g) = h_1(h_2 g)$

$$= (h_1 h_2) g$$

$$= (h_1 h_2) \cdot g.$$

② $G \curvearrowright G$ by conjugation. I.e.

$$g \cdot g' := g g' g^{-1}$$

(i) $e \cdot g' = e g' e^{-1} = g'$

(ii) $g_1 \cdot (g_2 \cdot g') = g_1 \cdot (g_2 g' g_2^{-1})$

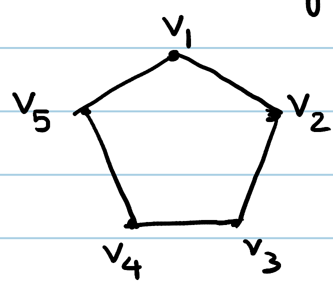
$$= g_1 (g_2 g' g_2^{-1}) g_1^{-1}$$

$$= (g_1 g_2)\, g' \,(g_2^{-1} g_1^{-1})$$

$$= (g_1 g_2)\, g' \,(g_1 g_2)^{-1}$$

$$= (g_1 g_2) \cdot g'$$

③ Symmetries of a regular pentagon.



Let G be the group of symmetries of a regular pentagn. Then $G \curvearrowright \{v_1, v_2, v_3, v_4, v_5\}$ (vertices)

$g \cdot v_i := g(v_i)$

$id \cdot v_i = v_i$

$g_1 \cdot (g_2 \cdot v_i) = g_1 \cdot (g_2(v_i))$

$= g_1(g_2(v_i))$

$= (g_1 \circ g_2)(v_i)$

$= (g_1 \circ g_2) \cdot v_i$

This is true for the group of <u>symmetries</u> of any object or structure.

④ $S_n \curvearrowright \{1, 2, \dots, n\}$

$\sigma \cdot i := \sigma(i)$.

⑤ $SL_2(\mathbb{R}) \curvearrowright$ upper half plane

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az+b}{cz+d}$$

(In the next week's HW assignment you will check that it is a (well-defined) action.)

[These are called Möbius transformations.]

⑥ Rotation about the origin acts on $\mathbb{R}^2$.

⑦ $SL_2(\mathbb{Z}) \curvearrowright \mathbb{Z}^2$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$

<u>Proposition</u> $G \underset{\theta}{\curvearrowright} X \iff \rho_\theta : G \longrightarrow S_X$

(Not discussed in the lecture; we will talk about later.)

$\rho_\theta(g)(x) := \theta(g, x)$

is well-defined and it is a <u>homomorphism</u>, i.e.

- $\rho_\theta(g_1 g_2) = \rho_\theta(g_1) \circ \rho_\theta(g_2)$.
- $\rho_\theta(e) = \mathrm{id}_X$

Pf. ($\Rightarrow$) Clearly $\rho_\theta(g): X \to X$. First we show

$$\rho_\theta(g_1 g_2) = \rho_\theta(g_1) \circ \rho_\theta(g_2).$$

And then we will prove $\rho_\theta(g)$ is a bijection, and so it is in S_X.

$$\begin{aligned}\rho_\theta(g_1 g_2)(x) &= (g_1 g_2) \cdot x \\ &= g_1 \cdot (g_2 \cdot x) \\ &= g_1 \cdot (\rho_\theta(g_2)(x)) \\ &= \rho_\theta(g_1)(\rho_\theta(g_2)(x)) \\ &= (\rho_\theta(g_1) \circ \rho_\theta(g_2))(x) \Rightarrow \rho_\theta(g_1 g_2) = \rho_\theta(g_1) \circ \rho_\theta(g_2)\end{aligned}$$

$$\begin{aligned}\rho_\theta(e)(x) &= e \cdot x \\ &= x \qquad \Rightarrow \rho_\theta(e) = \mathrm{id}.\end{aligned}$$

So $\rho_\theta(g) \circ \rho_\theta(g^{-1}) = \rho_\theta(e) = \rho_\theta(g^{-1}) \circ \rho_\theta(g)$

$\Rightarrow \rho_\theta(g)$ is an invertible function from X to X.

$\Rightarrow \rho_\theta(g) \in S_X$. (In your first HW assignment you showed the case of $G \curvearrowright G$

by left multiplication.)

($\Leftarrow$) $e\cdot x = \rho_\theta(e)(x) = \mathrm{id.}(x) = x$.

$$
\begin{aligned}
g_1\cdot(g_2\cdot x) &= \rho_\theta(g_1)(g_2\cdot x)\\
&= \rho_\theta(g_1)(\rho_\theta(g_2)(x))\\
&= (\rho_\theta(g_1)\circ\rho_\theta(g_2))(x)\\
&= \rho_\theta(g_1g_2)(x)\\
&= (g_1g_2)\cdot x.
\end{aligned}
$$
$\blacksquare$

When we are given a group action, we should try to understand <u>its orbits</u> and <u>the space of orbits</u>.

<u>Def</u>.. Suppose $G \curvearrowright_\theta X$. The orbit of $x\in X$ under this group action is

$$O(x) := \{g\cdot x \mid g\in G\}.$$

. The set of all the orbits is denoted by ${}_G\backslash^X$.

$${}_G\backslash^X := \{O(x) \mid x\in X\}.$$

(If the group acts from right, we write $X/_G$.)

$\underline{\text{Exp.}}$ $G \curvearrowright G$ left multiplication $\Rightarrow$

① $O(g) = G$ for any $g \in G$

② $G \backslash G = \{O(e)\}$ has only one element.

$\underline{\text{Exp.}}$ $n\mathbb{Z} \curvearrowright \mathbb{Z}$ left addition $\Rightarrow$

① $O(x) = n\mathbb{Z} + x$ for any $x \in \mathbb{Z}$

② $n\mathbb{Z} \backslash \mathbb{Z} = \{n\mathbb{Z}, n\mathbb{Z}+1, \dots, n\mathbb{Z}+n-1\}$

$= \mathbb{Z}_n$.

$\underline{\text{Exp.}}$ Rotations about the origin $\curvearrowright \mathbb{R}^2$

① $O(\vec{v}) = \{\vec{w} \in \mathbb{R}^2 \mid \|\vec{v}\| = \|\vec{w}\|\}$

② $\text{rotat.} \backslash \mathbb{R}^2 = \{\text{circles centered at the origin}\}$

$\updownarrow$

$\mathbb{R}^{\geq 0}$ (nice parametrization)

[Polar coordinates].

$\underline{\text{Exp.}}$ $H \curvearrowright G$ by left multiplication.

① $O(g) = \{hg \mid h \in H\} =: Hg$ (right coset)

② $H\backslash G = \{ Hg \mid g \in G \}.$

$\Rightarrow$ $x_1 = g_0^{-1} \cdot x_2$