Lecture 17: orbits and the action of cyclic groups

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8:55 AM

Proposition $G \curvearrowright X$ and $x_0 \in X$. Then

(i) $G_{x_0} := \{g \in G \mid g \cdot x_0 = x_0\}$ is a subgroup.

(ii) $\theta : G/G_{x_0} \longrightarrow O(x_0)$, $\theta(g\, G_{x_0}) = g \cdot x_0$

is a well-defined bijection.

Pf. (i) $e \cdot x_0 = x_0 \Rightarrow e \in G_{x_0} \Rightarrow G_{x_0} \neq \emptyset$

So by Subgroup Criteria we have to check the following

$$g_1, g_2 \in G_{x_0} \overset{?}{\Rightarrow} g_1^{-1} g_2 \in G_{x_0}.$$

$$\left.\begin{array}{l} g_1 \cdot x_0 = x_0 \\ g_2 \cdot x_0 = x_0 \end{array}\right\} \Rightarrow g_1 \cdot x_0 = g_2 \cdot x_0 \Rightarrow (g_1^{-1} g_2) \cdot x_0 = x_0$$

$$\Rightarrow g_1^{-1} g_2 \in G_{x_0}.$$

(ii) <u>well-defined</u>.

$$g_1 G_{x_0} = g_2 G_{x_0} \Rightarrow g_1 = g_2 h \quad \text{for some } h \in G_{x_0}$$

$$\Rightarrow g_1 \cdot x_0 = (g_2 h) \cdot x_0$$

$$\Rightarrow g_1 \cdot x_0 = g_2 \cdot (h \cdot x_0)$$

$$\Rightarrow g_1 \cdot x_0 = g_2 \cdot x_0 .$$

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$$g_1 \cdot x_0 = g_2 \cdot x_0 \Rightarrow (g_1^{-1} g_2) \cdot x_0 = x_0$$
$$\Rightarrow g_1^{-1} g_2 = h \in G_{x_0}$$
$$\Rightarrow g_2 = g_1 h \in g_1 G_{x_0}$$
$$\Rightarrow g_2 G_{x_0} = g_1 G_{x_0} .$$

Onto It is clear from the definition of $O(x_0)$. ∎

Cor. If G is a finite group, then

$$|O(x_0)| = [G : G_{x_0}] \mid |G| .$$

Pf. By the previous Proposition, $|O(x_0)| = |G/G_{x_0}|$ which is $[G : G_{x_0}]$ by definition. And we have already proved $|G| = |G_{x_0}| [G : G_{x_0}] \Rightarrow$

$$|O(x_0)| \mid |G| .$$ ∎

Since the set of left cosets is of particular importance, let's summarize its properties:

- $g_1 H = g_2 H \iff g_1^{-1} g_2 \in H$.
- $H g_1 = H g_2 \iff g_1 g_2^{-1} \in H$.

How does a cyclic group act on a set?

Let's assume $\langle a \rangle$ is a finite group of order d.

Suppose $\langle a \rangle \curvearrowright X$. How does orbits "look like"?

$$\overset{x_0}{\bullet} \longrightarrow \overset{a \cdot x_0}{\bullet} \longrightarrow \overset{a^2 \cdot x_0}{\bullet} \longrightarrow \overset{a^3 \cdot x_0}{\bullet} \quad \cdots$$

At some point we should come back as $a^d = e$

and so $a^d \cdot x_0 = x_0$. And so we get a cycle.

① Size of this cycle divides d.

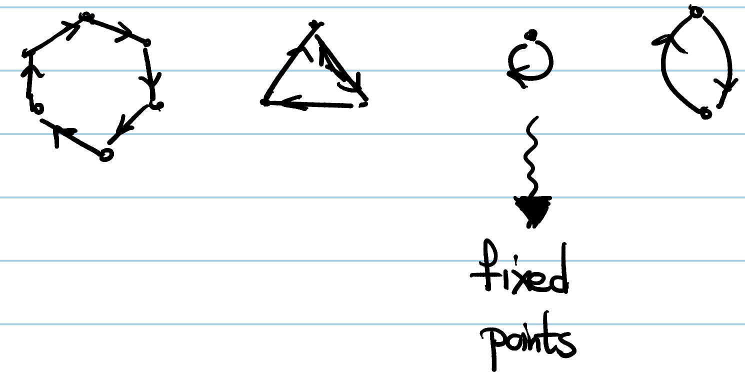
② Either this cycle is the entire X,

or take x_1 in X outside this cycle

and repeat.

So X is disjoint union of bunch of cycles

(whose size divides d) and a just "rotates" points on these cycles.



<u>Schreier directed graphs</u>: $G = \langle S \rangle \curvearrowright X$

vertices $= X$

(x_1, x_2) is an edge if $\exists\, s \in S$ s.t. $x_2 = s \cdot x_1$

So in the case of <u>finite cyclic group</u> we get the above cycles.

- We also discussed the following examples:

① $S_n \curvearrowright \{1, 2, \ldots, n\}$.

$G_n :=$ stablizer of n

$$= \{\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \mid \sigma(n) = n\}$$

So $|G_n| = (n-1)!$. G_n is <u>more or less</u> S_{n-1}.

$O(n) = \{1, 2, \ldots, n\}$.

$$[S_n : G_n] = |S_n| / |G_n| = n! / (n-1)! = |O(n)|.$$

② $G \curvearrowright G$ by conjugation, i.e.

$$g \cdot g' := g g' g^{-1}.$$

- $O(g') = \{ g g' g^{-1} \mid g \in G \} =: Cl(g')$

  is called the conjugacy class of g'.

- $g g' g^{-1}$ is called a conjugate of g'.

- Stablizer of $g' = \{ g \in G \mid g g' g^{-1} = g' \}$

$$C_G(g') = \{ g \in G \mid g g' = g' g \}$$

is called the centralizer of g' in G.

- So we have $|Cl(g')| = [G : C_G(g')]$.