Lecture 19: transpositions; Conjugation; Computation.

Monday, November 17, 2014
9:30 AM

Proposition. $(1) \neq \pi \in S_n$ can be uniquely written as a product of disjoint cycles.

Pf. (Let me go over the proof that I presented in the previous lecture with a different wording.)

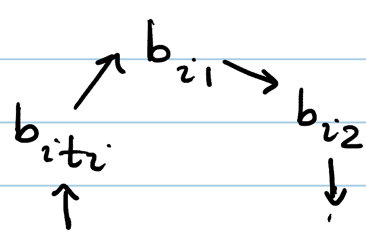
$$\pi = (a_{11}, a_{12}, \dots, a_{1k_1}) \cdots (a_{\ell 1}, a_{\ell 2}, \dots, a_{\ell k_\ell})$$

$$= \underbrace{(b_{11}, b_{12}, \dots, b_{1t_1})}_{c_1} \cdots \underbrace{(b_{s1}, \dots, b_{st_s})}_{c_s}.$$

We would like to show $(a_{11}, \dots, a_{1k_1}) = (b_{i1}, \dots, b_{it_i})$ for some i. Then after cancelation we can use induction on ℓ.

$$\pi(a_{11}) = a_{12} \neq a_{11} \Rightarrow a_{11} = b_{ij} \text{ for some } i \text{ and } j.$$

Notice that



$(b_{i1}, \dots, b_{it_i})$

$= (b_{ij}, b_{ij+1}, \dots, b_{it_i}, b_{i1}, \dots, b_{ij-1})$

So w.l.o.g. we can assume that $j=1$.

We also notice that c_k's commute. And so w.l.o.g. we can assume $i=1$. I.e. $a_{11}=b_{11}$.

$\Rightarrow \; a_{12}=\pi(a_{11})=\pi(b_{11})=b_{12}$

$\Rightarrow$ Repeating this we get $(a_{11},\dots,a_{1\ell_1})=(b_{11},\dots,b_{1t_1})$.

So we can cancel them and proceed by induction on ℓ. $\blacksquare$

<u>Def.</u> A 2-cycle is called a transposition.

<u>Cor.</u> Any permutation is a product of transpositions.

<u>Pf.</u> It is enough to show any cycle is a product of transpositions:

$$(a_1,a_2,\dots,a_n)\overset{?}{=}(a_1,a_2)(a_2,a_3)\cdots(a_{n-1},a_n)$$

$$(a_1,a_2)(a_2,a_3)\cdots(a_{n-1},a_n)\,\underline{a_j}= \quad \boxed{\text{if } j<n}$$

$$(a_1,a_2)\cdots(a_{j-1},a_j)(a_j,a_{j+1})\,\underline{a_j}=\underline{a_{j+1}}$$

$$(a_1,a_2)\cdots(a_{n-1},a_n)\,\underline{a_n}=\underline{a_1}.$$

$\square$

Cor. A k-cycle is a product of $\underline{k-1}$ transpositions.

Lemma. $\tau(a_1,\dots,a_n)\tau^{-1} = (\tau(a_1),\dots,\tau(a_n))$

Pf. $\tau(a_1,\dots,a_n)\tau^{-1}\ \underline{\tau(a_i)} = \tau(a_1,\dots,a_n)\ \underline{a_i}$

$$= \begin{cases} \tau(a_{i+1}) & \text{if } i\neq n \\ \tau(a_1) & \text{if } i=n. \end{cases}$$

$$x\notin\{\tau(a_1),\dots,\tau(a_n)\} \Rightarrow \tau^{-1}(x)\notin\{a_1,\dots,a_n\}$$

$$\Rightarrow (a_1,\dots,a_n)\ \underline{\tau^{-1}(x)} = \underline{\tau^{-1}(x)}$$

$$\Rightarrow \tau(a_1,\dots,a_n)\tau^{-1}\ \underline{x} = \underline{x}.$$

■

Def. The cyclic type of π is $2\leq \ell(c_1)\leq \ell(c_2)\leq\dots\leq \ell(c_\ell)$

where $\pi = c_1\cdot\dots\cdot c_\ell$ and c_i's are disjoint cycles.

Proposition. π_1 and π_2 are conjugate in S_n

if and only if they have the same <u>cyclic type</u>.

($\Rightarrow$)

Pf. $\pi_1 = c_1\cdot\dots\cdot c_\ell \Rightarrow \tau\pi_1\tau^{-1} = \tau(c_1\cdots c_\ell)\cdot\tau^{-1}$

$$= (\tau c_1\tau^{-1})\cdot(\tau c_2\tau^{-1})\cdot\dots\cdot(\tau c_\ell\tau^{-1})$$

By the above lemma, $\tau c_i\tau^{-1}$'s are disjoint cycles and $\underline{\text{length of } c_i = \text{length } \tau c_i \tau^{-1}}$. So π_1 and $\tau\pi_1\tau^{-1} = \pi_2$ have the same cyclic type.

$(\Leftarrow)$ $(a_{11}, \dots, a_{1\ell_1})(a_{21}, \dots, a_{2\ell_2}) \cdots (a_{k1}, \dots, a_{k\ell_k}) = \pi_1$

and $(b_{11}, \dots, b_{1\ell_1})(b_{21}, \dots, b_{2\ell_2}) \cdots (b_{k1}, \dots, b_{k\ell_k}) = \pi_2$

are products of disjoint cycles.

So $\exists\, \tau \in S_n$ s.t. $a_{ij} \mapsto b_{ij}$

$\Rightarrow \tau\pi_1\tau^{-1} = \pi_2$. $\blacksquare$

<u>Proposition</u>. $\pi = \tau_1 \cdot \dots \cdot \tau_r = \rho_1 \cdot \dots \cdot \rho_s$

and τ_i's and ρ_j's are transpositions

$\Longrightarrow r \equiv s \pmod 2$.

<u>Pf.</u> $\tau_1 \cdot \dots \cdot \tau_r = \rho_1 \cdot \dots \cdot \rho_s \Rightarrow \tau_1 \cdot \dots \cdot \tau_r \cdot \rho_s \cdot \dots \cdot \rho_1 = (1)$

So (1) is written as product of $r+s$ transpositions, and we would like to prove $r+s$ is <u>even</u>.

We introduce a process through which at each step the <u>parity of the number</u> of transpositions does not change and at the same time we eventually end up getting $\underline{\underline{0}}$ transpositions.

We bring $\underline{\underline{a}}$ to the left terms:

$$(a,x)(y,z) = (y,z)(a,x)$$

$$(x,y)(a,x) = (a,y)(x,y)$$

We get rid of $\underline{\underline{a}}$:

$$(a,x)(a,y) = (a,y)(y,x)$$

There cannot be only $\underline{\underline{1}}$ transposition involving $\underline{a}$.

$$(a,x)(a,x) = (1)$$

Through this process, we get rid of all the numbers without changing the parity of the number of transpositions so $2 \mid r+s \implies r \equiv s \pmod 2$. $\square$

<u>Def/Cor.</u> • $\pi \in S_n$ is called even if it can be written as a product of even number of transpositions. otherwise it is called odd.

- The sign map $\mathrm{sgn}: S_n \to \{\pm 1\}$,

$$\mathrm{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

__Proposition.__ ① $\mathrm{sgn}(\pi_1 \pi_2) = \mathrm{sgn}(\pi_1)\,\mathrm{sgn}(\pi_2)$ — it is a homomorphism.

② $\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi^{-1})$.

③ $\mathrm{sgn}(c) = (-1)^{k-1}$ if c is a k-cycle.

__Def./Cor.__ $A_n := \{\pi \in S_n \mid \pi \text{ is even}\}$ is a subgroup.

it is called __the alternating group__ on n elements.